LINEAR FUNCTIONAL ANALYSIS

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Chapter 3

NORMED VECTOR SPACES

3.1. Review of Vector Spaces

In this section, we shall review some of the ideas we have encountered in linear algebra concerning real vector spaces and extend them to complex vector spaces. Throughout, $\mathbb{F}$ denotes either the set $\mathbb{R}$ of all real numbers or the set $\mathbb{C}$ of all complex numbers.

The confident reader may choose to skip this part, while any reader who feels a little uncomfortable with the treatment here is advised to revisit appropriate material for the details.

A vector space over $\mathbb{F}$ is a set V, together with vector addition and scalar multiplication, and satisfying the following conditions:

(VA1) For every $\mathbf{x},\mathbf{y}\in V$, we have $\mathbf{x}+\mathbf{y}\in V$.
(VA2) For every $\mathbf{x},\mathbf{y},\mathbf{z}\in V$, we have $\mathbf{x}+(\mathbf{y}+\mathbf{z})=(\mathbf{x}+\mathbf{y})+\mathbf{z}$.
(VA3) There exists an element $\mathbf{0}\in V$ such that for every $\mathbf{x}\in V$, we have $\mathbf{x}+\mathbf{0}=\mathbf{0}+\mathbf{x}=\mathbf{x}$.
(VA4) For every $\mathbf{x}\in V$, there exists $-\mathbf{x}\in V$ such that $\mathbf{x}+(-\mathbf{x})=\mathbf{0}$.
(VA5) For every $\mathbf{x},\mathbf{y}\in V$, we have $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$.
(SM1) For every $c\in\mathbb{F}$ and $\mathbf{x}\in V$, we have $c\mathbf{x}\in V$.
(SM2) For every $c\in\mathbb{F}$ and $\mathbf{x},\mathbf{y}\in V$, we have $c(\mathbf{x}+\mathbf{y})=c\mathbf{x}+c\mathbf{y}$.
(SM3) For every $a,b\in\mathbb{F}$ and $\mathbf{x}\in V$, we have $(a+b)\mathbf{x}=a\mathbf{x}+b\mathbf{x}$.
(SM4) For every $a,b\in\mathbb{F}$ and $\mathbf{x}\in V$, we have $(ab)\mathbf{x}=a(b\mathbf{x})$.
(SM5) For every $\mathbf{x}\in V$, we have $1\mathbf{x}=\mathbf{x}$.

REMARK. We shall frequently omit reference to vector addition and scalar multiplication, and simply refer to V as a vector space over $\mathbb{F}$. Furthermore, if $\mathbb{F}=\mathbb{R}$, then we say that V is a real vector space; whereas if $\mathbb{F}=\mathbb{C}$, then we say that V is a complex vector space.

Suppose that V is a vector space over $\mathbb{F}$, and that W is a non-empty subset of V. Then we say that W is a linear subspace of V if it forms a vector space over $\mathbb{F}$ under the vector addition and scalar multiplication defined for V. It is easy to see that W is a linear subspace of V if $\mathbf{x}+\mathbf{y}\in W$ and $c\mathbf{x}\in W$ for every $\mathbf{x},\mathbf{y}\in W$ and $c\in\mathbb{F}$.

Suppose that V is a vector space over $\mathbb{F}$. By a linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r \in V$, we mean an expression of the type $c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r$, where $c_1, \ldots, c_r \in \mathbb{F}$. The set

$$\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_r\} = \{c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r : c_1, \ldots, c_r \in \mathbb{F}\}$$

is called the linear span of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$, and is a linear subspace of V.

The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r \in V$ are linearly independent over $\mathbb{F}$ if the only solution of the equation $c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r = \mathbf{0}$ in $c_1, \ldots, c_r \in \mathbb{F}$ is given by $c_1 = \ldots = c_r = 0$. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subseteq V$ forms a basis for V if $\mathbf{v}_1, \ldots, \mathbf{v}_r \in V$ are linearly independent over $\mathbb{F}$ and $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_r\} = V$. In this case, every element $\mathbf{x} \in V$ can be expressed uniquely in the form $\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r$, where $c_1, \ldots, c_r \in \mathbb{F}$.

A vector space V over $\mathbb{F}$ is said to be finite dimensional if it has a basis containing only finitely many elements. In this case, any two bases for V have the same number of elements. This number is called the dimension of V, and denoted by $\dim V$. Indeed, any finite set of linearly independent vectors in V can be expanded, if necessary, to a basis for V. Furthermore, any set of $\dim V$ linearly independent vectors in V is a basis for V.

A vector space V over $\mathbb{F}$ is said to be infinite dimensional if it does not have a basis containing only finitely many elements.

3.2. Norm in a Vector Space

In this section, we study the problem of endowing a vector space with a norm which gives a notion of length to the vectors.

Definition. A normed vector space is a vector space V over $\mathbb{F}$, together with a real valued function $\|\cdot\| : V \to \mathbb{R}$, called a norm, and satisfying the following conditions:

(NS1) For every $\mathbf{x} \in V$, we have $\|\mathbf{x}\| \ge 0$.
(NS2) For every $\mathbf{x} \in V$, we have $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.
(NS3) For every $\mathbf{x} \in V$ and every $c \in \mathbb{F}$, we have $\|c\mathbf{x}\| = |c|\|\mathbf{x}\|$.
(NS4) (TRIANGLE INEQUALITY) For every $\mathbf{x}, \mathbf{y} \in V$, we have $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$.

Remark. The condition (NS2) above is in fact superfluous. It follows immediately from condition (NS3) by taking $c = 0$. We have included it here for comparison with the properties of metrics which we shall discuss later.

Example 3.2.1. Suppose that $r \in \mathbb{N}$. Consider the real euclidean vector space $\mathbb{R}^r$. For every vector $\mathbf{x} = (x_1, \ldots, x_r) \in \mathbb{R}^r$, let

$$\|\mathbf{x}\| = \left(\sum_{i=1}^r |x_i|^2\right)^{1/2}.$$

It can be shown that conditions (NS1)–(NS4) are satisfied. The function $\|\cdot\| : \mathbb{R}^r \to \mathbb{R}$ is known as the euclidean norm or usual norm in $\mathbb{R}^r$. We do not include the details here, in view of our next example.

Example 3.2.2. Suppose that V is a finite dimensional vector space over $\mathbb{F}$, with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$. For every vector $\mathbf{x} \in V$, there exist unique $c_1, \ldots, c_r \in \mathbb{F}$ such that $\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r$, and let

$$\|\mathbf{x}\| = \left(\sum_{i=1}^r |c_i|^2\right)^{1/2}.$$

It is not difficult to check conditions (NS1)–(NS3). To check condition (NS4), let $\mathbf{y} = a_1\mathbf{v}_1 + \ldots + a_r\mathbf{v}_r$ be another vector in V. Then

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 &= \sum_{i=1}^r |c_i + a_i|^2 = \sum_{i=1}^r (c_i + a_i)(\overline{c_i} + \overline{a_i}) = \sum_{i=1}^r (c_i\overline{c_i} + c_i\overline{a_i} + a_i\overline{c_i} + a_i\overline{a_i}) \\ &\le \sum_{i=1}^r |c_i|^2 + 2\sum_{i=1}^r |c_i||a_i| + \sum_{i=1}^r |a_i|^2 = \|\mathbf{x}\|^2 + 2\sum_{i=1}^r |c_i||a_i| + \|\mathbf{y}\|^2.\end{aligned}$$

By the Cauchy-Schwarz inequality

$$\sum_{i=1}^{r} |c_i||a_i| \le \left(\sum_{i=1}^{r} |c_i|^2\right)^{1/2} \left(\sum_{i=1}^{r} |a_i|^2\right)^{1/2} = \|\mathbf{x}\|\|\mathbf{y}\|,$$

we conclude that

$$\|\mathbf{x}+\mathbf{y}\|^2 \le \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.$$

Condition (NS4) now follows on taking square roots.

EXAMPLE 3.2.3. Consider the set ℓ^∞ of all bounded infinite sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ of complex numbers. It is not difficult to show that ℓ^∞ is a complex vector space, with vector addition $\mathbf{x}+\mathbf{y}$ and scalar multiplication $c\mathbf{x}$ defined respectively by

$$\mathbf{x}+\mathbf{y} = (x_1+y_1, x_2+y_2, x_3+y_3, \ldots) \qquad \text{and} \qquad c\mathbf{x} = (cx_1, cx_2, cx_3, \ldots).$$

For every $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \ell^\infty$, let

$$\|\mathbf{x}\|_\infty = \sup_{i\in\mathbb{N}} |x_i|.$$

It is not difficult to check conditions (NS1)–(NS3). To check condition (NS4), note that

$$\|\mathbf{x}+\mathbf{y}\|_\infty = \sup_{i\in\mathbb{N}} |x_i+y_i| \le \sup_{i\in\mathbb{N}}(|x_i|+|y_i|) \le \sup_{i\in\mathbb{N}} |x_i| + \sup_{i\in\mathbb{N}} |y_i| = \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty.$$

The function $\|\cdot\|_\infty : \ell^\infty \to \mathbb{R}$ is called the supremum norm or usual norm in ℓ^∞.

EXAMPLE 3.2.4. Suppose that $a, b \in \mathbb{R}$ and $a < b$. Consider the set $C[a,b]$ of all continuous complex valued functions on $[a,b]$. It is not difficult to show that $C[a,b]$ is a complex vector space, with vector addition $f+g$ and scalar multiplication cf defined respectively by

$$(f+g)(t) = f(t)+g(t) \qquad \text{and} \qquad (cf)(t) = cf(t) \qquad \text{for every } t \in [a,b].$$

For every function $f \in C[a,b]$, let

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|.$$

It is not difficult to check conditions (NS1)–(NS3). To check condition (NS4), note that

$$\|f+g\|_\infty = \sup_{t\in[a,b]} |f(t)+g(t)| \le \sup_{t\in[a,b]} (|f(t)|+|g(t)|) \le \sup_{t\in[a,b]} |f(t)| + \sup_{t\in[a,b]} |g(t)| = \|f\|_\infty + \|g\|_\infty.$$

The function $\|\cdot\|_\infty : C[a,b] \to \mathbb{R}$ is called the supremum norm in $C[a,b]$.

EXAMPLE 3.2.5. Consider the complex vector space $C[0,1]$ of all continuous complex valued functions on $[0,1]$. For every function $f \in C[0,1]$, let

$$\|f\| = \left(\int_0^1 |f(t)|^2 \,\mathrm{d}t\right)^{1/2}.$$

It is not difficult to check conditions (NS1) and (NS3). For condition (NS2), note that $\|f\| = 0$, together with the continuity of f in $[0,1]$, ensures that $f(t) = 0$ for every $t \in [0,1]$. To check condition (NS4), note that

$$\begin{aligned}
\|f+g\|^2 &= \int_0^1 |f(t)+g(t)|^2 \,\mathrm{d}t = \int_0^1 (f(t)+g(t))(\overline{f(t)}+\overline{g(t)}) \,\mathrm{d}t \\
&= \int_0^1 \left(f(t)\overline{f(t)} + f(t)\overline{g(t)} + g(t)\overline{f(t)} + g(t)\overline{g(t)}\right) \mathrm{d}t \\
&\le \int_0^1 |f(t)|^2 \,\mathrm{d}t + 2\int_0^1 |f(t)||g(t)| \,\mathrm{d}t + \int_0^1 |g(t)|^2 \,\mathrm{d}t \\
&= \|f\|^2 + 2\int_0^1 |f(t)||g(t)| \,\mathrm{d}t + \|g\|^2.
\end{aligned}$$

By the Cauchy-Schwarz inequality

$$\int_0^1 |f(t)||g(t)|\,\mathrm{d}t \le \left(\int_0^1 |f(t)|^2\,\mathrm{d}t\right)^{1/2} \left(\int_0^1 |g(t)|^2\,\mathrm{d}t\right)^{1/2} = \|f\|\|g\|,$$

we conclude that

$$\|f+g\|^2 \le \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 = (\|f\| + \|g\|)^2.$$

Condition (NS4) now follows on taking square roots.

REMARK. It is easily shown that a norm in a vector space V over $\mathbb{F}$ gives rise to a metric on V. To see this, note that we can define a real valued function $\rho : V \times V \to \mathbb{R}$ by writing

$$\rho(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| \qquad \text{for every } \mathbf{x},\mathbf{y} \in V.$$

Then the following conditions are satisfied:
- (MS1) For every $\mathbf{x},\mathbf{y} \in V$, we have $\rho(\mathbf{x},\mathbf{y}) \ge 0$.
- (MS2) For every $\mathbf{x},\mathbf{y} \in V$, we have $\rho(\mathbf{x},\mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$.
- (MS3) For every $\mathbf{x},\mathbf{y} \in V$, we have $\rho(\mathbf{x},\mathbf{y}) = \rho(\mathbf{y},\mathbf{x})$.
- (MS4) (TRIANGLE INEQUALITY) For every $\mathbf{x},\mathbf{y},\mathbf{z} \in V$, we have $\rho(\mathbf{x},\mathbf{z}) \le \rho(\mathbf{x},\mathbf{y}) + \rho(\mathbf{y},\mathbf{z})$.

Furthermore, ρ is a translation-invariant metric on V, since $\rho(\mathbf{x}+\mathbf{z},\mathbf{y}+\mathbf{z}) = \rho(\mathbf{x},\mathbf{y})$ for every $\mathbf{x},\mathbf{y},\mathbf{z} \in V$.

EXAMPLE 3.2.6. In the real normed vector space $\mathbb{R}^r$ in Example 3.2.1, we obtain the euclidean metric or usual metric

$$\rho(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \left(\sum_{i=1}^{r} |x_i - y_i|^2\right)^{1/2},$$

as discussed in Example 1.1.6.

EXAMPLE 3.2.7. In the complex normed vector space ℓ^∞ in Example 3.2.3, we obtain the supremum metric or usual metric

$$\rho(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \sup_{i\in\mathbb{N}} |x_i - y_i|,$$

as discussed in Example 1.1.13.

EXAMPLE 3.2.8. In the complex normed vector space $C[0,1]$ in Example 3.2.5, we obtain the metric

$$\rho_2(f,g) = \|f-g\| = \left(\int_0^1 |f(t)-g(t)|^2\,\mathrm{d}t\right)^{1/2},$$

as discussed in Example 1.1.17 with $[a,b] = [0,1]$.

3.3. Continuity Properties

In this section, we establish some continuity properties of norm and with respect to algebraic operations.

THEOREM 3A. *Any norm $\|\cdot\| : V \to \mathbb{R}$ on a vector space V over $\mathbb{F}$ is continuous on V.*

PROOF. Suppose that $\mathbf{x}_0 \in V$ is fixed. Then for every $\mathbf{x} \in V$, the Triangle inequality gives

$$\|\mathbf{x}\| - \|\mathbf{x}_0\| \le \|\mathbf{x}-\mathbf{x}_0\| \qquad \text{and} \qquad \|\mathbf{x}_0\| - \|\mathbf{x}\| \le \|\mathbf{x}-\mathbf{x}_0\|,$$

so that $|\|\mathbf{x}\| - \|\mathbf{x}_0\|| \le \|\mathbf{x}-\mathbf{x}_0\|$. Hence $|\|\mathbf{x}\| - \|\mathbf{x}_0\|| < \epsilon$ for every $\mathbf{x} \in V$ satisfying $\|\mathbf{x}-\mathbf{x}_0\| < \epsilon$. ♣

THEOREM 3B. *In any normed vector space V over $\mathbb{F}$, vector addition and scalar multiplication are continuous.*

Proof. To show that vector addition is continuous, suppose that $\mathbf{x}_0, \mathbf{y}_0 \in V$ are fixed. For every $\mathbf{x}, \mathbf{y} \in V$, the Triangle inequality gives

$$\|(\mathbf{x}+\mathbf{y}) - (\mathbf{x}_0+\mathbf{y}_0)\| = \|(\mathbf{x}-\mathbf{x}_0) + (\mathbf{y}-\mathbf{y}_0)\| \leq \|\mathbf{x}-\mathbf{x}_0\| + \|\mathbf{y}-\mathbf{y}_0\|.$$

Hence $\|(\mathbf{x}+\mathbf{y}) - (\mathbf{x}_0+\mathbf{y}_0)\| < \epsilon$ for every $\mathbf{x}, \mathbf{y} \in V$ satisfying $\|\mathbf{x}-\mathbf{x}_0\| < \frac{1}{2}\epsilon$ and $\|\mathbf{y}-\mathbf{y}_0\| < \frac{1}{2}\epsilon$. To show that scalar multiplication is continuous, suppose that $\mathbf{x}_0 \in V$ and $c_0 \in \mathbb{F}$ are fixed. For every $\mathbf{x} \in V$ and $c \in \mathbb{F}$, the Triangle inequality and condition (NS3) give

$$\|c\mathbf{x} - c_0\mathbf{x}_0\| = \|c\mathbf{x} - c\mathbf{x}_0 + c\mathbf{x}_0 - c_0\mathbf{x}_0\| \leq \|c\mathbf{x} - c\mathbf{x}_0\| + \|c\mathbf{x}_0 - c_0\mathbf{x}_0\| = |c|\|\mathbf{x}-\mathbf{x}_0\| + |c-c_0|\|\mathbf{x}_0\|.$$

Suppose first of all that $|c - c_0| < 1$. Then $|c| < 1 + |c_0|$, and so

$$\|c\mathbf{x} - c_0\mathbf{x}_0\| \leq (1+|c_0|)\|\mathbf{x}-\mathbf{x}_0\| + |c-c_0|\|\mathbf{x}_0\|.$$

Hence $\|c\mathbf{x} - c_0\mathbf{x}_0\| < \epsilon$ for every $\mathbf{x} \in V$ and $c \in \mathbb{F}$ satisfying

$$\|\mathbf{x}-\mathbf{x}_0\| < \frac{\epsilon}{2(1+|c_0|)} \qquad \text{and} \qquad |c-c_0| < \min\left\{1, \frac{\epsilon}{2(1+\|\mathbf{x}_0\|)}\right\}.$$

This completes the proof. ♣

3.4. Finite Dimensional Normed Vector Spaces

The purpose of this section is to show that every finite dimensional normed vector space is complete. The strategy that we shall adopt is to show that in any finite dimensional vector space V, there is at least one norm on V under which V is complete with respect to the induced metric. Since completeness is characterized by the convergence of all Cauchy sequences, we shall show that convergence properties are not affected when we change from one norm to another in V, so that the change of norm preserves completeness. The change of norm is best studied by making the following crucial definition.

Definition. Suppose that V is a vector space over $\mathbb{F}$. We say that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are equivalent if there exist positive real numbers $k, K \in \mathbb{R}$ such that

$$k\|\mathbf{x}\|_1 \leq \|\mathbf{x}\|_2 \leq K\|\mathbf{x}\|_1 \qquad \text{for every } \mathbf{x} \in V.$$

Remark. Note that the above implies $K^{-1}\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq k^{-1}\|\mathbf{x}\|_2$ for every $\mathbf{x} \in V$, so that equivalence of norms is a symmetric relation on the set of all norms on V. In fact, it is easy to check that equivalence of norms is an equivalence relation on the set of all norms on V. We should therefore wish to show that if V is a finite dimensional vector space, then this equivalence relation has only a single equivalence class.

The following result shows that convergence properties are not affected under equivalence of norms.

THEOREM 3C. *Suppose that V is a vector space over $\mathbb{F}$. Suppose further that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on V, with induced metrics ρ_1 and ρ_2 respectively. Then the following assertions hold:*
(a) A sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ converges to $\mathbf{x}$ in the metric space (V, ρ_1) if and only if it converges to $\mathbf{x}$ in the metric space (V, ρ_2).
(b) A sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the metric space (V, ρ_1) if and only if it is a Cauchy sequence in the metric space (V, ρ_2).
(c) The metric space (V, ρ_1) is complete if and only if the metric space (V, ρ_2) is complete.

PROOF. Parts (a) and (b) follow from the observation that

$$\rho_2(\mathbf{x},\mathbf{y}) \le K\rho_1(\mathbf{x},\mathbf{y}) \qquad \text{and} \qquad \rho_1(\mathbf{x},\mathbf{y}) \le k^{-1}\rho_2(\mathbf{x},\mathbf{y}) \qquad \text{for every } \mathbf{x},\mathbf{y} \in V.$$

To prove part (c), suppose that (V,ρ_1) is complete. If $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (V,ρ_2), then it follows from part (b) that it is a Cauchy sequence in (V,ρ_1). Since (V,ρ_1) is complete, it follows that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ converges in (V,ρ_1), and so it follows from part (a) that it converges in (V,ρ_2). Hence (V,ρ_2) is complete. ♣

We next show that in a finite dimensional vector space V, there is at least one norm on V under which V is complete with respect to the induced metric. The norm we use is the one described in Example 3.2.2.

THEOREM 3D. *Suppose that V is a finite dimensional vector space over $\mathbb{F}$, with basis $\{\mathbf{v}_1,\ldots,\mathbf{v}_r\}$ and norm $\|\cdot\|_0 : V \to \mathbb{R}$, defined for every $\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r$, where $c_1,\ldots,c_r \in \mathbb{F}$, by*

$$\|\mathbf{x}\|_0 = \left(\sum_{i=1}^r |c_i|^2\right)^{1/2}.$$

Then V is complete under the metric ρ_0 induced by $\|\cdot\|_0$.

PROOF. Suppose that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in V. For each $n \in \mathbb{N}$, let $\mathbf{x}_n = c_{n1}\mathbf{v}_1 + \ldots + c_{nr}\mathbf{v}_r$, where $c_{n1},\ldots,c_{nr} \in \mathbb{F}$ are unique. Note that

$$\rho_0(\mathbf{x}_m,\mathbf{x}_n) = \|\mathbf{x}_m - \mathbf{x}_n\|_0 = \left(\sum_{i=1}^r |c_{mi} - c_{ni}|^2\right)^{1/2},$$

so that for every fixed $i = 1,\ldots,r$, we have

$$|c_{mi} - c_{ni}| \le \rho_0(\mathbf{x}_m,\mathbf{x}_n),$$

and so $(c_{ni})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{F}$. Since $\mathbb{F}$ is complete, it follows that for every fixed $i = 1,\ldots,r$, there exists $c_i \in \mathbb{F}$ such that $c_{ni} \to c_i$ as $n \to \infty$, so that given any $\epsilon > 0$, there exists N_i such that

$$|c_{ni} - c_i| < \frac{\epsilon}{\sqrt{r}} \qquad \text{whenever } n > N_i.$$

Now let $\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r$. Then it is easy to see that

$$\rho_0(\mathbf{x}_n,\mathbf{x}) = \|\mathbf{x}_n - \mathbf{x}\|_0 = \left(\sum_{i=1}^r |c_{ni} - c_i|^2\right)^{1/2} < \epsilon \qquad \text{whenever } n > \max\{N_1,\ldots,N_r\}.$$

It follows that $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$. Hence V is complete. ♣

The last piece of our jigsaw is the following result.

THEOREM 3E. *Suppose that V is a finite dimensional vector space over $\mathbb{F}$, with basis $\{\mathbf{v}_1,\ldots,\mathbf{v}_r\}$. Then any norm $\|\cdot\| : V \to \mathbb{R}$ is equivalent to the norm $\|\cdot\|_0 : V \to \mathbb{R}$ in Theorem 3D.*

Combining Theorems 3C, 3D and 3E, we obtain the following result.

THEOREM 3F. *Every finite dimensional normed vector space over $\mathbb{F}$ is complete with respect to the metric induced by its norm.*

PROOF OF THEOREM 3E. Clearly

$$K = \left(\sum_{i=1}^{r} \|\mathbf{v}_i\|^2 \right)^{1/2} > 0,$$

since $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a basis of V. Suppose that $\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r \in V$. Then it follows from (NS4), (NS3) and the Cauchy-Schwarz inequality that

$$\|\mathbf{x}\| \le \sum_{i=1}^{r} \|c_i\mathbf{v}_i\| = \sum_{i=1}^{r} |c_i|\|\mathbf{v}_i\| \le \left(\sum_{i=1}^{r} |c_i|^2 \right)^{1/2} \left(\sum_{i=1}^{r} \|\mathbf{v}_i\|^2 \right)^{1/2} = K\|\mathbf{x}\|_0.$$

Next, consider the function $f : \mathbb{F}^r \to \mathbb{R}$, defined by

$$f(c_1, \ldots, c_r) = \left\| \sum_{i=1}^{r} c_i\mathbf{v}_i \right\| \qquad \text{for every } (c_1, \ldots, c_r) \in \mathbb{F}^r.$$

It follows from Theorem 3A that f is continuous with respect to the euclidean metric on $\mathbb{F}^n$. The unit circle

$$S = \left\{ (\lambda_1, \ldots, \lambda_r) \in \mathbb{F}^r : \sum_{i=1}^{r} |\lambda_i|^2 = 1 \right\}$$

is compact, and so it follows from Theorem 2N that there exists $(a_1, \ldots, a_r) \in S$ such that

$$f(a_1, \ldots, a_r) \le f(\lambda_1, \ldots, \lambda_r) \qquad \text{for every } (\lambda_1, \ldots, \lambda_r) \in S.$$

Let $k = f(a_1, \ldots, a_r)$. Then clearly $k \ge 0$. Furthermore, we must have $k \ne 0$, for otherwise it follows from (NS2) that

$$\sum_{i=1}^{r} a_i\mathbf{v}_i = \mathbf{0},$$

so that $a_1 = \ldots = a_r = 0$, contradicting the assumption that $(a_1, \ldots, a_r) \in S$. Hence $k > 0$. Now for every non-zero $\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r \in V$, we clearly have

$$\left(\frac{c_1}{\|\mathbf{x}\|_0}, \ldots, \frac{c_r}{\|\mathbf{x}\|_0} \right) \in S, \qquad \text{and so} \qquad k \le f\left(\frac{c_1}{\|\mathbf{x}\|_0}, \ldots, \frac{c_r}{\|\mathbf{x}\|_0} \right) = \left\| \frac{\mathbf{x}}{\|\mathbf{x}\|_0} \right\|.$$

It follows from (NS3) that $\|\mathbf{x}\| \ge k\|\mathbf{x}\|_0$. ♣

We conclude this section by establishing the following corollary of Theorem 3F.

THEOREM 3G. *Any finite dimensional linear subspace of a normed vector space is closed.*

PROOF. Any finite dimensional linear subspace of a normed vector space is a finite dimensional normed vector space, and is therefore complete in view of Theorem 3F. It follows from Theorem 2G that it is closed. ♣

3.5. Linear Subspaces of Normed Vector Spaces

While any finite dimensional linear subspace of a normed vector space is closed, the same cannot be said for infinite dimensional linear subspaces, as illustrated by the two examples below.

EXAMPLE 3.5.1. Recall Example 3.2.3, and consider again the normed vector space ℓ^∞ of all bounded infinite sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ of complex numbers, with supremum norm

$$\|\mathbf{x}\|_\infty = \sup_{i\in\mathbb{N}} |x_i|.$$

Consider now the subset $\ell_0 \subseteq \ell^\infty$ consisting of all infinite sequences of complex numbers which have only finitely many non-zero terms. It is not difficult to show that ℓ_0 is a linear subspace of ℓ^∞. Clearly

$$\mathbf{a} = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right) \in \ell^\infty.$$

For every $n \in \mathbb{N}$, let

$$\mathbf{x}_n = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, 0, \ldots\right) \in \ell_0.$$

Then

$$\|\mathbf{x}_n - \mathbf{a}\|_\infty = \left\|\left(\underbrace{0, \ldots, 0}_{n}, \frac{1}{n+1}, \frac{1}{n+2}, \frac{1}{n+3}, \ldots\right)\right\| = \frac{1}{n+1} \to 0 \qquad \text{as } n \to \infty.$$

It follows that the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in ℓ_0 converges in ℓ^∞, but the limit $\mathbf{a}$ does not belong to ℓ_0. Hence the linear subspace ℓ_0 is not closed in ℓ^∞.

EXAMPLE 3.5.2. Recall Example 3.2.5, and consider again the normed vector space $C[0,1]$ of all continuous complex valued functions f on $[0,1]$, with norm

$$\|f\| = \left(\int_0^1 |f(t)|^2 \,\mathrm{d}t\right)^{1/2}.$$

Consider now the subset $S = \{f \in C[0,1] : f(0) = 0\} \subseteq C[0,1]$. It is not difficult to show that S is a linear subspace of $C[0,1]$. Let $g \in C[0,1]$ be defined by $g(t) = 1$ for every $t \in [0,1]$. For every $n \in \mathbb{N}$, let $f_n \in S$ be defined by

$$f_n(t) = \begin{cases} nt & \text{if } 0 \le t \le n^{-1}, \\ 1 & \text{if } n^{-1} \le t \le 1. \end{cases}$$

Then

$$f_n(t) - g(t) = \begin{cases} nt - 1 & \text{if } 0 \le t \le n^{-1}, \\ 0 & \text{if } n^{-1} \le t \le 1, \end{cases}$$

and so

$$\|f_n - g\| = \left(\int_0^{n^{-1}} (nt-1)^2 \,\mathrm{d}t\right)^{1/2} = \left(\frac{1}{3n}\right)^{1/2} \to 0 \qquad \text{as } n \to \infty.$$

It follows that the sequence $(f_n)_{n\in\mathbb{N}}$ in S converges in $C[0,1]$, but the limit g does not belong to S. Hence the linear subspace S is not closed in $C[0,1]$.

As we shall see later, closed linear subspaces of a normed vector space are more important than linear subspaces that are not closed. In order to create closed linear subspaces, we consider the closure of W. Recall that this is the set

$$\overline{W} = W \cup \{\mathbf{x} \in V : \text{there exists a sequence } (\mathbf{x}_n)_{n\in\mathbb{N}} \text{ in } W \text{ such that } \mathbf{x}_n \to \mathbf{x} \text{ as } n \to \infty\},$$

the union of W with the set of all limit points of W. Since any $\mathbf{x} \in W$ is the limit of the constant sequence $\mathbf{x}, \mathbf{x}, \mathbf{x}, \ldots$ in W, it follows easily that

$$\overline{W} = \{\mathbf{x} \in V : \text{there exists a sequence } (\mathbf{x}_n)_{n\in\mathbb{N}} \text{ in } W \text{ such that } \mathbf{x}_n \to \mathbf{x} \text{ as } n \to \infty\}.$$

THEOREM 3H. *Suppose that V is a normed vector space over $\mathbb{F}$, and that W is a linear subspace of V. Then the closure $\overline{W}$ of W is a closed linear subspace of V.*

PROOF. Suppose that $\mathbf{x}, \mathbf{y} \in \overline{W}$. Then there exist sequences $(\mathbf{x}_n)_{n\in\mathbb{N}}$ and $(\mathbf{y}_n)_{n\in\mathbb{N}}$ in W such that $\mathbf{x}_n \to \mathbf{x}$ and $\mathbf{y}_n \to \mathbf{y}$ as $n \to \infty$. Since W is a linear subspace of V, it follows that the sequence $(\mathbf{x}_n + \mathbf{y}_n)_{n\in\mathbb{N}}$ belongs to W, and that for every $c \in \mathbb{F}$, the sequence $(c\mathbf{x}_n)_{n\in\mathbb{N}}$ belongs to W. In view of Theorem 3B, we have $\mathbf{x}_n + \mathbf{y}_n \to \mathbf{x} + \mathbf{y}$ and $c\mathbf{x}_n \to c\mathbf{x}$ as $n \to \infty$. It follows that $\mathbf{x} + \mathbf{y} \in \overline{W}$ and $c\mathbf{x} \in \overline{W}$. Hence $\overline{W}$ is a linear subspace of V. To show that $\overline{W}$ is closed in V, we simply note that V is a metric space and then use Theorem 1E. ♣

We are in particular interested in linear subspaces of a normed vector space spanned by elements of a given subset.

DEFINITION. Suppose that V is a normed vector space over $\mathbb{F}$. For any subset $A \subseteq V$, the linear span $\mathrm{span}(A)$ of A is defined to be the set of all linear combinations of elements of A. More precisely,

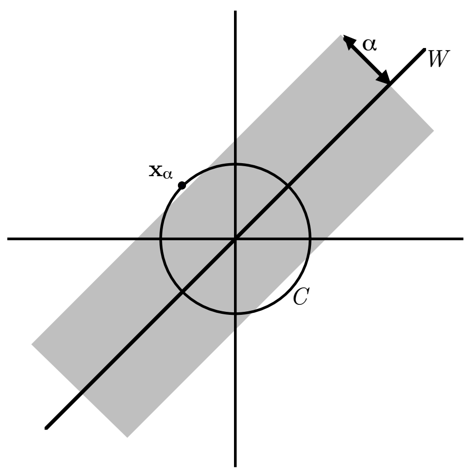
$$\mathrm{span}(A) = \{c_1\mathbf{x}_1 + \ldots + c_r\mathbf{x}_r : r \in \mathbb{N},\ c_1, \ldots, c_r \in \mathbb{F} \text{ and } \mathbf{x}_1, \ldots, \mathbf{x}_r \in A\}.$$

THEOREM 3J. *Suppose that V is a normed vector space over $\mathbb{F}$. For any subset $A \subseteq V$, the set $\mathrm{span}(A)$ is a linear subspace of V. Furthermore, $\mathrm{span}(A)$ is the intersection of all linear subspaces of V which contain A, and its closure $\overline{\mathrm{span}(A)}$ is the intersection of all closed linear subspaces of V which contain A.*

PROOF. It is not difficult to show that $\mathrm{span}(A)$ is closed under vector addition and scalar multiplication, and is therefore a linear subspace of V. Next, let E denote the intersection of all linear subspaces of V which contain A. Clearly, for every linear subspace W in the intersection E, we have $A \subseteq W$. Since the linear subspace W is closed under vector addition and scalar multiplication, it follows that $\mathrm{span}(A) \subseteq W$. Hence $\mathrm{span}(A) \subseteq E$. On the other hand, $\mathrm{span}(A)$ is a linear subspace of V which clearly contains A, and so $E \subseteq \mathrm{span}(A)$. The second assertion follows. Finally, let F denote the intersection of all closed linear subspaces of V which contain A. Clearly, for every closed linear subspace W in the intersection F, we have $A \subseteq W$, so that $\mathrm{span}(A) \subseteq W$. Since W is closed, it follows that $\overline{\mathrm{span}(A)} \subseteq W$. Hence $\overline{\mathrm{span}(A)} \subseteq F$. On the other hand, it follows from Theorem 3H that $\overline{\mathrm{span}(A)}$ is a closed linear subspace of V which clearly contains A, and so $F \subseteq \overline{\mathrm{span}(A)}$. The last assertion follows immediately. ♣

We illustrate the importance of closed linear subspaces by considering the following example.

EXAMPLE 3.5.3. Consider first of all the real euclidean space $\mathbb{R}^2$. Here any one-dimensional linear subspace W is simply a line through the origin, as shown in the picture below.



Consider now the unit circle $C = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| = 1\}$. Suppose that $\alpha \in \mathbb{R}$ satisfies $0 < \alpha < 1$. The shaded strip in the picture represents a width of 2α. It is easy to see that we can find a point $\mathbf{x}_\alpha$ on the unit circle C which does not lie in the shaded strip, and so its distance from the line W must be greater than α. In general, we have the following result:
RIESZ'S LEMMA: *Suppose that V is a normed vector space, and that W is a closed linear subspace such that $W \neq V$. Then for every real number $\alpha \in (0,1)$, there exists $\mathbf{x}_\alpha \in V$ satisfying $\|\mathbf{x}_\alpha\| = 1$ and such that $\|\mathbf{x}_\alpha - \mathbf{y}\| > \alpha$ for every $\mathbf{y} \in W$.*
To sketch a proof of Riesz's lemma, note that there exists non-zero $\mathbf{x} \in V \setminus W$. Suppose that ρ denotes the metric induced by the norm of V. Let $\rho(\mathbf{x}, W) = \inf\{\|\mathbf{x} - \mathbf{w}\| : \mathbf{w} \in W\}$. Then it can be shown, using the ideas in Problem 17 in Chapter 1, that $\rho(\mathbf{x}, W) = 0$ if and only if $\mathbf{x} \in \overline{W}$. Since $\mathbf{x} \notin W$ and W is closed, we have $W = \overline{W}$ and $\rho(\mathbf{x}, W) > 0$. Let $d = \rho(\mathbf{x}, W)$. Suppose now that $\alpha \in (0,1)$. Then $d < d\alpha^{-1}$, and so there exists $\mathbf{w} \in W$ such that $\|\mathbf{x} - \mathbf{w}\| < d\alpha^{-1}$. Let

$$\mathbf{x}_\alpha = \frac{\mathbf{x} - \mathbf{w}}{\|\mathbf{x} - \mathbf{w}\|}.$$

Clearly $\|\mathbf{x}_\alpha\| = 1$. Furthermore, for every $\mathbf{y} \in W$, we have

$$\begin{aligned}\|\mathbf{x}_\alpha - \mathbf{y}\| &= \left\| \frac{\mathbf{x} - \mathbf{w}}{\|\mathbf{x} - \mathbf{w}\|} - \mathbf{y} \right\| = \left\| \frac{\mathbf{x}}{\|\mathbf{x} - \mathbf{w}\|} - \frac{\mathbf{w}}{\|\mathbf{x} - \mathbf{w}\|} - \mathbf{y} \right\| \\ &= \frac{1}{\|\mathbf{x} - \mathbf{w}\|} \|\mathbf{x} - (\mathbf{w} + \|\mathbf{x} - \mathbf{w}\|\mathbf{y})\| > (d\alpha^{-1})^{-1} d = \alpha,\end{aligned}$$

since $\mathbf{w} + \|\mathbf{x} - \mathbf{w}\|\mathbf{y} \in W$. This proves Riesz's lemma. Note that the crucial part of the proof is the observation that $d = \rho(\mathbf{x}, W) > 0$. This cannot be guaranteed if the linear subspace W is not closed.

We next highlight a deficiency in infinite dimensional normed vector spaces.

EXAMPLE 3.5.4. Suppose that V is an infinite dimensional normed vector space over $\mathbb{F}$. We shall show that the unit circle $C = \{\mathbf{x} \in V : \|\mathbf{x}\| = 1\}$ in V is not compact. Let $\mathbf{x}_1 \in C$. The finite dimensional linear subspace $\operatorname{span}\{\mathbf{x}_1\}$ is not equal to V and is closed, in view of Theorem 3G. It follows from Riesz's lemma that there exists $\mathbf{x}_2 \in C$ such that

$$\|\mathbf{x}_2 - \mathbf{y}\| > \tfrac{1}{2} \qquad \text{for every } \mathbf{y} \in \operatorname{span}\{\mathbf{x}_1\}.$$

The finite dimensional linear subspace $\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2\}$ is not equal to V and is also closed. It follows from Riesz's lemma that there exists $\mathbf{x}_3 \in C$ such that

$$\|\mathbf{x}_3 - \mathbf{y}\| > \tfrac{1}{2} \qquad \text{for every } \mathbf{y} \in \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2\}.$$

Proceeding inductively, we construct a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in C such that

$$\|\mathbf{x}_{n+1} - \mathbf{y}\| > \tfrac{1}{2} \qquad \text{for every } \mathbf{y} \in \operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}.$$

It follows that $\|\mathbf{x}_m - \mathbf{x}_n\| > \frac{1}{2}$ whenever $m, n \in \mathbb{N}$ satisfy $m \neq n$, and so the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ cannot have a convergent subsequence. Hence C is not compact.

3.6. Banach Spaces

Recall that in finite dimensional real euclidean space $\mathbb{R}^r$, a set is compact if it is bounded and closed, so that in particular, the unit circle is compact. The same conclusion can be drawn for finite dimensional complex euclidean space $\mathbb{C}^r$. Indeed, our proof of Theorem 3E makes use of this crucial observation, and leads to the completeness of all finite dimensional normed vector spaces. On the other hand, we have shown in Example 3.5.4 that in any infinite dimensional normed vector space over $\mathbb{F}$, the unit circle is not compact. This represents a major difference between finite dimensional and infinite dimensional normed vector spaces.

To obtain deeper results for infinite dimensional normed vector spaces, we must therefore make extra restrictions on the normed vector spaces. One such restriction is completeness. Recall that any normed vector space $(V, \|\cdot\|)$ over $\mathbb{F}$ is a metric space, with induced metric

$$\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

DEFINITION. Suppose that V is normed vector space over $\mathbb{F}$. Suppose further that V is a complete metric space under the metric induced by its norm. Then we say that V is a Banach space.

EXAMPLE 3.6.1. In view of Theorem 3F, every finite dimensional normed vector space over $\mathbb{F}$ is a Banach space. These include the real enclidean space $\mathbb{R}^r$ and the complex euclidean space $\mathbb{C}^r$ for every $r \in \mathbb{N}$.

EXAMPLE 3.6.2. The norm vector space ℓ^∞ of all bounded infinite sequences of complex numbers is complete, so that ℓ^∞ is a Banach space. Recall that ℓ^∞ has norm

$$\|\mathbf{x}\|_\infty = \sup_{i \in \mathbb{N}} |x_i|,$$

giving rise to the metric

$$\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_\infty = \sup_{i \in \mathbb{N}} |x_i - y_i|.$$

Suppose that $(\mathbf{x}_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in ℓ^∞. For every $n \in \mathbb{N}$, let

$$\mathbf{x}_n = (x_{n1}, x_{n2}, x_{n3}, \ldots, x_{ni}, \ldots).$$

Then

$$\begin{aligned}
\mathbf{x}_1 &= (x_{11}, x_{12}, x_{13}, \ldots, x_{1i}, \ldots),\\
\mathbf{x}_2 &= (x_{21}, x_{22}, x_{23}, \ldots, x_{2i}, \ldots),\\
\mathbf{x}_3 &= (x_{31}, x_{32}, x_{33}, \ldots, x_{3i}, \ldots),\\
&\ \vdots\\
\mathbf{x}_n &= (x_{n1}, x_{n2}, x_{n3}, \ldots, x_{ni}, \ldots),\\
&\ \vdots
\end{aligned}$$

For any fixed $i \in \mathbb{N}$, let us consider the sequence $x_{1i}, x_{2i}, x_{3i}, \ldots, x_{ni}, \ldots$. It is clear that for every $m, n \in \mathbb{N}$, we have

$$|x_{mi} - x_{ni}| \le \|\mathbf{x}_m - \mathbf{x}_n\| = \rho(\mathbf{x}_m, \mathbf{x}_n).$$

Since $(\mathbf{x}_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in ℓ^∞, it follows that $(x_{ni})_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, it follows that there exists $a_i \in \mathbb{C}$ such that $x_{ni} \to a_i$ as $n \to \infty$. Let

$$\mathbf{a} = (a_1, a_2, a_3, \ldots, a_i, \ldots).$$

We shall show that $\mathbf{a} \in \ell^\infty$, and that $\mathbf{x}_n \to \mathbf{a}$ as $n \to \infty$ in ℓ^∞. Given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\rho(\mathbf{x}_m, \mathbf{x}_n) < \epsilon \qquad \text{for every } m, n \in \mathbb{N} \text{ satisfying } m > n \ge N. \tag{1}$$

Taking $n = N$, we have

$$\rho(\mathbf{x}_m, \mathbf{x}_N) < \epsilon \qquad \text{for every } m \in \mathbb{N} \text{ satisfying } m > N.$$

It follows that for every $I \in \mathbb{N}$, we have

$$\sup_{1 \le i \le I} |x_{mi} - x_{Ni}| < \epsilon \qquad \text{for every } m \in \mathbb{N} \text{ satisfying } m > N.$$

Keeping I fixed and letting $m \to \infty$, we obtain

$$\sup_{1 \le i \le I} |a_i - x_{Ni}| \le \epsilon.$$

Since $I \in \mathbb{N}$ is arbitrary, we conclude that

$$\rho(\mathbf{a}, \mathbf{x}_N) = \sup_{i \in \mathbb{N}} |a_i - x_{Ni}| \le \epsilon.$$

Note next that

$$\|\mathbf{a}\| = \|\mathbf{x}_N + \mathbf{a} - \mathbf{x}_N\| \le \|\mathbf{x}_N\| + \|\mathbf{a} - \mathbf{x}_N\| = \|\mathbf{x}_N\| + \rho(\mathbf{a}, \mathbf{x}_N) \le \|\mathbf{x}_N\| + \epsilon,$$

so that $\mathbf{a} \in \ell^\infty$. To show that $\mathbf{x}_n \to \mathbf{a}$ as $n \to \infty$ in ℓ^∞, we observe that it follows from (1) that for every $n \ge N$, we have

$$\rho(\mathbf{x}_m, \mathbf{x}_n) < \epsilon \qquad \text{for every } m \in \mathbb{N} \text{ satisfying } m > n.$$

It follows that for every $I \in \mathbb{N}$, we have

$$\sup_{1 \le i \le I} |x_{mi} - x_{ni}| < \epsilon \qquad \text{for every } m \in \mathbb{N} \text{ satisfying } m > n.$$

Keeping I fixed and letting $m \to \infty$, we obtain

$$\sup_{1 \le i \le I} |a_i - x_{ni}| \le \epsilon.$$

Since $I \in \mathbb{N}$ is arbitrary, we conclude that

$$\rho(\mathbf{a}, \mathbf{x}_n) = \sup_{i \in \mathbb{N}} |a_i - x_{ni}| \le \epsilon.$$

To summarize, we have shown that given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\rho(\mathbf{a}, \mathbf{x}_n) \le \epsilon$ for every $n \in \mathbb{N}$ satisfying $n \ge N$. Hence $\mathbf{x}_n \to \mathbf{a}$ as $n \to \infty$ in ℓ^∞.

Problems for Chapter 3

1. Suppose that $\mathbf{x}$ is a non-zero vector in a normed vector space V over $\mathbb{F}$, and that $\alpha > 0$. Find a real number $c \in \mathbb{R}$ such that $\|c\mathbf{x}\| = \alpha$.

2. Suppose that V and W are vector spaces over $\mathbb{F}$. Consider the cartesian product $V \times W$, with vector addition and scalar multiplication defined by

$$(\mathbf{v}_1, \mathbf{w}_1) + (\mathbf{v}_2, \mathbf{w}_2) = (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_1 + \mathbf{w}_2) \qquad \text{and} \qquad c(\mathbf{v}_1, \mathbf{w}_1) = (c\mathbf{v}_1, c\mathbf{w}_1)$$

for every $(\mathbf{v}_1, \mathbf{w}_1), (\mathbf{v}_2, \mathbf{w}_2) \in V \times W$ and $c \in \mathbb{F}$.

a) Show that $V \times W$ is a vector space over $\mathbb{F}$.
b) Suppose that $\|\cdot\|_V$ is a norm on V and $\|\cdot\|_W$ is a norm on W. Show that

$$\|(\mathbf{v}, \mathbf{w})\| = \|\mathbf{v}\|_V + \|\mathbf{w}\|_W$$

defines a norm on $V \times W$.
c) Show that a sequence $(\mathbf{v}_n, \mathbf{w}_n)_{n \in \mathbb{N}}$ in $V \times W$ converges to $(\mathbf{v}, \mathbf{w}) \in V \times W$ as $n \to \infty$ if and only if the sequence $(\mathbf{v}_n)_{n \in \mathbb{N}}$ in V converges to $\mathbf{v} \in V$ and the sequence $(\mathbf{w}_n)_{n \in \mathbb{N}}$ in W converges to $\mathbf{w} \in W$ as $n \to \infty$.
d) Comment on Cauchy sequences.

3. Consider the set
$$\ell^1 = \left\{ \mathbf{x} = (x_1, x_2, x_3, \ldots) \in \mathbb{C}^\infty : \sum_{i=1}^{\infty} |x_i| < \infty \right\},$$
consisting of all absolutely summable infinite sequences of complex numbers.
   a) Show that ℓ^1 is a complex vector space.
   b) Show that
$$\|\mathbf{x}\|_1 = \sum_{i=1}^{\infty} |x_i|$$
   defines a norm on ℓ^1.

4. Consider the real vector space $\mathbb{R}^2$. For every $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, let $\|\mathbf{x}\| = |x_1| + |x_2|$.
   a) Show that $\|\cdot\|$ defines a norm on $\mathbb{R}^2$.
   b) Sketch the unit circle $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| = 1\}$.

5. Consider the complex vector space $C[0, 1]$ of all continuous complex valued functions defined on $[0, 1]$. For every $n \in \mathbb{N}$, let $f_n \in C[0, 1]$ be defined by
$$f_n(t) = \begin{cases} 1 - nt & \text{if } 0 \le t \le n^{-1}, \\ 0 & \text{if } n^{-1} \le t \le 1. \end{cases}$$
   a) Determine $\|f_n\|_\infty$, where $\|\cdot\|_\infty$ is the supremum norm in Example 3.2.4 with $[a, b] = [0, 1]$.
   b) Determine $\|f_n\|$, where $\|\cdot\|$ is the norm in Example 3.2.5.
   c) Comment on your observations.

6. Let c_0 denote the set of all infinite sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ of complex numbers such that $x_i \to 0$ as $i \to \infty$.
   a) Show that c_0 is a linear subspace of ℓ^∞.
   b) Show that c_0 is closed in ℓ^∞ with respect to the supremum norm $\|\cdot\|_\infty$ in Example 3.2.3.

7. Let W be the linear subspace of all polynomials in $C[0, 1]$.
   a) Show that W is not closed in $C[0, 1]$ with respect to the supremum norm in Example 3.2.4 with $[a, b] = [0, 1]$.
   b) Show that W is not closed in $C[0, 1]$ with respect to the norm in Example 3.2.5.
   [Hint: Consider the exponential function e^x.]

8. Let $P[0, 1]$ denote the complex vector space of all complex valued polynomials defined on $[0, 1]$. This can be viewed as a linear subspace of $C[0, 1]$. Show that the two norms
$$\|f\|_\infty = \sup_{t \in [0,1]} |f(t)| \qquad \text{and} \qquad \|f\|_1 = \int_0^1 |f(t)|\, \mathrm{d}t$$
are not equivalent on $P[0, 1]$.

9. Consider the complex vector space ℓ^∞ of all bounded infinite sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ of complex numbers. Let $A = \{\mathrm{e}_j : j \in \mathbb{N}\}$, where
$$\mathrm{e}_j = (\underbrace{0, \ldots, 0}_{j-1}, 1, 0, 0, 0, \ldots) \qquad \text{for every } j \in \mathbb{N}.$$
Show that $\overline{\mathrm{span}(A)} = c_0$, where c_0 is defined in Problem 6.

10. Suppose that V is a normed vector space over $\mathbb{F}$. Let

$$S = \{\mathbf{x} \in V : \|\mathbf{x}\| < 1\} \qquad \text{and} \qquad T = \{\mathbf{x} \in V : \|\mathbf{x}\| \le 1\}.$$

a) Show that T is closed.
b) Show that every vector $\mathbf{x} \in V$ satisfying $\|\mathbf{x}\| = 1$ is an accumulation point of S.
c) Show that every vector $\mathbf{x} \in V$ satisfying $\|\mathbf{x}\| > 1$ is not an accumulation point of S.
d) Deduce that $\overline{S} = T$.

11. Let c_0 denote the set of all infinite sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ of complex numbers such that $x_i \to 0$ as $i \to \infty$. Explain why c_0 is a Banach space under the norm

$$\|\mathbf{x}\|_\infty = \sup_{i \in \mathbb{N}} |x_i|.$$

[Hint: Use Theorem 2G and Problem 6.]

12. Suppose that $a, b \in \mathbb{R}$ and $a < b$. Show that the vector space $C[a, b]$ of all continuous complex valued functions defined on $[a, b]$, with supremum norm

$$\|f\|_\infty = \sup_{t \in [a,b]} |f(t)|,$$

is a Banach space.

13. In the notation of Problem 2, suppose that V is a Banach space with respect to the norm $\|\cdot\|_V$ and W is a Banach space with respect to the norm $\|\cdot\|_W$. Show that $V \times W$ is a Banach space with respect to the norm $\|\cdot\|$.

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