LINEAR FUNCTIONAL ANALYSIS

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Chapter 7

INTRODUCTION TO LINEAR TRANSFORMATIONS

7.1. Introduction

In this chapter, we are concerned with linear mappings from one vector space to another. Recall that the special case when the second vector space is the underlying field of the first vector space is the subject of discussion in the last chapter. Our purpose here is to elaborate on the ideas there and generalize the results.

Definition. Suppose that V and W are vector spaces over $\mathbb{F}$. By a linear transformation from V to W, we mean a mapping $T : V \to W$ satisfying the following conditions:
(LT1) For every $\mathbf{x}, \mathbf{y} \in V$, we have $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$.
(LT2) For every $c \in \mathbb{F}$ and $\mathbf{x} \in V$, we have $T(c\mathbf{x}) = cT(\mathbf{x})$.

Definition. Suppose that V is a vector space over $\mathbb{F}$. A linear transformation $T : V \to V$ is called a linear operator on V.

Remark. Note that in the special case when $W = \mathbb{F}$, a linear transformation $T : V \to \mathbb{F}$ is simply a linear functional on V.

Definition. Suppose that V and W are normed vector spaces over $\mathbb{F}$. A linear transformation $T : V \to W$ is said to be bounded if there exists a real number $M \geq 0$ such that $\|T(\mathbf{x})\| \leq M\|\mathbf{x}\|$ for every $\mathbf{x} \in V$.

Example 7.1.1. Consider the normed vector space $C[0, 1]$ of all continuous complex valued functions on $[0, 1]$, with supremum norm
$$\|f\| = \sup_{t \in [0,1]} |f(t)|.$$
It is easy to check that the mapping $T : C[0, 1] \to \mathbb{C}$, given by $T(f) = f(0)$ for every $f \in C[0, 1]$, is a linear transformation. Furthermore, for every $f \in C[0, 1]$, we have $|T(f)| = |f(0)| \leq \|f\|$. It follows that $T : C[0, 1] \to \mathbb{C}$ is bounded, with $|T(f)| \leq M\|f\|$ for every $f \in C[0, 1]$, where $M = 1$.

EXAMPLE 7.1.2. Consider the normed vector space ℓ^2 of all square summable infinite sequences of complex numbers, with norm

$$\|\mathbf{x}\| = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2}.$$

Let $\mathbf{c} = (c_1, c_2, c_3, \ldots) \in \ell^\infty$ be chosen and fixed, and consider a mapping $T : \ell^2 \to \ell^2$, where for every $\mathbf{x} = (x_1, x_2, x_3, \ldots)$, we have $T(\mathbf{x}) = (c_1x_1, c_2x_2, c_3x_3, \ldots)$. It is not difficult to check that $T : \ell^2 \to \ell^2$ is a linear operator. Furthermore, for every $\mathbf{x} \in \ell^2$, we have

$$\|T(\mathbf{x})\|^2 = \sum_{i=1}^{\infty} |c_ix_i|^2 \le \left(\sup_{i\in\mathbb{N}} |c_i|\right)^2 \sum_{i=1}^{\infty} |x_i|^2 = \left(\sup_{i\in\mathbb{N}} |c_i|\right)^2 \|\mathbf{x}\|^2.$$

It follows that $T : \ell^2 \to \ell^2$ is bounded, with $\|T(\mathbf{x})\| \le M\|\mathbf{x}\|$ for every $\mathbf{x} \in \ell^2$, where

$$M = \sup_{i\in\mathbb{N}} |c_i| = \|\mathbf{c}\|_\infty,$$

the supremum norm of $\mathbf{c}$ in ℓ^∞.

EXAMPLE 7.1.3. Suppose that $a, b \in \mathbb{R}$ and $a < b$. Consider the normed vector space $L^2[a, b]$ of all complex valued Lebesgue measurable functions that are square integrable on $[a, b]$, with norm

$$\|f\| = \left(\int_a^b |f(t)|^2 \,\mathrm{d}t\right)^{1/2}.$$

Let $\phi \in C[a, b]$ be chosen and fixed, and consider a mapping $T : L^2[a, b] \to L^2[a, b]$, where for every $f \in L^2[a, b]$, the function $T(f) \in L^2[a, b]$ is defined by

$$(T(f))(t) = \phi(t)f(t) \qquad \text{for every } t \in [a, b].$$

It is not difficult to check that $T : L^2[a, b] \to L^2[a, b]$ is a linear operator. Furthermore, for every $f \in L^2[a, b]$, we have

$$\|T(f)\|^2 = \int_a^b |\phi(t)f(t)|^2 \,\mathrm{d}t \le \left(\sup_{t\in[a,b]} |\phi(t)|\right)^2 \int_a^b |f(t)|^2 \,\mathrm{d}t = \left(\sup_{t\in[a,b]} |\phi(t)|\right)^2 \|f\|^2.$$

It follows that $T : L^2[a, b] \to L^2[a, b]$ is bounded, with $|T(f)| \le M\|f\|$ for every $f \in L^2[a, b]$, where

$$M = \sup_{t\in[a,b]} |\phi(t)| = \|\phi\|_\infty,$$

the supremum norm of ϕ in $C[a, b]$.

EXAMPLE 7.1.4. Suppose that $a, b, c, d \in \mathbb{R}$, with $a < b$ and $c < d$. Consider the normed vector spaces $L^2[a, b]$ and $L^2[c, d]$, with respective norms

$$\|f\| = \left(\int_a^b |f(t)|^2 \,\mathrm{d}t\right)^{1/2} \qquad \text{and} \qquad \|h\| = \left(\int_c^d |h(s)|^2 \,\mathrm{d}s\right)^{1/2}.$$

Let $\phi : [c, d] \times [a, b] \to \mathbb{C}$ be a fixed continuous function, and consider a mapping $T : L^2[a, b] \to L^2[c, d]$, where for every $f \in L^2[a, b]$, the function $T(f) \in L^2[c, d]$ is given by

$$(T(f))(s) = \int_a^b \phi(s, t)f(t) \,\mathrm{d}t \qquad \text{for every } s \in [c, d].$$

It is not difficult to check that $T : L^2[a,b] \to L^2[c,d]$ is a linear transformation. Furthermore, for every $f \in L^2[a,b]$ and every $s \in [c,d]$, the Cauchy-Schwarz inequality gives

$$|(T(f))(s)|^2 \le \left(\int_a^b |\phi(s,t)|^2 \,\mathrm{d}t\right)\left(\int_a^b |f(t)|^2 \,\mathrm{d}t\right) = \left(\int_a^b |\phi(s,t)|^2 \,\mathrm{d}t\right)\|f\|^2,$$

and so

$$\|T(f)\|^2 = \int_c^d |(T(f))(s)|^2 \,\mathrm{d}s \le \left(\int_c^d\int_a^b |\phi(s,t)|^2 \,\mathrm{d}t\mathrm{d}s\right)\|f\|^2.$$

It follows that $T : L^2[a,b] \to L^2[c,d]$ is bounded, with $|T(f)| \le M\|f\|$ for every $f \in L^2[a,b]$, where

$$M = \left(\int_c^d\int_a^b |\phi(s,t)|^2 \,\mathrm{d}t\mathrm{d}s\right)^{1/2}.$$

Example 7.1.5. Consider the normed vector space $L^2(\mathbb{R})$ of all complex valued Lebesgue measurable functions that are square integrable on $\mathbb{R}$, with norm

$$\|f\| = \left(\int_{-\infty}^{\infty} |f(t)|^2 \,\mathrm{d}t\right)^{1/2}.$$

Let V denote the subset of $L^2(\mathbb{R})$ consisting of all those functions $f \in L^2(\mathbb{R})$ that are differentiable and such that the derivative $f' \in L^2(\mathbb{R})$. We shall first show that V is a linear subspace of $L^2(\mathbb{R})$. Suppose that $f, g \in V$ and $c \in \mathbb{C}$. Then clearly $f + g, cf \in L^2(\mathbb{R})$ are differentiable. Furthermore, we have $f', g' \in L^2(\mathbb{R})$, and for every $t \in \mathbb{R}$, we have

$$|(f+g)'(t)|^2 = |f'(t) + g'(t)|^2 \le (|f'(t)| + |g'(t)|)^2 \le 2|f'(t)|^2 + 2|g'(t)|^2$$

and

$$|(cf)'(t)|^2 = |cf'(t)|^2 = |c|^2|f'(t)|^2,$$

so that $(f+g)', (cf)' \in L^2(\mathbb{R})$. It follows that $f + g, cf \in V$, and so V is a linear subspace of $L^2(\mathbb{R})$. Consider now a mapping $T : V \to L^2(\mathbb{R})$, where $T(f) = f'$ for every $f \in V$. It is not difficult to check that $T : V \to L^2(\mathbb{R})$ is a linear transformation. However, this linear transformation is not bounded. To see this, consider the sequence $(f_n)_{n\in\mathbb{N}}$ of functions in V, where for every $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$f_n(t) = \begin{cases} 0 & \text{if } t \le -n^{-1}, \\ 2(nt+1)^2 & \text{if } -n^{-1} \le t \le -(2n)^{-1}, \\ 1 - 2n^2t^2 & \text{if } -(2n)^{-1} \le t \le (2n)^{-1}, \\ 2(nt-1)^2 & \text{if } (2n)^{-1} \le t \le n^{-1}, \\ 0 & \text{if } t \ge n^{-1}. \end{cases}$$

Here each function f_n is made up of two half-lines and parts of three parabolas. It can be checked that $\|f_n\| \to 0$ and $\|f_n'\| \to \infty$ as $n \to \infty$.

Example 7.1.6. Consider the normed vector space ℓ^2 of all square summable infinite sequences of complex numbers, with norm

$$\|\mathbf{x}\| = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2}.$$

Consider a mapping $T : \ell^2 \to \ell^2$, where for every $\mathbf{x} = (x_1, x_2, x_3, \ldots)$, we have $T(\mathbf{x}) = (0, x_1, x_2, x_3, \ldots)$. It is not difficult to check that $T : \ell^2 \to \ell^2$ is a linear operator. Furthermore, we have $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ for every $\mathbf{x} \in \ell^2$, so that $T : \ell^2 \to \ell^2$ is bounded. Note that this linear operator is norm preserving.

An important property of linear transformations from one normed vector space to another is that continuity and boundedness are essentially the same. The following is a generalization of Theorem 6A.

THEOREM 7A. *Suppose that V and W are normed vector spaces over $\mathbb{F}$. Then for any linear transformation $T : V \to W$, the following statements are equivalent:*
(a) T is continuous in V.
(b) T is continuous at $\mathbf{x} = \mathbf{0}$.
(c) T is bounded.

EXAMPLE 7.1.7. Suppose that V is a finite dimensional inner product space over $\mathbb{F}$, with orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$. Then for any $\mathbf{x} \in V$, there exist $c_1, \ldots, c_r \in \mathbb{F}$ such that $\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r$, and it follows (see the proof of Theorem 5C) that

$$\|\mathbf{x}\|^2 = \sum_{i=1}^{r} |c_i|^2.$$

Consider now a linear transformation $T : V \to W$, where W is a normed vector space over $\mathbb{F}$. Then it is easy to see that

$$\|T(\mathbf{x})\| = \|T(c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r)\| \le \sum_{i=1}^{r} \|c_iT(\mathbf{v}_i)\| = \sum_{i=1}^{r} |c_i|\|T(\mathbf{v}_i)\|.$$

Using the Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^{r} |c_i|\|T(\mathbf{v}_i)\| = \left(\sum_{i=1}^{r} |c_i|^2\right)^{1/2} \left(\sum_{i=1}^{r} \|T(\mathbf{v}_i)\|^2\right)^{1/2}.$$

It follows that

$$\|T(\mathbf{x})\| \le \left(\sum_{i=1}^{r} \|T(\mathbf{v}_i)\|^2\right)^{1/2} \|\mathbf{x}\|,$$

and so the linear transformation $T : V \to W$ is bounded, with $\|T(\mathbf{x})\| \le M\|\mathbf{x}\|$ for every $\mathbf{x} \in V$, where

$$M = \left(\sum_{i=1}^{r} \|T(\mathbf{v}_i)\|^2\right)^{1/2}.$$

It now follows from Theorem 7A that the linear transformation $T : V \to W$ is continuous in V. We have therefore shown that any linear transformation $T : V \to W$ from a finite dimensional inner product space V over $\mathbb{F}$ to a normed vector space W over $\mathbb{F}$ is continuous in V.

In fact, the above is a special case of the following result.

THEOREM 7B. *Suppose that V and W are normed vector spaces over $\mathbb{F}$. Suppose further that V is finite dimensional. Then any linear transformation $T : V \to W$ is continuous in V.*

PROOF. Recall Theorem 3E, that any two norms in a finite dimensional vector space V over $\mathbb{F}$ are equivalent. This enables us to make use of any norm on V which we can construct, so let us attempt to construct a new norm from the given norm $\|\cdot\|_V$ on V and the given norm $\|\cdot\|_W$ on W. We shall show that the function $\|\cdot\| : V \to \mathbb{R}$, given by

$$\|\mathbf{x}\| = \|\mathbf{x}\|_V + \|T(\mathbf{x})\|_W \qquad \text{for every } \mathbf{x} \in V,$$

is a norm on V. Clearly $\|\mathbf{x}\| \ge 0$ for every $\mathbf{x} \in V$, so that condition (NS1) is satisfied. On the other hand, it follows from $T(\mathbf{0}) = \mathbf{0}$ that $\|\mathbf{0}\| = 0$, while $\|\mathbf{x}\| = 0$ clearly implies $\|\mathbf{x}\|_V = 0$, so that $\mathbf{x} = \mathbf{0}$, and so condition (NS2) is satisfied. Condition (NS3) follows on noting that for every $c \in \mathbb{F}$ and $\mathbf{x} \in V$, we have

$$\|c\mathbf{x}\| = \|c\mathbf{x}\|_V + \|T(c\mathbf{x})\|_W = \|c\mathbf{x}\|_V + \|cT(\mathbf{x})\|_W = |c|\|\mathbf{x}\|_V + |c|\|T(\mathbf{x})\|_W = |c|\|\mathbf{x}\|.$$

Condition (NS4) follows on noting that for every $\mathbf{x}, \mathbf{y} \in V$, we have

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\| &= \|\mathbf{x}+\mathbf{y}\|_V + \|T(\mathbf{x}+\mathbf{y})\|_W = \|\mathbf{x}+\mathbf{y}\|_V + \|T(\mathbf{x})+T(\mathbf{y})\|_W \\ &\le \|\mathbf{x}\|_V + \|\mathbf{y}\|_V + \|T(\mathbf{x})\|_W + \|T(\mathbf{y})\|_W = \|\mathbf{x}\| + \|\mathbf{y}\|.\end{aligned}$$

Now since $\|\cdot\|$ is equivalent to $\|\cdot\|_V$, it follows that there exists a real number $K > 0$ such that

$$\|\mathbf{x}\| \le K\|\mathbf{x}\|_V \qquad \text{for every } \mathbf{x} \in V.$$

It follows that

$$\|T(\mathbf{x})\|_W \le \|\mathbf{x}\| \le K\|\mathbf{x}\|_V \qquad \text{for every } \mathbf{x} \in V,$$

so that $T : V \to W$ is bounded. It follows from Theorem 7A that $T : V \to W$ is continuous in V. ♣

7.2. Space of Linear Transformations

We now extend the argument in Section 6.2 to the collection of all continuous linear transformations from a normed vector space to another. The following is a generalization of Theorem 6B.

Definition. Suppose that V and W are normed vector spaces over $\mathbb{F}$. We denote by $B(V, W)$ the collection of all continuous linear transformations from V to W, and by $B(V)$ the collection of all continuous linear operators on V.

THEOREM 7C. *Suppose that V and W are normed vector spaces over $\mathbb{F}$. Then the following assertions hold:*

(a) $B(V, W)$ is a vector space over $\mathbb{F}$ with respect to pointwise vector addition and scalar multiplication.

(b) The function $\|\cdot\| : B(V, W) \to \mathbb{R}$, defined for every $T \in B(V, W)$ by

$$\|T\| = \sup_{\mathbf{x} \in V,\ \|\mathbf{x}\| \le 1} \|T(\mathbf{x})\|,$$

is a norm on $B(V, W)$.

(c) Suppose further that W is a Banach space. Then $B(V, W)$ is a Banach space.

Remark. Note that $\|T(\mathbf{x})\| \le \|T\|\|\mathbf{x}\|$ for every $\mathbf{x} \in V$.

Example 7.2.1. Note that $\|T\| \le 1$ in Example 7.1.1. On the other hand, the function $g \in C[0, 1]$, given by $g(t) = 1$ for every $t \in [0, 1]$, satisfies $\|T(g)\| = \|g\| = 1$, and shows that we must have $\|T\| = 1$.

7.3. Composition of Linear Transformations

We begin by establishing the following simple result.

THEOREM 7D. *Suppose that V, W and U are normed vector spaces over $\mathbb{F}$. Then for every $T \in B(V, W)$ and $S \in B(W, U)$, we have $ST \in B(V, U)$, and $\|ST\| \le \|S\|\|T\|$.*

PROOF. It is easy to check that ST is a linear transformation. Furthermore, ST is continuous, since the composition of continuous mappings is continuous. Hence $ST \in B(V,U)$. On the other hand, we have

$$\|(ST)(\mathbf{x})\| = \|S(T(\mathbf{x}))\| \le \|S\|\|T(\mathbf{x})\| \le \|S\|\|T\|\|\mathbf{x}\| \qquad \text{for every } \mathbf{x} \in V.$$

The last assertion follows immediately. ♣

With compositions, we are now able to discuss inverses. For the remainder of this chapter, we shall restrict our discussion to continuous linear operators from a normed vector space to itself.

DEFINITION. Suppose that V is a normed vector space over $\mathbb{F}$. A linear operator $T \in B(V)$ is said to be invertible if there exists a linear operator $S \in B(V)$ such that

$$ST = I = TS,$$

where $I : V \to V$ is the identity linear operator on V, so that $I(\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \in V$. In this case, we say that S is the inverse linear operator of T, and write $S = T^{-1}$.

REMARK. In general, a mapping is invertible if it is one-to-one and onto. Here, we need more, namely that T^{-1} is required to belong to $B(V)$; in other words, we require T^{-1} to be continuous, or bounded.

EXAMPLE 7.3.1. Let us return to Example 7.1.6. Consider first the linear operator $T : \ell^2 \to \ell^2$, where for every $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \ell^2$, we have $T(\mathbf{x}) = (0, x_1, x_2, x_3, \ldots)$. This continuous linear operator is not invertible, since it is clearly not onto. Consider next the linear operator $S : \ell^2 \to \ell^2$, where for every $\mathbf{x} = (x_1, x_2, x_3, x_4, \ldots)$, we have $S(\mathbf{x}) = (x_2, x_3, x_4, \ldots)$. This continuous linear operator is also not invertible, since it is clearly not one-to-one. Note here that $ST = I \ne TS$.

EXAMPLE 7.3.2. Let us return to Example 7.1.3. Consider the linear operator $T : L^2[0,1] \to L^2[0,1]$, where for every $f \in L^2[0,1]$, the function $T(f) \in L^2[0,1]$ is defined by

$$(T(f))(t) = tf(t) \qquad \text{for every } t \in [0,1].$$

In other words, we take $\phi(t) = t$ in the notation of Example 7.1.3. The continuous linear operator T is one-to-one, for if $tf(t) = tg(t)$ for almost all $t \in [0,1]$, then $f(t) = g(t)$ for almost all $t \in [0,1]$. However, it is easy to see that T is not onto, since the only possible candidate for f such that $(T(f))(t) = 1$ for almost all $t \in [0,1]$ must satisfy $f(t) = t^{-1}$ for almost all $t \in [0,1]$. However, this candidate f is not square integrable over $[0,1]$, so that $f \notin L^2[0,1]$. Hence the linear operator T is not invertible.

EXAMPLE 7.3.3. Let us again return to Example 7.1.3. However, consider instead the continuous linear operator $T : L^2[0,1] \to L^2[0,1]$, where for every $f \in L^2[0,1]$, the function $T(f) \in L^2[0,1]$ is defined by

$$(T(f))(t) = (1+t)f(t) \qquad \text{for every } t \in [0,1].$$

For a possible inverse to T, let us consider the continuous linear operator $S : L^2[0,1] \to L^2[0,1]$, where for every $g \in L^2[0,1]$, the function $S(g) \in L^2[0,1]$ is defined by

$$(S(g))(t) = (1+t)^{-1}g(t) \qquad \text{for every } t \in [0,1].$$

Note that

$$((ST)(f))(t) = (S(T(f)))(t) = (1+t)^{-1}(T(f))(t) = (1+t)^{-1}(1+t)f(t) = f(t)$$

for every $f \in L^2[0,1]$ and $t \in [0,1]$, and that

$$((TS)(g))(t) = (T(S(g)))(t) = (1+t)(S(g))(t) = (1+t)(1+t)^{-1}g(t) = g(t)$$

for every $g \in L^2[0,1]$ and $t \in [0,1]$. Hence $ST = I = TS$, so that T and S are both invertible.

To describe the collection of all invertible continuous linear operators on a normed vector space, we need to make an extra assumption.

THEOREM 7E. *Suppose that V is a Banach space over $\mathbb{F}$. Then the set $\mathfrak{I}$ of all invertible operators in $B(V)$ is an open set in $B(V)$.*

We need two intermediate results. The first of these is almost trivial.

THEOREM 7F. *Suppose that V is a normed vector space over $\mathbb{F}$, and that $T, S \in B(V)$ are invertible. Then $ST \in B(V)$ is invertible, and $(ST)^{-1} = T^{-1}S^{-1}$.*

THEOREM 7G. *Suppose that V is a Banach space over $\mathbb{F}$, and that $T \in B(V)$. Suppose further that $\|T\| < 1$. Then $I - T \in B(V)$ is invertible, and*

$$(I - T)^{-1} = \lim_{n\to\infty}(I + T + T^2 + \ldots + T^n)$$

in the normed vector space $B(V)$.

PROOF. Suppose that $\mathbf{x} \in V$. Then the sequence $((I + T + T^2 + \ldots + T^n)(\mathbf{x}))_{n\in\mathbb{N}}$ is a Cauchy sequence in V, since for every $m, n \in \mathbb{N}$ satisfying $m > n$, we have

$$\begin{aligned}\|(I + T + T^2 + \ldots + T^m)(\mathbf{x}) &- (I + T + T^2 + \ldots + T^n)(\mathbf{x})\| = \|T^{n+1}(\mathbf{x}) + \ldots + T^m(\mathbf{x})\| \\ &\le \|T^{n+1}(\mathbf{x})\| + \ldots \|T^m(\mathbf{x})\| \le (\|T\|^{n+1} + \ldots \|T\|^m)\|\mathbf{x}\| \le \left(\sum_{i=n+1}^{\infty}\|T\|^i\right)\|\mathbf{x}\| = \frac{\|T\|^{n+1}}{1 - \|T\|}\|\mathbf{x}\| \to 0\end{aligned}$$

as $n \to \infty$. Since V is a Banach space over $\mathbb{F}$, the sequence converges to a limit $\mathbf{u} \in V$. Let $\mathbf{u} = A\mathbf{x}$. It is not difficult to show that $A : V \to V$ is a linear operator on V. Furthermore, letting $m \to \infty$, we have

$$\|A(\mathbf{x}) - (I + T + T^2 + \ldots + T^n)(\mathbf{x})\| \le \frac{\|T\|^{n+1}}{1 - \|T\|}\|\mathbf{x}\| \qquad \text{for every } \mathbf{x} \in V,$$

so that $A - (I + T + T^2 + \ldots + T^n) \in B(V)$, and so $A \in B(V)$. Clearly we have

$$\|A - (I + T + T^2 + \ldots + T^n)\| \le \frac{\|T\|^{n+1}}{1 - \|T\|} \to 0 \qquad \text{as } n \to \infty,$$

and so $I + T + T^2 + \ldots + T^n \to A$ as $n \to \infty$. It remains to show that $A = (I - T)^{-1}$. For every $\mathbf{x} \in V$, we have

$$\begin{aligned}((I - T)A)(\mathbf{x}) &= \left((I - T)\lim_{n\to\infty}(I + T + T^2 + \ldots + T^n)\right)(\mathbf{x}) \\ &= \lim_{n\to\infty}((I - T)(I + T + T^2 + \ldots + T^n))(\mathbf{x}) \\ &= \lim_{n\to\infty}(I - T^{n+1})(\mathbf{x}) = \lim_{n\to\infty}(\mathbf{x} - T^{n+1}(\mathbf{x})).\end{aligned}$$

But $\|T^{n+1}(\mathbf{x})\| \le \|T\|^{n+1}\|\mathbf{x}\| \to 0$ as $n \to \infty$, so that $T^{n+1}(\mathbf{x}) \to \mathbf{0}$ as $n \to \infty$, and so $((I-T)A)(\mathbf{x}) = \mathbf{x}$. A similar argument shows that $(A(I - T))(\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \in V$. Hence $A = (I - T)^{-1}$ as required. ♣

PROOF OF THEOREM 7E. Suppose that $T \in \mathfrak{I}$. Then clearly $\|T^{-1}\| \ne 0$. We shall show that the open ball

$$\mathfrak{B} = \{S \in B(V) : \|T - S\| < \|T^{-1}\|^{-1}\}$$

is a subset of $\mathfrak{I}$. To do this, it suffices to show that every linear operator $S \in \mathfrak{B}$ is invertible. Note from Theorem 7D that $\|(T - S)T^{-1}\| \le |T - S\|\|T^{-1}\| < 1$, so it follows from Theorem 7G that $ST^{-1} = I - (T - S)T^{-1}$ is invertible, and from Theorem 7F that $S = (ST^{-1})T$ is invertible. ♣

Problems for Chapter 7

1. Consider the normed vector space ℓ^2 of all square summable infinite sequences of complex numbers, with norm

$$\|\mathbf{x}\| = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2}.$$

For every $\mathbf{x} = (x_1, x_2, x_3, x_4, \ldots) \in \ell^2$, let $T(\mathbf{x}) = (0, 2x_1, x_2, 2x_3, x_4, \ldots)$.
a) Show that $T(\mathbf{x}) \in \ell^2$ for every $\mathbf{x} \in \ell^2$.
b) Show that $T : \ell^2 \to \ell^2$ is a bounded linear operator.
c) Find the norm $\|T\|$.
d) Find $T^2(\mathbf{x})$ for every $\mathbf{x} \in \ell^2$.
e) Compare $\|T^2\|$ with $\|T\|^2$.

2. Suppose that a linear operator $T : V \to V$ on a normed vector space V satisfies $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ for every $\mathbf{x} \in V$. Show that T is bounded, with $\|T\| = 1$.

3. Suppose that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is an orthonormal basis in a Hilbert space V over $\mathbb{C}$, and that $(c_n)_{n\in\mathbb{N}}$ is a fixed sequence of complex numbers.
a) Show that there exists a bounded linear operator $T : V \to V$ such that $T(\mathbf{x}_n) = c_n\mathbf{x}_n$ for every $n \in \mathbb{N}$ if and only if the sequence $(c_n)_{n\in\mathbb{N}}$ is bounded.
b) Determine the norm $\|T\|$ if such a bounded linear operator $T : V \to V$ exists.

4. Consider Example 7.1.5.
a) Show that $f_n \in V$ for every $n \in \mathbb{N}$.
b) Show that $\|f_n\| \to 0$ and $\|f_n'\| \to \infty$ as $n \to \infty$.

5. Show that the linear transformation $T : V \to L^2(\mathbb{R})$, defined by $T(f) = f'$ for every $f \in V$ in the notation of Example 7.1.5, is bounded with respect to the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} \left(f(t)\overline{g(t)} + f'(t)\overline{g'(t)} \right) \mathrm{d}t$$

on V.

6. Consider the normed vector space $L^2[0,1]$ of all complex valued Lebesgue measurable functions that are square integrable on $[0,1]$, with norm

$$\|f\| = \left(\int_0^1 |f(t)|^2 \,\mathrm{d}t \right)^{1/2}.$$

Show that the Volterra operator $T : L^2[0,1] \to L^2[0,1]$, defined for every $f \in L^2[0,1]$ by

$$(T(f))(t) = \int_0^t f(s)\,\mathrm{d}s \qquad \text{for every } t \in [0,1],$$

is bounded, with norm $\|T\| = 1/\sqrt{2}$.

7. Suppose that V is a complex Hilbert space, and that $\mathbf{x}_0 \in V$ is fixed. Show that the linear transformation $T : V \to \mathbb{C}$, defined by $T(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x}_0\rangle$ for every $\mathbf{x} \in V$, is bounded, and find the norm $\|T\|$.

8. Suppose that V is a complex Hilbert space, and that $\mathbf{x}_1, \mathbf{x}_2 \in V$ are fixed. For every $\mathbf{x} \in V$, let $T(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x}_1\rangle\mathbf{x}_2$. Show that $T : V \to V$ is a bounded linear operator, and find the norm $\|T\|$.

9. Suppose that V is a normed vector space over $\mathbb{F}$, and that $P, Q \in B(V)$ are fixed. Show that $T : B(V) \to B(V)$, defined for every $R \in B(V)$ by $T(R) = PRQ$, is a bounded linear operator on $B(V)$.

10. Prove Theorem 7A.
[HINT: Study carefully the proof of Theorem 6A.]

11. Prove Theorem 7C.
[HINT: Study carefully the proof of Theorem 6B.]

12. Suppose that V and W are normed vector spaces over $\mathbb{F}$, and that $T : V \to W$ is a bounded linear transformation. Show that $\ker(T) = \{\mathbf{x} \in V : T(\mathbf{x}) = \mathbf{0}\}$ is a closed set in V.

13. Suppose that V and W are normed vector spaces over $\mathbb{F}$, and that $T : V \to W$ is a bounded linear transformation.
    a) Show that $G(T) = \{(\mathbf{x}, T(\mathbf{x})) : \mathbf{x} \in V\}$ is a linear subspace of $V \times W$.
    b) Suppose that $V \times W$ has norm defined in Problem 2 in Chapter 3. Show that $G(T)$ is a closed set in $V \times W$.

14. Suppose that V is a normed vector space over $\mathbb{F}$, and that $T, S \in B(V)$.
    a) Suppose that TS is invertible. Does this imply that T and S are invertible?
    b) Show that T and S are invertible if and only if TS and ST are invertible.
    c) Suppose that $I - TS$ is invertible. Show that $I - ST$ is invertible, with inverse $I + S(I - TS)^{-1}T$.

15. Suppose that V is a Banach space over $\mathbb{F}$, and that $(T_n)_{n\in\mathbb{N}}$ is a sequence of invertible linear operators in $B(V)$. Suppose further that $\|T_n^{-1}\| < 1$ for every $n \in \mathbb{N}$, and that $T_n \to T$ as $n \to \infty$. Show that $T \in B(V)$, and that it is invertible.

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