LINEAR FUNCTIONAL ANALYSIS

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Chapter 6

LINEAR FUNCTIONALS

6.1. Introduction

We shall be concerned with linear mappings from one vector space to another. Many of the ideas that we shall encounter in this study can be introduced by studying the special case when the second vector space is the underlying field of the first vector space. Since we are mainly concerned with vector spaces over $\mathbb{F}$, we shall therefore first study mappings from a vector space over $\mathbb{F}$ to $\mathbb{F}$.

DEFINITION. Suppose that V is a vector space over $\mathbb{F}$. By a linear functional on V, we mean a mapping $T : V \to \mathbb{F}$ satisfying the following conditions:
(LF1) For every $\mathbf{x}, \mathbf{y} \in V$, we have $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$.
(LF2) For every $c \in \mathbb{F}$ and $\mathbf{x} \in V$, we have $T(c\mathbf{x}) = cT(\mathbf{x})$.

EXAMPLE 6.1.1. Suppose that $\lambda_1, \ldots, \lambda_r \in \mathbb{F}$ are fixed. It is not difficult to show that the mapping $T : \mathbb{F}^r \to \mathbb{F}$, defined for every $\mathbf{x} = (x_1, \ldots, x_r) \in \mathbb{F}^r$ by writing

$$T(\mathbf{x}) = \lambda_1 x_1 + \ldots + \lambda_r x_r,$$

is a linear functional on $\mathbb{F}^r$.

EXAMPLE 6.1.2. Suppose that $\lambda \in C[0,1]$ is fixed. It is not difficult to show that the mapping $T : C[0,1] \to \mathbb{C}$, defined for every $f \in C[0,1]$ by

$$T(f) = \int_0^1 f(t)\lambda(t)\,\mathrm{d}t,$$

is a linear functional on $C[0,1]$.

EXAMPLE 6.1.3. Suppose that $(\lambda_i)_{i\in\mathbb{N}}$ is a bounded infinite sequence of complex numbers. We can define a mapping $T : \ell^1 \to \mathbb{C}$ as follows. For every $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ in ℓ^1, we write

$$T(\mathbf{x}) = \sum_{i=1}^{\infty} \lambda_i x_i.$$

Clearly $T(\mathbf{x})$ is well defined, since the sequence $(x_i)_{i\in\mathbb{N}}$ is absolutely summable. It is then easy to check that (LF1) and (LF2) are satisfied. Hence $T : \ell^1 \to \mathbb{C}$ is a linear functional.

EXAMPLE 6.1.4. Suppose that V is a Hilbert space over $\mathbb{F}$, and that $\mathbf{x}_0 \in V$ is fixed. We can define a mapping $T : V \to \mathbb{F}$ by writing $T(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x}_0 \rangle$ for every $\mathbf{x} \in V$. It is easy to check that (LF1) and (LF2) follow from the linearity of the inner product. Hence $T : V \to \mathbb{F}$ is a linear functional. This example motivates the Riesz-Fréchet theorem in Section 6.3.

An important property of linear functionals is that continuity and boundedness in a normed vector space are essentially the same. More precisely, we establish the result below.

THEOREM 6A. *Suppose that V is a normed vector space over $\mathbb{F}$. Then for any linear functional $T : V \to \mathbb{F}$, the following statements are equivalent:*
(a) T is continuous in V.
(b) T is continuous at $\mathbf{x} = \mathbf{0}$.
(c) The set $\{|T(\mathbf{x})| : \mathbf{x} \in V \text{ and } \|\mathbf{x}\| \le 1\}$ is bounded.

PROOF. ((a)⇒(b)) Trivial.

((b)⇒(c)) Suppose that T is continuous at $\mathbf{x} = \mathbf{0}$. Then there exists $\delta > 0$ such that

$$|T(\mathbf{x})| = |T(\mathbf{x}) - T(\mathbf{0})| < 1 \qquad \text{whenever } \mathbf{x} \in V \text{ and } \|\mathbf{x}\| = \|\mathbf{x} - \mathbf{0}\| < \delta.$$

For every $\mathbf{x} \in V$ satisfying $\|\mathbf{x}\| \le 1$, we have $\|\frac{1}{2}\delta\mathbf{x}\| < \delta$ and so $|T(\frac{1}{2}\delta\mathbf{x})| < 1$. It follows from the linearity of the functional T that

$$|T(\mathbf{x})| < \frac{2}{\delta} \qquad \text{whenever } \mathbf{x} \in V \text{ and } \|\mathbf{x}\| \le 1.$$

Hence the set $\{|T(\mathbf{x})| : \mathbf{x} \in V \text{ and } \|\mathbf{x}\| \le 1\}$ is bounded.

((c)⇒(a)) Suppose that $|T(\mathbf{x})| \le M$ whenever $\mathbf{x} \in V$ and $\|\mathbf{x}\| \le 1$. For distinct $\mathbf{x}, \mathbf{y} \in V$, we have

$$\left\| \frac{\mathbf{x} - \mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|} \right\| = 1, \qquad \text{and so} \qquad \left| T\left(\frac{\mathbf{x} - \mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|} \right) \right| \le M.$$

It follows from the linearity of the functional T that for distinct $\mathbf{x}, \mathbf{y} \in V$, we have

$$|T(\mathbf{x}) - T(\mathbf{y})| = |T(\mathbf{x} - \mathbf{y})| \le M\|\mathbf{x} - \mathbf{y}\|.$$

The continuity of T in V follows easily from this. ♣

EXAMPLE 6.1.5. Consider the normed space $C^1[0,1]$ of all continuously differentiable complex valued functions in $[0,1]$, with supremum norm

$$\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|.$$

It is easy to check that the mapping $T : C^1[0,1] \to \mathbb{C}$, where $T(f) = f'(1)$ for every $f \in C^1[0,1]$, is a linear functional on $C^1[0,1]$. For every $n \in \mathbb{N}$, the function $f_n : [0,1] \to \mathbb{C}$, defined for every $t \in [0,1]$ by $f_n(t) = t^n$, belongs to $C^1[0,1]$, and satisfies

$$f_n'(1) = n \qquad \text{and} \qquad \|f_n\|_\infty = \sup_{t\in[0,1]} |t^n| = 1 \qquad \text{for every } n \in \mathbb{N}.$$

It follows that the set $\{|T(f)| : f \in C^1[0,1] \text{ and } \|f\|_\infty \le 1\}$ is not bounded. Hence T is discontinuous in $C^1[0,1]$ with respect to the supremum norm.

6.2. Dual Spaces

The purpose of this section is to show that the collection of all continuous linear functionals on a normed vector space has a very nice algebraic structure.

THEOREM 6B. *Suppose that V is a normed vector space over $\mathbb{F}$, and that V^* is the set of all continuous linear functionals on V. Then V^* is a Banach space over $\mathbb{F}$, with norm*

$$\|T\| = \sup_{\mathbf{x}\in V,\ \|\mathbf{x}\|\le 1} |T(\mathbf{x})| \qquad \text{for every } T \in V^*. \tag{1}$$

Furthermore, we have

$$|T(\mathbf{x})| \le \|T\|\|\mathbf{x}\| \qquad \text{for every } \mathbf{x} \in V \text{ and } T \in V^*. \tag{2}$$

REMARK. The normed vector space V^* is called the dual space of V. Note that V^* is a Banach space irrespective of whether V is or not.

PROOF OF THEOREM 6B. Suppose that $T \in V^*$. Then the inequality in (2) is clearly valid if $\mathbf{x} = \mathbf{0}$. For any non-zero $\mathbf{x} \in V$, the vector $\mathbf{x}/\|\mathbf{x}\|$ has unit norm, so that

$$|T(\mathbf{x})| = \left|\|\mathbf{x}\| T\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)\right| = \left|T\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)\right| \|\mathbf{x}\| \le \|T\|\|\mathbf{x}\|.$$

To show that V^* is a Banach space over $\mathbb{F}$, we must show that (i) V^* is a vector space over $\mathbb{F}$; (ii) the function $\|\cdot\| : V^* \to \mathbb{R}$, defined by (1), is a norm; and (iii) V^* is complete. The proof of (i) is lengthy but straightforward. To prove (ii), note that for any continuous linear functional $T : V \to \mathbb{F}$, it follows immediately from Theorem 6A that the supremum in (1) exists, so that $\|T\|$ is a real number. It is then easy to check conditions (NS1)–(NS4). To prove (iii), suppose that $(T_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in V^*. Then given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|T_m - T_n\| = \sup_{\mathbf{x}\in V,\ \|\mathbf{x}\|\le 1} |T_m(\mathbf{x}) - T_n(\mathbf{x})| < \epsilon \qquad \text{whenever } m > n \ge N. \tag{3}$$

It follows from (2) with T replaced by $T_m - T_n$ that

$$|T_m(\mathbf{x}) - T_n(\mathbf{x})| \le \|T_m - T_n\|\|\mathbf{x}\| < \epsilon\|\mathbf{x}\| \qquad \text{whenever } m > n \ge N,$$

and so the sequence $(T_n(\mathbf{x}))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{F}$. Since $\mathbb{F}$ is complete, the sequence $(T_n(\mathbf{x}))_{n\in\mathbb{N}}$ converges in $\mathbb{F}$. Suppose that $T_n(\mathbf{x}) \to T(\mathbf{x})$ as $n \to \infty$. Then it is easy to show that $T : V \to \mathbb{F}$ is a linear functional. It remains to show that T is continuous in V, and that $\|T_n - T\| \to 0$ as $n \to \infty$. It is a consequence of (3) that for every $\mathbf{x} \in V$ satisfying $\|\mathbf{x}\| \le 1$, we have

$$|T_m(\mathbf{x}) - T_n(\mathbf{x})| < \epsilon \qquad \text{whenever } m > n \ge N.$$

Letting $m \to \infty$, we conclude that for every $\mathbf{x} \in V$ satisfying $\|\mathbf{x}\| \le 1$, we have

$$|T(\mathbf{x}) - T_n(\mathbf{x})| < \epsilon \qquad \text{whenever } n \ge N. \tag{4}$$

It follows that

$$\|T_n - T\| = \sup_{\mathbf{x}\in V,\ \|\mathbf{x}\|\le 1} |T_n(\mathbf{x}) - T(\mathbf{x})| \le \epsilon \qquad \text{whenever } n \ge N.$$

Hence $\|T_n - T\| \to 0$ as $n \to \infty$. On the other hand, note in particular that the inequality in (4) holds when $n = N$. This means that $T - T_N$ is bounded on the closed unit ball $\{\mathbf{x} \in V : \|\mathbf{x}\| \le 1\}$, and therefore is continuous in V in view of Theorem 6A. Hence $T = (T - T_N) + T_N$ is continuous in V. ♣

EXAMPLE 6.2.1. We shall show that the dual space $(\ell^1)^*$ of the space ℓ^1 of all absolutely summable infinite sequences $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ of complex numbers is isomorphic to the space ℓ^∞ of all bounded infinite sequences $\boldsymbol{\lambda} = (\lambda_i)_{i\in\mathbb{N}}$ of complex numbers. To do so, we shall construct a unitary transformation of the type

$$\phi : \ell^\infty \to (\ell^1)^* : \boldsymbol{\lambda} \mapsto T_{\boldsymbol{\lambda}},$$

where

$$T_{\boldsymbol{\lambda}}(\mathbf{x}) = \sum_{i=1}^{\infty} \lambda_i x_i \qquad \text{for every } \mathbf{x} = (x_i)_{i\in\mathbb{N}} \in \ell^1;$$

see Example 6.1.3. In other words, we need to show that ϕ is linear, onto and norm preserving. Note that we are somewhat lacking in rigour, as we have only considered unitary transformations for Hilbert spaces. However, the discussion here will suffice for the purpose of gaining some insight into the situation. To show that ϕ is linear is straightforward, and we shall omit the details. To show that ϕ is onto, suppose that $\Lambda \in (\ell^1)^*$ is given. We need to find $\boldsymbol{\lambda} \in \ell^\infty$ such that $\Lambda = T_{\boldsymbol{\lambda}}$. Suppose that $(\mathbf{e}_i)_{i\in\mathbb{N}}$ is the standard basis in ℓ^1. For every $i \in \mathbb{N}$, let $\lambda_i = \Lambda(\mathbf{e}_i)$. Then clearly

$$|\lambda_i| \le \sup_{\mathbf{x}\in\ell^1,\ \|\mathbf{x}\|\le 1} |\Lambda(\mathbf{x})| = \|\Lambda\|. \tag{5}$$

It follows that $\boldsymbol{\lambda} = (\lambda_i)_{i\in\mathbb{N}}$ is a bounded infinite sequence of complex numbers and so belongs to ℓ^∞. Furthermore, for every $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ in ℓ^1, we have

$$\Lambda(\mathbf{x}) = \Lambda\left(\sum_{i=1}^{\infty} x_i \mathrm{e}_i\right) \qquad \text{and} \qquad T_{\boldsymbol{\lambda}}(\mathbf{x}) = \sum_{i=1}^{\infty} \lambda_i x_i = \sum_{i=1}^{\infty} x_i \Lambda(\mathrm{e}_i).$$

For every $N \in \mathbb{N}$, we clearly have

$$\Lambda\left(\sum_{i=1}^{N} x_i \mathrm{e}_i\right) = \sum_{i=1}^{N} x_i \Lambda(\mathrm{e}_i).$$

Letting $N \to \infty$ preserves equality, and so $\Lambda(\mathbf{x}) = T_{\boldsymbol{\lambda}}(\mathbf{x})$ as required. Finally, note that

$$|T_{\boldsymbol{\lambda}}(\mathbf{x})| = \left|\sum_{i=1}^{\infty} \lambda_i x_i\right| \le \left(\sup_{i\in\mathbb{N}} |\lambda_i|\right) \left(\sum_{i=1}^{\infty} |x_i|\right) = \|\boldsymbol{\lambda}\|_\infty \|\mathbf{x}\|.$$

Hence $\|T_{\boldsymbol{\lambda}}\| \le \|\boldsymbol{\lambda}\|_\infty$. On the other hand, (5) gives the opposite inequality $\|\boldsymbol{\lambda}\|_\infty \le \|T_{\boldsymbol{\lambda}}\|$. We must therefore have $\|T_{\boldsymbol{\lambda}}\| = \|\boldsymbol{\lambda}\|_\infty$, so that ϕ is norm preserving.

6.3. Self Duality of Hilbert Spaces

The following result is motivated by Example 6.1.4.

THEOREM 6C. (RIESZ-FRÉCHET) *Suppose that V is a Hilbert space over $\mathbb{F}$. Then for every continuous linear functional $T : V \to \mathbb{F}$, there exists a unique $\mathbf{x}_0 \in V$ such that $\|T\| = \|\mathbf{x}_0\|$ and*

$$T(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x}_0 \rangle \qquad \textit{for every } \mathbf{x} \in V. \tag{6}$$

PROOF. (Uniqueness) This follows immediately from (IP1) and Theorem 4B.

(Existence) The result is obvious if $T : V \to \mathbb{F}$ is the zero functional, since we simply take $\mathbf{x}_0 = \mathbf{0}$. Suppose now that $T : V \to \mathbb{F}$ is not the zero functional. Then it is easy to show that

$$W = \ker T = \{\mathbf{x} \in V : T(\mathbf{x}) = 0\}$$

is a proper closed linear subspace of V. It follows from Theorem 5F that $V = W \oplus W^\perp$, where $W^\perp \neq \{\mathbf{0}\}$, so that there exists a non-zero vector $\mathbf{z} \in W^\perp$. Multiplying by a suitable non-zero element of $\mathbb{F}$ if necessary, we may further assume that $T(\mathbf{z}) = 1$. Suppose now that $\mathbf{x} \in V$. We can write

$$\mathbf{x} = (\mathbf{x} - T(\mathbf{x})\mathbf{z}) + T(\mathbf{x})\mathbf{z}.$$

Then it is easy to check that $T(\mathbf{x} - T(\mathbf{x})\mathbf{z}) = 0$, so that $\mathbf{x} - T(\mathbf{x})\mathbf{z} \in W$ and so $T(\mathbf{x})\mathbf{z} \in W^\perp$. It follows that

$$\langle \mathbf{x}, \mathbf{z} \rangle = \langle T(\mathbf{x})\mathbf{z}, \mathbf{z} \rangle = T(\mathbf{x})\|\mathbf{z}\|^2 \qquad \text{for every } \mathbf{x} \in V.$$

Taking $\mathbf{x}_0 = \mathbf{z}/\|\mathbf{z}\|^2$ now gives (6). Finally, note that for every $\mathbf{x} \in V$ satisfying $\|\mathbf{x}\| \leq 1$, we have

$$|T(\mathbf{x})| = |\langle \mathbf{x}, \mathbf{x}_0 \rangle| \leq \|\mathbf{x}\|\|\mathbf{x}_0\|,$$

so that $\|T\| \leq \|\mathbf{x}_0\|$. On the other hand, $\mathbf{x} = \mathbf{x}_0/\|\mathbf{x}_0\|$ is a unit vector, and so

$$\|T\| \geq |T(\mathbf{x})| = \frac{|T(\mathbf{x}_0)|}{\|\mathbf{x}_0\|} = \frac{|\langle \mathbf{x}_0, \mathbf{x}_0 \rangle|}{\|\mathbf{x}_0\|} = \|\mathbf{x}_0\|.$$

Hence $\|T\| = \|\mathbf{x}_0\|$ as required. ♣

Remark. Theorem 6C shows that there exists an onto and norm preserving mapping $\psi : V \to V^\perp$, given by $\psi(\mathbf{x}_0) = \langle \cdot, \mathbf{x}_0 \rangle$ for every $\mathbf{x}_0 \in V$. It is for this reason that we say that Hilbert spaces are self dual. Note that ψ is conjugate linear; in other words, we have $\psi(\mathbf{x}_0 + \mathbf{y}_0) = \psi(\mathbf{x}_0) + \psi(\mathbf{y}_0)$ and $\psi(c\mathbf{x}_0) = \overline{c}\psi(\mathbf{x}_0)$ for every $\mathbf{x}_0, \mathbf{y}_0 \in V$ and $c \in \mathbb{F}$.

Problems for Chapter 6

1. Consider a linear functional $T : C[0,1] \to \mathbb{C}$, defined for every $f \in C[0,1]$ by $T(f) = f(1)$.
   a) Show that T is continuous in $C[0,1]$ with respect to the supremum norm
   $$\|f\| = \sup_{t \in [0,1]} |f(t)|.$$
   b) Determine whether T is continuous in $C[0,1]$ with respect to the norm
   $$\|f\| = \int_0^1 |f(t)|^2 \, \mathrm{d}t,$$
   and justify your assertion.

2. Consider the vector space $P[0,1]$ of all polynomials (in variable t) with complex coefficients defined on $[0,1]$. For every $k \in \mathbb{N} \cup \{0\}$, consider the mapping $T_k : P[0,1] \to \mathbb{C}$ defined for every $f \in P[0,1]$ by $T_k(f) = a_k$, where a_k is the coefficient of t^k in f.
   a) Show that for every $k \in \mathbb{N} \cup \{0\}$, the mapping $T_k : P[0,1] \to \mathbb{C}$ is a linear functional.
   b) By considering polynomials of the form
   $$f_n(t) = (1-t)^n$$
   for natural numbers $n \geq k$, show that $T_k : P[0,1] \to \mathbb{C}$ is not continuous in $P[0,1]$ with respect to the supremum norm
   $$\|f\| = \sup_{t \in [0,1]} |f(t)|.$$

3. a) Consider the vector space $C[0,1]$, with supremum norm

$$\|f\| = \sup_{t\in[0,1]} |f(t)|,$$

and a linear functional $T : C[0,1] \to \mathbb{C}$, defined for every $f \in C[0,1]$ by

$$T(f) = \int_0^1 tf(t)\,\mathrm{d}t.$$

(i) Determine $\|T\|$.
(ii) Find an element $f \in C[0,1]$ such that $|T(f)| = \|T\|$.

b) Consider the linear subspace $W = \{f \in C[0,1] : f(1) = 0\}$ of $C[0,1]$, with the same supremum norm, and the restriction $T_W : W \to \mathbb{C}$ of the linear functional T to W.
(i) Show that $\|T_W\| = \|T\|$.
(ii) Show that there does not exist any element $g \in W$ such that $|T_W(g)| = \|T_W\|$.

4. Suppose that $T : V \to \mathbb{F}$ is a non-zero continuous linear functional on a Banach space V over $\mathbb{F}$.
a) Show that $W = \{\mathbf{x} \in V : T(\mathbf{x}) = 1\}$ is a non-empty, closed and convex linear subspace of V.
b) Show that

$$\inf_{\mathbf{x}\in W} \|\mathbf{x}\| = \frac{1}{\|T\|}.$$

c) What can you say about the minimum distance of W to the origin $\mathbf{0} \in V$ if the norm $\|T\|$ is not attained? Comment in view of Theorem 4G.

5. By following the ideas in Example 6.2.1, show that the dual space $(c_0)^*$ of the space c_0 of all infinite sequences $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ of complex numbers such that $x_i \to 0$ as $i \to \infty$, with supremum norm

$$\|\mathbf{x}\|_\infty = \sup_{i\in\mathbb{N}} |x_i|,$$

is isomorphic to the space ℓ^1 of all absolutely summable infinite sequences of complex numbers.

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