LINEAR FUNCTIONAL ANALYSIS

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Chapter 5

ORTHOGONAL EXPANSIONS

5.1. Orthogonal and Orthonormal Systems

Definition. Two vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space V over $\mathbb{F}$ are said to be orthogonal to each other if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Definition. A system $(\mathbf{x}_\alpha)_{\alpha \in I}$ of vectors in an inner product space V over $\mathbb{F}$ is said to be an orthogonal system if $\langle \mathbf{x}_\alpha, \mathbf{x}_\beta \rangle = 0$ for every $\alpha, \beta \in I$ satisfying $\alpha \neq \beta$; in other words, if the system consists of vectors in V that are orthogonal to each other.

Definition. An orthogonal system $(\mathbf{x}_\alpha)_{\alpha \in I}$ of vectors in an inner product space V over $\mathbb{F}$ is said to be an orthonormal system if $\|\mathbf{x}_\alpha\| = 1$ for every $\alpha \in I$.

Remark. In the special case when $I = \mathbb{N}$, an orthonormal system $(\mathbf{x}_n)_{n \in \mathbb{N}}$ is sometimes also known as an orthonormal sequence. Note also that the set $\mathbb{Z}$ has the same cardinality as the set $\mathbb{N}$. It is convenient to think of orthonormal systems $(\mathbf{x}_n)_{n \in \mathbb{Z}}$ also as orthonormal sequences. Any result valid for a general orthonormal sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ has an analogue that is valid for a general orthonormal sequence $(\mathbf{x}_n)_{n \in \mathbb{Z}}$.

Example 5.1.1. In the inner product space ℓ^2 of all square summable infinite sequences of complex numbers, with inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i},$$

the system $(\mathbf{x}_n)_{n \in \mathbb{N}}$, given by

$$\mathbf{x}_n = (\underbrace{0, \ldots, 0}_{n-1}, 1, 0, 0, 0, \ldots) \qquad \text{for every } n \in \mathbb{N},$$

is an orthonormal sequence, sometimes known as the standard orthonormal sequence in ℓ^2.

EXAMPLE 5.1.2. In the inner product space $L^2[-\pi,\pi]$ of all Lebesgue measurable complex valued functions that are square integrable on $[-\pi,\pi]$, with inner product

$$\langle f,g\rangle=\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,\mathrm{d}t,$$

consider the system $(f_n)_{n\in\mathbb{Z}}$, where for every $n\in\mathbb{Z}$, the function $f_n:[-\pi,\pi]\to\mathbb{C}$ is defined by

$$f_n(t)=\frac{1}{(2\pi)^{1/2}}\mathrm{e}^{\mathrm{i}nt}\qquad\text{for every } t\in[-\pi,\pi].$$

Then for every $n,m\in\mathbb{Z}$, we have

$$\langle f_n,f_m\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}(n-m)t}\,\mathrm{d}t=\begin{cases}1 & \text{if } n=m,\\ 0 & \text{if } n\neq m.\end{cases}$$

It follows that $(f_n)_{n\in\mathbb{Z}}$ is an orthonormal sequence. Consider also the system $(g_0,g_1,h_1,g_2,h_2,\ldots)$, where the function $g_0:[-\pi,\pi]\to\mathbb{C}$ is defined by

$$g_0(t)=\frac{1}{(2\pi)^{1/2}}\qquad\text{for every } t\in[-\pi,\pi],$$

and where for every $n\in\mathbb{N}$, the functions $g_n:[-\pi,\pi]\to\mathbb{C}$ and $h_n:[-\pi,\pi]\to\mathbb{C}$ are defined by

$$g_n(t)=\frac{1}{\pi^{1/2}}\cos nt\qquad\text{and}\qquad h_n(t)=\frac{1}{\pi^{1/2}}\sin nt\qquad\text{for every } t\in[-\pi,\pi].$$

With a bit of work, it can be shown that $(g_0,g_1,h_1,g_2,h_2,\ldots)$ is also an orthonormal system.

Suppose that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is an orthonormal sequence in an inner product space V over $\mathbb{F}$. Consider the linear span $\mathrm{span}(\{\mathbf{x}_n:n\in\mathbb{N}\})$, consisting of all finite linear combinations of the terms of the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$. Suppose that $\mathbf{x}$ is an element in this linear span. Then there exists a sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ such that

$$\mathbf{x}=\sum_{n=1}^{\infty}c_n\mathbf{x}_n.$$

Note that we do not have to worry about convergence here, as all but finitely many of the numbers c_n are equal to zero. Then for every $n\in\mathbb{N}$, we have

$$\langle\mathbf{x},\mathbf{x}_n\rangle=\left\langle\sum_{m=1}^{\infty}c_m\mathbf{x}_m,\mathbf{x}_n\right\rangle=\sum_{m=1}^{\infty}c_m\langle\mathbf{x}_m,\mathbf{x}_n\rangle=c_n.$$

We now generalize this simple idea.

DEFINITION. Suppose that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is an orthonormal sequence in a Hilbert space V over $\mathbb{F}$. Then for every vector $\mathbf{x}\in V$, the number $c_n=\langle\mathbf{x},\mathbf{x}_n\rangle\in\mathbb{F}$ is called the n-th Fourier coefficient of $\mathbf{x}$ with respect to the orthonormal sequence, and the series

$$\mathbf{x}\sim\sum_{n\in\mathbb{N}}c_n\mathbf{x}_n=\sum_{n\in\mathbb{N}}\langle\mathbf{x},\mathbf{x}_n\rangle\mathbf{x}_n$$

is called the Fourier series of $\mathbf{x}$ with respect to the orthonormal sequence.

REMARK. Note that Example 5.1.2 should be familiar to anyone with any knowledge of Fourier series. Note also that at this point, we have not discussed the convergence or otherwise of the Fourier series.

EXAMPLE 5.1.3. Suppose that $f \in L^2[-\pi,\pi]$, the inner product space discussed in Example 5.1.2. Then for every $n \in \mathbb{Z}$, we have

$$c_n = \langle f, f_n \rangle = \frac{1}{(2\pi)^{1/2}} \int_{-\pi}^{\pi} f(t)\mathrm{e}^{-int}\,\mathrm{d}t,$$

and the Fourier series for f with respect to the orthonormal system $(f_n)_{n\in\mathbb{Z}}$ is given by

$$f(t) \sim \sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{int}.$$

On the other hand,

$$a_0 = \langle f, g_0 \rangle = \frac{1}{(2\pi)^{1/2}} \int_{-\pi}^{\pi} f(t)\,\mathrm{d}t.$$

Furthermore, for every $n \in \mathbb{N}$, we have

$$a_n = \langle f, g_n \rangle = \frac{1}{\pi^{1/2}} \int_{-\pi}^{\pi} f(t)\cos nt\,\mathrm{d}t \qquad \text{and} \qquad b_n = \langle f, h_n \rangle = \frac{1}{\pi^{1/2}} \int_{-\pi}^{\pi} f(t)\sin nt\,\mathrm{d}t,$$

and the Fourier series for f with respect to the orthonormal system $(g_0, g_1, h_1, g_2, h_2, \ldots)$ is given by

$$f(t) \sim \frac{a_0}{(2\pi)^{1/2}} + \frac{1}{\pi^{1/2}} \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

5.2. Convergence of Fourier Series

We have yet to determine whether the Fourier series of a given vector actually represents the vector in any way. In this section, we study first the problem of the convergence of the Fourier series of a given vector with respect to an orthonormal sequence. We begin by showing that the sequence of Fourier coefficients is square summable.

THEOREM 5A. (BESSEL'S INEQUALITY) *Suppose that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is an orthonormal sequence in an inner product space V over $\mathbb{F}$. Then for every vector $\mathbf{x} \in V$, we have*

$$\sum_{n=1}^{\infty} |\langle \mathbf{x}, \mathbf{x}_n \rangle|^2 \le \|\mathbf{x}\|^2.$$

PROOF. For every $N \in \mathbb{N}$, let

$$\mathbf{s}_N = \sum_{n=1}^{N} \langle \mathbf{x}, \mathbf{x}_n \rangle \mathbf{x}_n$$

denote the N-th partial sum of the Fourier series for $\mathbf{x}$. Then (see Problem 4)

$$\sum_{n=1}^{N} |\langle \mathbf{x}, \mathbf{x}_n \rangle|^2 = \|\mathbf{x}\|^2 - \|\mathbf{s}_N - \mathbf{x}\|^2 \le \|\mathbf{x}\|^2.$$

The result follows on letting $N \to \infty$. ♣

Recall that in a metric space (V,ρ), we say that a series

$$\sum_{n=1}^{\infty} \mathbf{y}_n$$

converges to $\mathbf{y}$, denoted by

$$\sum_{n=1}^{\infty} \mathbf{y}_n = \mathbf{y},$$

if the sequence of real numbers

$$\rho\left(\sum_{n=1}^{N} \mathbf{y}_n, \mathbf{y}\right) \to 0 \qquad \text{as } N \to \infty. \tag{1}$$

If V is an inner product space over $\mathbb{F}$, then the metric ρ is induced by the inner product via the norm. Hence the condition (1) is equivalent to

$$\left\|\sum_{n=1}^{N} \mathbf{y}_n - \mathbf{y}\right\| \to 0 \qquad \text{as } N \to \infty.$$

THEOREM 5B. *Suppose that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is an orthonormal sequence in a Hilbert space V over $\mathbb{F}$. Then for every sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$, the series*

$$\sum_{n=1}^{\infty} \lambda_n \mathbf{x}_n \tag{2}$$

converges in V if and only if the real series

$$\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty; \tag{3}$$

in other words, if and only if the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is square summable.

PROOF. $(\Rightarrow)$ Suppose that

$$\sum_{m=1}^{\infty} \lambda_m \mathbf{x}_m = \mathbf{x}.$$

For every $n \in \mathbb{N}$, choosing $N \in \mathbb{N}$ to satisfy $N \geq n$, we have

$$\left\langle \sum_{m=1}^{N} \lambda_m \mathbf{x}_m, \mathbf{x}_n \right\rangle = \sum_{m=1}^{N} \lambda_m \langle \mathbf{x}_m, \mathbf{x}_n \rangle = \lambda_n.$$

In view of the continuity of the inner product, it follows on letting $N \to \infty$ that

$$\langle \mathbf{x}, \mathbf{x}_n \rangle = \left\langle \sum_{m=1}^{\infty} \lambda_m \mathbf{x}_m, \mathbf{x}_n \right\rangle = \lambda_n.$$

The conclusion (3) now follows on applying Bessel's inequality.

($\Leftarrow$) For every $m \in \mathbb{N}$, let

$$\mathbf{t}_N = \sum_{n=1}^{N} \lambda_n \mathbf{x}_n$$

denote the N-th partial sum of the series (2). Then for every $M, N \in \mathbb{N}$ satisfying $M > N$, we have

$$\mathbf{t}_M - \mathbf{t}_N = \sum_{n=N+1}^{M} \lambda_n \mathbf{x}_n.$$

On the other hand, it follows from Pythagoras's theorem (see Problem 3) and the orthonormality of the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ that

$$\left\| \sum_{n=N+1}^{M} \lambda_n \mathbf{x}_n \right\|^2 = \sum_{n=N+1}^{M} \|\lambda_n \mathbf{x}_n\|^2 = \sum_{n=N+1}^{M} |\lambda_n|^2 \|\mathbf{x}_n\|^2 = \sum_{n=N+1}^{M} |\lambda_n|^2.$$

Hence the condition (3) implies that the sequence $(\mathbf{t}_N)_{N\in\mathbb{N}}$ is a Cauchy sequence in V. The convergence of the series (2) now follows from the completeness of the Hilbert space V. ♣

5.3. Orthonormal Bases

We now wish to study the problem of whether a given vector is equal to its Fourier series with respect to some orthonormal sequence. More precisely, we wish to determine whether the Fourier series of a given vector with respect to some orthonormal sequence converges to the vector itself. Note that by convergence in a Hilbert space, we mean convergence with respect to the norm induced by the inner product. In the case of Hilbert spaces of functions, this does not necessarily mean pointwise convergence.

To motivate our next definition, let us consider a Hilbert space V over $\mathbb{F}$, with a given orthonormal sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$. For any vector $\mathbf{x} \in V$, consider its Fourier series

$$\sum_{n=1}^{\infty} \langle \mathbf{x}, \mathbf{x}_n \rangle \mathbf{x}_n.$$

Let

$$\mathbf{y} = \mathbf{x} - \sum_{m=1}^{\infty} \langle \mathbf{x}, \mathbf{x}_m \rangle \mathbf{x}_m$$

denote the difference between $\mathbf{x}$ and its Fourier series. We would like to show that $\mathbf{y} = \mathbf{0}$.

First of all, for every $n \in \mathbb{N}$, we have

$$\langle \mathbf{y}, \mathbf{x}_n \rangle = \langle \mathbf{x}, \mathbf{x}_n \rangle - \left\langle \sum_{m=1}^{\infty} \langle \mathbf{x}, \mathbf{x}_m \rangle \mathbf{x}_m, \mathbf{x}_n \right\rangle = \langle \mathbf{x}, \mathbf{x}_n \rangle - \sum_{m=1}^{\infty} \langle \mathbf{x}, \mathbf{x}_m \rangle \langle \mathbf{x}_m, \mathbf{x}_n \rangle = \langle \mathbf{x}, \mathbf{x}_n \rangle - \langle \mathbf{x}, \mathbf{x}_n \rangle = \mathbf{0}.$$

On the other hand, in the inner product space ℓ^2 discussed in Example 5.1.1, if we remove $\mathbf{x}_1$ from the standard orthonormal sequence, the system $(\mathbf{x}_n)_{n\in\mathbb{N}\setminus\{1\}}$ still constitutes an orthonormal sequence. However, it is easy to see that the Fourier series of the vector $\mathbf{x}_1$ with respect to the orthonormal system $(\mathbf{x}_n)_{n\in\mathbb{N}\setminus\{1\}}$ converges to the zero vector $\mathbf{0}$. It follows that $\mathbf{y} = \mathbf{x}_1$ in this case.

Definition. An orthonormal sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in a Hilbert space V over $\mathbb{F}$ is said to be an orthonormal basis of V if the only vector $\mathbf{y} \in V$ that satisfies the condition

$$\langle \mathbf{y}, \mathbf{x}_n \rangle = 0 \qquad \text{for every } n \in \mathbb{N}$$

is the zero vector $\mathbf{y} = \mathbf{0}$.

Example 5.3.1. The standard orthonormal sequence in ℓ^2, given in Example 5.1.1 clearly forms an orthonormal basis of ℓ^2. Suppose that $\mathbf{y} = (y_1, y_2, y_3, \ldots) \in \ell^2$ satisfies $\langle \mathbf{y}, \mathbf{x}_n \rangle = 0$ for every $n \in \mathbb{N}$. Then since $\langle \mathbf{y}, \mathbf{x}_n \rangle = y_n$ for every $n \in \mathbb{N}$, it follows that $\mathbf{y} = \mathbf{0}$ is the zero sequence.

THEOREM 5C. *Suppose that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is an orthonormal basis in a Hilbert space V over $\mathbb{F}$. Then for every vector $\mathbf{x} \in V$, we have*

$$\mathbf{x} = \sum_{n=1}^{\infty} \langle \mathbf{x}, \mathbf{x}_n \rangle \mathbf{x}_n \qquad \textit{and} \qquad \|\mathbf{x}\|^2 = \sum_{n=1}^{\infty} |\langle \mathbf{x}, \mathbf{x}_n \rangle|^2.$$

Proof. The first assertion has already been established earlier. On the other hand, as in the proof of Theorem 5B, it follows from Pythagoras's theorem and the orthonormality of the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ that for every $N \in \mathbb{N}$, we have

$$\left\| \sum_{n=1}^{N} \langle \mathbf{x}, \mathbf{x}_n \rangle \mathbf{x}_n \right\|^2 = \sum_{n=1}^{N} \|\langle \mathbf{x}, \mathbf{x}_n \rangle \mathbf{x}_n\|^2 = \sum_{n=1}^{N} |\langle \mathbf{x}, \mathbf{x}_n \rangle|^2 \|\mathbf{x}_n\|^2 = \sum_{n=1}^{N} |\langle \mathbf{x}, \mathbf{x}_n \rangle|^2.$$

The second assertion now follows on letting $N \to \infty$. ♣

Remark. The orthonormal systems in Example 5.1.2 form orthonormal bases of the Hilbert space $L^2[-\pi, \pi]$. We shall not prove this assertion here. Suffice to say that this forms the foundation of the theory of classical Fourier series.

An orthonormal basis in a Hilbert space is sometimes also called a complete orthonormal sequence by some authors. The result below is an attempt to explain this terminology.

THEOREM 5D. *Suppose that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is an orthonormal sequence in a Hilbert space V over $\mathbb{F}$. Then the following statements are equivalent:*
(a) $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is an orthonormal basis in V.
(b) The linear span $\mathrm{span}(\{\mathbf{x}_n : n \in \mathbb{N}\})$ has closure V.
(c) For every $\mathbf{x} \in V$, we have

$$\|\mathbf{x}\|^2 = \sum_{n=1}^{\infty} |\langle \mathbf{x}, \mathbf{x}_n \rangle|^2.$$

Proof. In view of Theorem 5C, it remains to prove that (b)⇒(a) and (c)⇒(a).

((b)⇒(a)) Suppose that $\mathbf{y} \in V$ satisfies $\langle \mathbf{y}, \mathbf{x}_n \rangle = 0$ for every $n \in \mathbb{N}$. Consider the set

$$S = \{\mathbf{x} \in V : \langle \mathbf{y}, \mathbf{x} \rangle = 0\}.$$

It is easy to check that S is a linear subspace of V. Since $\mathbf{x}_n \in S$ for every $n \in \mathbb{N}$, it follows that S must contain the linear span $\mathrm{span}(\{\mathbf{x}_n : n \in \mathbb{N}\})$. On the other hand, S is closed in view of the continuity of the inner product, and so S must contain the closure of the linear span $\mathrm{span}(\{\mathbf{x}_n : n \in \mathbb{N}\})$. Hence $S = V$. In particular, we have $\mathbf{y} \in S$, and so $\langle \mathbf{y}, \mathbf{y} \rangle = 0$, whence $\mathbf{y} = \mathbf{0}$ as required.

((c)⇒(a)) Suppose on the contrary that the orthonormal sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ does not form an orthonormal basis in V. Then there exists a non-zero $\mathbf{x} \in V$ such that $\langle \mathbf{x}, \mathbf{x}_n \rangle = 0$ for every $n \in \mathbb{N}$. Then $\|\mathbf{x}\| \neq 0$, but

$$\sum_{n=1}^{\infty} |\langle \mathbf{x}, \mathbf{x}_n \rangle|^2 = 0,$$

clearly a contradiction. ♣

5.4. Separable Hilbert Spaces

We now turn to the problem of the existence of an orthonormal basis in a Hilbert space. Suppose first of all that V is a finite dimensional inner product space over $\mathbb{F}$. Then any basis of V can be converted to an orthogonal basis by the Gram-Schmidt process and then normalized. Hence every finite dimensional Hilbert space V over $\mathbb{F}$ has an orthonormal basis.

DEFINITION. A Hilbert space V over $\mathbb{F}$ is said to be separable if it is finite dimensional or if it has an orthonormal sequence that forms an orthonormal basis. In other words, V is separable if it has a countable orthonormal basis.

The purpose of this section is to show that we have already studied all the separable Hilbert spaces over $\mathbb{F}$. More precisely, we shall show that every separable Hilbert space over $\mathbb{F}$ is similar to one of the examples that we have studied. To do so, we must first give a meaning to the word "similar".

DEFINITION. Suppose that V and W are Hilbert spaces over $\mathbb{F}$. A mapping $\phi : V \to W$ is said to be a unitary transformation if the following conditions are satisfied:
(UT1) $\phi : V \to W$ is linear: For every $\mathbf{x}, \mathbf{y} \in V$ and $\alpha, \beta \in \mathbb{F}$, we have $\phi(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\phi(\mathbf{x}) + \beta\phi(\mathbf{y})$.
(UT2) $\phi : V \to W$ is onto: For every $\mathbf{z} \in W$, there exists $\mathbf{x} \in V$ such that $\phi(\mathbf{x}) = \mathbf{z}$.
(UT3) $\phi : V \to W$ is one-to-one: For every $\mathbf{x}, \mathbf{y} \in V$, we have $\mathbf{x} = \mathbf{y}$ whenever $\phi(\mathbf{x}) = \phi(\mathbf{y})$.
(UT4) $\phi : V \to W$ preserves inner product: For every $\mathbf{x}, \mathbf{y} \in V$, we have $\langle \phi(\mathbf{x}), \phi(\mathbf{y}) \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$.

REMARK. The conditions (UT3) and (UT4) can be replaced by the following:
(UT5) $\phi : V \to W$ preserves norm: For every $\mathbf{x} \in V$, we have $\|\phi(\mathbf{x})\| = \|\mathbf{x}\|$.
Clearly (UT5) follows from (UT4). On the other hand, if $\phi(\mathbf{x}) = \phi(\mathbf{y})$, then it follows from (UT1) that $\phi(\mathbf{x} - \mathbf{y}) = \mathbf{0}$, so that $\|\phi(\mathbf{x} - \mathbf{y})\| = \|\mathbf{0}\| = 0$. It then follows from (UT5) that $\|\mathbf{x} - \mathbf{y}\| = 0$, so that $\mathbf{x} - \mathbf{y} = \mathbf{0}$, giving (UT3). Finally, (UT4) follows from (UT5) in view of (UT1) and the Polarization identity (see Problem 9(b) in Chapter 4) in the case when $\mathbb{F} = \mathbb{C}$. In the case when $\mathbb{F} = \mathbb{R}$, the Polarization identity is replaced by a simpler identity (see Problem 8(b) in Chapter 4).

DEFINITION. Two Hilbert spaces V and W over $\mathbb{F}$ are said to be isomorphic if there exists a unitary transformation $\phi : V \to W$ from V to W.

THEOREM 5E. *Suppose that V is a separable complex Hilbert space. Then either V is isomorphic to $\mathbb{C}^r$ for some $r \in \mathbb{N}$, or V is isomorphic to the separable Hilbert space ℓ^2.*

PROOF. Suppose first of all that V is finite dimensional, of dimension r, say. Then V has a basis of r elements which can be orthogonalized by the Gram-Schmidt process and then normalized to give an orthonormal basis $(\mathbf{x}_1, \ldots, \mathbf{x}_r)$. Consider now a mapping $\phi : V \to \mathbb{C}^r$, defined as follows. For every vector $\mathbf{x} \in V$ with unique Fourier series

$$\mathbf{x} = \sum_{i=1}^{r} \langle \mathbf{x}, \mathbf{x}_i \rangle \mathbf{x}_i = \sum_{i=1}^{r} c_i \mathbf{x}_i,$$

we write

$$\phi(\mathbf{x}) = (c_1, \ldots, c_r) \in \mathbb{C}^r.$$

It is not difficult to check that $\phi : V \to \mathbb{C}^r$ is linear. On the other hand, for every $(c_1, \ldots, c_r) \in \mathbb{C}^r$, the vector $\mathbf{x} = c_1\mathbf{x}_1 + \ldots + c_r\mathbf{x}_r$ satisfies $\phi(\mathbf{x}) = (c_1, \ldots, c_r)$, so that $\phi : V \to \mathbb{C}^r$ is onto. Finally, note from Pythagoras's theorem that

$$\|\mathbf{x}\|^2 = \sum_{i=1}^{r} |\langle \mathbf{x}, \mathbf{x}_i \rangle|^2 = \sum_{i=1}^{r} |c_i|^2 = \|\phi(\mathbf{x})\|^2,$$

so that $\phi : V \to \mathbb{C}^r$ is norm preserving. Hence $\phi : V \to \mathbb{C}^r$ is a unitary transformation, and so V is isomorphic to $\mathbb{C}^r$.

Suppose next that V has an orthonormal sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ that forms an orthonormal basis. Consider now a mapping $\phi : V \to \ell^2$, defined as follows. For every vector $\mathbf{x} \in V$ with unique Fourier series

$$\mathbf{x} = \sum_{n=1}^{\infty} \langle \mathbf{x}, \mathbf{x}_n \rangle \mathbf{x}_n = \sum_{n=1}^{\infty} c_n \mathbf{x}_n,$$

we write

$$\phi(\mathbf{x}) = (c_1, c_2, c_3, \ldots).$$

In view of Theorem 5D, we have $\phi(\mathbf{x}) \in \ell^2$ and $\|\phi(\mathbf{x})\| = \|\mathbf{x}\|$, so that $\phi : V \to \ell^2$ well defined and norm preserving. On the other hand, it is not difficult to check that $\phi : V \to \ell^2$ is linear. Finally, for every $(c_1, c_2, c_3, \ldots) \in \ell^2$, it follows from Theorem 5B that

$$\sum_{n=1}^{\infty} c_n \mathbf{x}_n$$

converges to a vector $\mathbf{x} \in V$, and that $\phi(\mathbf{x}) = (c_1, c_2, c_3, \ldots)$, and so $\phi : V \to \ell^2$ is onto. Hence $\phi : V \to \ell^2$ is a unitary transformation, and so V is isomorphic to ℓ^2.

That the Hilbert space ℓ^2 is separable is an immediate consequence of Example 5.3.1. ♣

Remark. Similarly, one can show that if V is a separable real Hilbert space, then either V is isomorphic to $\mathbb{R}^r$ for some $r \in \mathbb{N}$, or V is isomorphic to $\ell^2(\mathbb{R})$, the space of all square summable infinite sequences of real numbers.

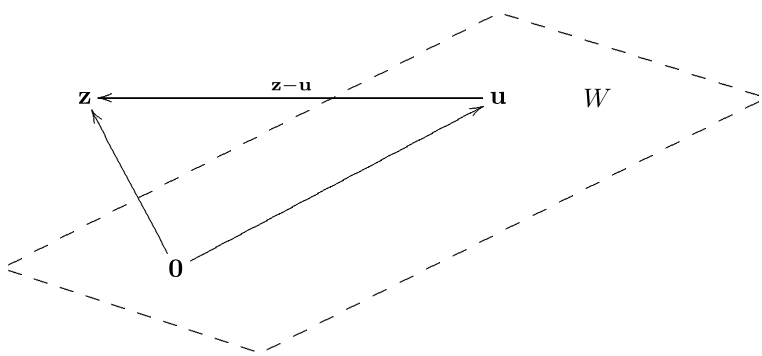
5.5. Splitting up a Hilbert Space

Suppose that V is a vector space. We often seek to split V into smaller parts. More precisely, suppose that W and U are subspaces of V such that $W \cap U = \{\mathbf{0}\}$ and every $\mathbf{x} \in V$ can be written in the form $\mathbf{x} = \mathbf{y} + \mathbf{z}$ for some $\mathbf{y} \in W$ and $\mathbf{z} \in U$. Then we say that V is a direct sum of W and U, and write $V = W \oplus U$. Suppose further that V has an inner product, and that every vector in W is orthogonal to every vector in U. Then we say that V is an orthogonal direct sum of W and U. In this section, we shall study this problem when V is a Hilbert space over $\mathbb{F}$ and W is a closed subspace of V.

Definition. Suppose that V is a Hilbert space over $\mathbb{F}$, and that W is a non-empty subset of V. By the orthogonal complement of W, we mean the set

$$W^\perp = \{\mathbf{x} \in V : \langle \mathbf{x}, \mathbf{y} \rangle = 0 \text{ for every } \mathbf{y} \in W\}.$$

Remark. It is not difficult to show that for every Hilbert space V over $\mathbb{F}$ and subset $W \subseteq V$, the orthogonal complement $W^\perp$ is a closed subspace of V. The picture below shows a very interesting property of $W^\perp$ in the special case when W is a closed subspace of V.



Here the vector $\mathbf{z}$ is orthogonal to every vector in W. Note that

$$\|\mathbf{z}\| \leq \|\mathbf{z}-\mathbf{u}\| \qquad \text{for every } \mathbf{u} \in W.$$

We shall need this idea in the proof of our next result.

THEOREM 5F. *Suppose that V is a Hilbert space over $\mathbb{F}$ and W is a closed subspace of V. Then for every $\mathbf{x} \in V$, there exist unique $\mathbf{y} \in W$ and $\mathbf{z} \in W^{\perp}$ such that $\mathbf{x} = \mathbf{y} + \mathbf{z}$.*

PROOF. Let $\mathbf{x} \in V$ be chosen. We shall make use of the closest point property. It is not difficult to show that W is convex in V, and so it follows from Theorem 4G that there exists a vector $\mathbf{y} \in W$ such that

$$\|\mathbf{x}-\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{v}\| \qquad \text{for every } \mathbf{v} \in W.$$

Every $\mathbf{u} \in W$ can be written in the form $\mathbf{u} = \mathbf{v} - \mathbf{y}$, where $\mathbf{v} \in W$. Hence

$$\|\mathbf{x}-\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{y}-\mathbf{u}\| \qquad \text{for every } \mathbf{u} \in W.$$

Let $\mathbf{z} = \mathbf{x} - \mathbf{y}$. Then

$$\|\mathbf{z}\| \leq \|\mathbf{z}-\mathbf{u}\| \qquad \text{for every } \mathbf{u} \in W.$$

We shall show that $\mathbf{z} \in W^{\perp}$. For every $\mathbf{u} \in W$ and $\lambda \in \mathbb{F}$, we have $\lambda\mathbf{u} \in W$, and so

$$\|\mathbf{z}\|^2 \leq \|\mathbf{z}-\lambda\mathbf{u}\|^2 = \langle \mathbf{z}-\lambda\mathbf{u}, \mathbf{z}-\lambda\mathbf{u}\rangle = \|\mathbf{z}\|^2 - \lambda\langle \mathbf{u},\mathbf{z}\rangle - \overline{\lambda}\langle \mathbf{z},\mathbf{u}\rangle + |\lambda|^2\|\mathbf{u}\|^2.$$

It follows that

$$2\mathfrak{Re}\left(\overline{\lambda}\langle \mathbf{z},\mathbf{u}\rangle\right) \leq |\lambda|^2\|\mathbf{u}\|^2.$$

We now choose $\lambda = tc$, where $t > 0$ and $c \in \mathbb{F}$ satisfies $|c| = 1$ and $\overline{c}\langle \mathbf{z},\mathbf{u}\rangle = |\langle \mathbf{z},\mathbf{u}\rangle|$. Then

$$2t|\langle \mathbf{z},\mathbf{u}\rangle| \leq t^2\|\mathbf{u}\|^2, \qquad \text{and so} \qquad |\langle \mathbf{z},\mathbf{u}\rangle| \leq \tfrac{1}{2}t\|\mathbf{u}\|^2.$$

But $\lambda \in \mathbb{F}$ is arbitrary, and therefore so is $t > 0$. We must therefore have $|\langle \mathbf{z},\mathbf{u}\rangle| = 0$, so that $\langle \mathbf{z},\mathbf{u}\rangle = 0$, and so $\mathbf{z}$ and $\mathbf{u}$ are orthogonal. Since $\mathbf{u} \in W$ is arbitrary, it follows that $\mathbf{z} \in W^{\perp}$. Finally, suppose that

$$\mathbf{x} = \mathbf{y} + \mathbf{z} = \mathbf{y}' + \mathbf{z}', \qquad \text{where } \mathbf{y},\mathbf{y}' \in W \text{ and } \mathbf{z},\mathbf{z}' \in W^{\perp}.$$

Then $\mathbf{y} - \mathbf{y}' = \mathbf{z}' - \mathbf{z} \in W \cap W^{\perp}$. It is not difficult to show that $W \cap W^{\perp} = \{\mathbf{0}\}$. It follows that $\mathbf{y} = \mathbf{y}'$ and $\mathbf{z} = \mathbf{z}'$, giving uniqueness. ♣

THEOREM 5G. *Suppose that V is a Hilbert space over $\mathbb{F}$ and W is a closed subspace of V. Then $(W^{\perp})^{\perp} = W$.*

PROOF. It follows from the definition of orthogonal complement that $W \subseteq (W^{\perp})^{\perp}$. To show the opposite inclusion, suppose that $\mathbf{x} \in (W^{\perp})^{\perp}$. Write $\mathbf{x} = \mathbf{y} + \mathbf{z}$, where $\mathbf{y} \in W$ and $\mathbf{z} \in W^{\perp}$. Clearly $\langle \mathbf{y},\mathbf{z}\rangle = 0$. Furthermore $\langle \mathbf{x},\mathbf{z}\rangle = 0$. But then $\langle \mathbf{x},\mathbf{z}\rangle = \langle \mathbf{y}+\mathbf{z},\mathbf{z}\rangle = \langle \mathbf{y},\mathbf{z}\rangle + \langle \mathbf{z},\mathbf{z}\rangle = \|\mathbf{z}\|^2$. It follows that $\mathbf{z} = \mathbf{0}$, and so $\mathbf{x} = \mathbf{y} \in W$. ♣

PROBLEMS FOR CHAPTER 5

1. Consider the system $(f_\alpha)_{\alpha\in\mathbb{R}}$, where for every $\alpha \in \mathbb{R}$, the function $f_\alpha : \mathbb{R} \to \mathbb{C}$ is defined by $f_\alpha(t) = \mathrm{e}^{\mathrm{i}\alpha t}$ for every $t \in \mathbb{R}$.
   a) Show that $(f_\alpha)_{\alpha\in\mathbb{R}}$ is an orthonormal system in the inner product space Ψ in Problem 3 in Chapter 4.
   b) Is $(f_\alpha)_{\alpha\in\mathbb{R}}$ an orthonormal sequence? Justify your assertion.

2. Show that the sequence $(g_0, g_1, h_1, g_2, h_2, \ldots)$ in Example 5.1.2 is an orthonormal system by following the steps below:
   a) Show that $\langle g_m, g_n\rangle = 0$ and $\langle h_m, h_n\rangle = 0$ for every $m, n \in \mathbb{N}$ satisfying $m \neq n$.
   b) Show that $\langle g_m, h_n\rangle = 0$ for every $m, n \in \mathbb{N}$.
   c) Show that $\langle g_0, g_n\rangle = 0$ and $\langle g_0, h_n\rangle = 0$ for every $n \in \mathbb{N}$.
   d) Show that $\langle g_0, g_0\rangle = 1$.
   e) Show that $\langle g_n, g_n\rangle = 1$ and $\langle h_n, h_n\rangle = 1$ for every $n \in \mathbb{N}$.

3. Suppose that I is a finite set, and that $(\mathbf{x}_\alpha)_{\alpha\in I}$ is an orthogonal system in an inner product space over $\mathbb{F}$. By writing the left hand side as an inner product and then expanding, prove Pythagoras's theorem, that
$$\left\|\sum_{\alpha\in I}\mathbf{x}_\alpha\right\|^2 = \sum_{\alpha\in I}\|\mathbf{x}_\alpha\|^2.$$

4. Suppose that I is a finite set, and that $(\mathbf{x}_\alpha)_{\alpha\in I}$ is an orthonormal system in an inner product space V over $\mathbb{F}$. Suppose further that $\mathbf{x} \in V$, and that $c_\alpha = \langle \mathbf{x}, \mathbf{x}_\alpha\rangle$ for every $\alpha \in I$.
   a) By writing the left hand side as an inner product and then expanding, show that for every system $(\lambda_\alpha)_{\alpha\in I}$ in $\mathbb{F}$, we have
$$\left\|\mathbf{x} - \sum_{\alpha\in I}\lambda_\alpha\mathbf{x}_\alpha\right\|^2 = \|\mathbf{x}\|^2 + \sum_{\alpha\in I}|\lambda_\alpha - c_\alpha|^2 - \sum_{\alpha\in I}|c_\alpha|^2.$$
   b) Show that the closest point $\mathbf{y}$ in the linear span $\mathrm{span}(\{\mathbf{x}_\alpha : \alpha \in I\})$ to $\mathbf{x}$ is given by
$$\mathbf{y} = \sum_{\alpha\in I}\langle \mathbf{x}, \mathbf{x}_\alpha\rangle\mathbf{x}_\alpha,$$
   and that
$$\sum_{\alpha\in I}|\langle \mathbf{x}, \mathbf{x}_\alpha\rangle|^2 = \|\mathbf{x}\|^2 - \|\mathbf{y} - \mathbf{x}\|^2.$$
   c) Show that if $\mathbf{x} \in \mathrm{span}(\{\mathbf{x}_\alpha : \alpha \in I\})$, then
$$\mathbf{x} = \sum_{\alpha\in I}\langle \mathbf{x}, \mathbf{x}_\alpha\rangle\mathbf{x}_\alpha.$$

5. Consider the system $(f_j)_{j\in\mathbb{Z}}$, where for every $j \in \mathbb{Z}$, the function $f_j(z) = z^j$ is a rational function analytic on the unit circle $T = \{z \in \mathbb{C} : |z| = 1\}$.
   a) Show that $(f_j)_{j\in\mathbb{Z}}$ is an orthonormal sequence in the inner product space $Q(T)$ studied in Example 4.2.3.
   b) Suppose that $\alpha \in \mathbb{C}$ satisfies $|\alpha| \neq 1$. Find the Fourier coefficients of the function $f \in Q(T)$, defined by $f(z) = (z - \alpha)^{-1}$, with respect to the orthonormal sequence in part (a). Take care to distinguish the cases $|\alpha| > 1$ and $|\alpha| < 1$.

6. Suppose that V is a Hilbert space over $\mathbb{F}$, and that W is a closed subspace of V. Suppose further that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is an orthonormal basis of W. Show that for every vector $\mathbf{x} \in V$, the vector
$$\mathbf{y} = \sum_{n=1}^{\infty}\langle \mathbf{x}, \mathbf{x}_n\rangle\mathbf{x}_n$$
satisfies
$$\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x} - \mathbf{u}\| \qquad \text{for every } \mathbf{u} \in W.$$

7. Suppose that W is a closed subspace of a Hilbert space over $\mathbb{F}$. Show that $W \cap W^\perp = \{\mathbf{0}\}$.

8. Consider the set $\ell^2_{\mathbb{Z}}$ of all doubly infinite sequences $\mathbf{x} = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots)$ of complex numbers such that

$$\sum_{i=-\infty}^{\infty} |x_i|^2 < \infty.$$

a) Show that for every $\mathbf{x}, \mathbf{y} \in \ell^2_{\mathbb{Z}}$, the quantity

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=-\infty}^{\infty} x_i \overline{y_i}$$

is well defined, and gives rise to an inner product on $\ell^2_{\mathbb{Z}}$.
b) Show that $\ell^2_{\mathbb{Z}}$ is isomorphic to ℓ^2 by writing down an isomorphism explicitly.

9. Suppose that V is a Hilbert space over $\mathbb{F}$, and that $\mathbf{y} \in V$ is non-zero. Suppose further that

$$W = \{\mathbf{x} \in V : \langle \mathbf{x}, \mathbf{y} \rangle = 0\}.$$

Describe the set $W^\perp$.

10. Consider the inner product space $L^2[-1,1]$ as discussed in Example 4.4.4.
a) Let $W = \{f \in L^2[-1,1] : f(t) = 0 \text{ for every } t \in [-1,0]\}$. Find $W^\perp$.
b) Let

$$W_{\text{odd}} = \{f \in L^2[-1,1] : f(-t) = -f(t) \text{ for every } t \in [-1,1]\}$$

and

$$W_{\text{even}} = \{f \in L^2[-1,1] : f(-t) = f(t) \text{ for every } t \in [-1,1]\}.$$

Show that $L^2[-1,1] = W_{\text{odd}} \oplus W_{\text{even}}$ represents an orthogonal direct sum.

11. Consider the inner product space ℓ_0 consisting of all infinite sequences of complex numbers with only finitely many non-zero terms, with the inner product of ℓ^2, as discussed in Section 4.1. Let $\mathbf{a} = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right) \in \ell^2$.
a) Show that $W = \{\mathbf{x} \in \ell_0 : \langle \mathbf{x}, \mathbf{a} \rangle = 0 \text{ in } \ell^2\}$ is a closed subspace of ℓ_0.
b) Show that $W^\perp = \{\mathbf{0}\}$, where $\mathbf{0}$ denotes the sequence with all terms zero.
c) Is $\ell_0 = W \oplus W^\perp$? Comment on your answer.

12. Suppose that $f, g \in L^2[-\pi, \pi]$, with Fourier series

$$f(t) \sim \sum_{n \in \mathbb{Z}} c_n \mathrm{e}^{\mathrm{i}nt} \qquad \text{and} \qquad g(t) \sim \sum_{n \in \mathbb{Z}} d_n \mathrm{e}^{\mathrm{i}nt}.$$

a) By establishing a suitable unitary transformation $\phi : L^2[-\pi, \pi] \to \ell^2_{\mathbb{Z}}$, where $\ell^2_{\mathbb{Z}}$ is the inner product space described in Problem 8, prove Parseval's formula, that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \overline{g(t)} \, \mathrm{d}t = \sum_{n \in \mathbb{Z}} c_n \overline{d_n}.$$

b) Deduce that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2 \, \mathrm{d}t = \sum_{n \in \mathbb{Z}} |c_n|^2.$$

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