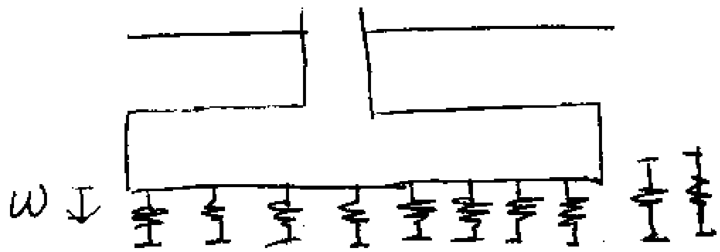
Vigas en un medio elástico
(Método de Winkler)

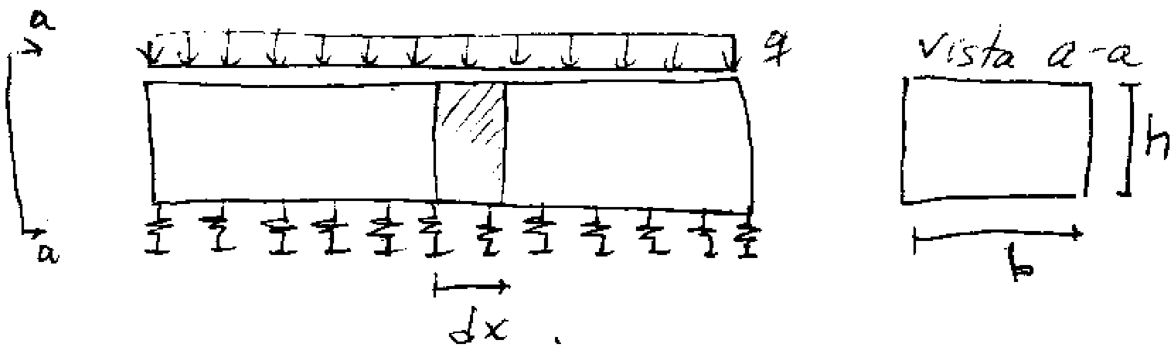


No hay continuidad de deformaciones.

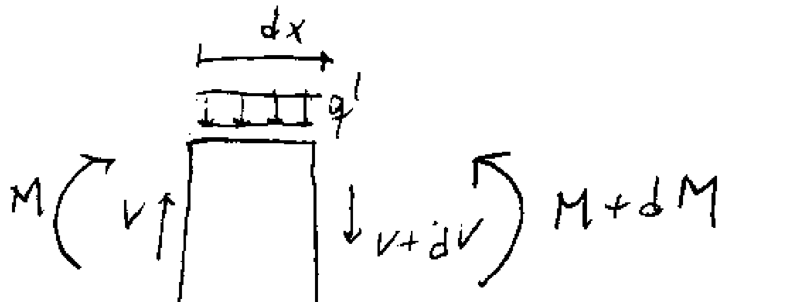
$\boxed{Q = K \cdot w}$ ec. válida punto a punto.

Q : Tensión [kg/cm²]
w = Asentamiento [cm]
k = cte. balasto [kg/cm³]

Ecuación diferencial de la viga



Sea $k' = k \cdot b$
$q' = q \cdot b$



Equilibrio

$$\Sigma F_V = 0 \Rightarrow -V + V + dV + q'dx - k'w\,dx = 0$$

$$\boxed{\frac{dV}{dx} = k'w - q'}$$

$\sum M = 0$. $V = \frac{dM}{dx} \Rightarrow \frac{dV}{dx} = \frac{d^2M}{dx^2}$

$$\Rightarrow \frac{d^2M}{dx^2} = k'w - q'$$

Ecuación de flexión de una viga (Navier-Bernoulli)

$$M = -EI\frac{d^2w}{dx^2}$$; con EI prop. de la viga.

se obtiene

$$-EI\frac{d^4w}{dx^4} = k'w - q'$$

$$\Rightarrow \frac{d^4w}{dx^4} + \frac{k'w}{EI} - \frac{q'}{EI} = 0$$

Solución de la ecuación homogénea

$$\frac{dw^4}{dx^4} + \frac{k'w}{EI} = 0 \quad (1)$$

- 1ra forma de solución

solución $w = e^{Dx}$

$$D^4 e^{Dx} + \frac{k' e^{Dx}}{EI} = 0$$

$$\Rightarrow D^4 = \frac{-k'}{EI}$$ cuya solución es compleja.

$$\Rightarrow \quad D_{1,2} = \pm \sqrt[4]{\frac{k'}{4EI}}\,(1+i)$$

$$D_{3,4} = \pm \sqrt{\frac{k'}{4EI}}\,(-1+i)$$

Se define $\lambda = \sqrt[4]{\frac{k'}{4EI}}$ $\left(\begin{array}{c}\text{longitud}\\ \text{elástica de}\\ \text{una viga}\end{array}\ 1/\lambda\right)$

$$[\lambda] = 1/cm$$

- 2^{do} forma

utilizando un cambio de variable $v = \lambda \cdot x$

(1) queda de la siguiente forma

$$\frac{d^4 w}{dv^4} + 4w = 0$$

La solución general es

$$w = z_1 e^{s_1 v} + z_2 e^{s_2 v} + z_3 e^{-s_1 v} + z_4 e^{-s_2 v}$$

donde s son las soluciones de la ecuación

$$s^4 + 4 = 0$$

$$s_1 = -s_4 = 1+i \;\; ; \;\; s_2 = -s_3 = 1-i$$

Por lo tanto la solución queda expresada por

$$w = A_1 e^{(1+i)\lambda x} + A_2 e^{-(1+i)\lambda x} + A_3 e^{(-1+i)\lambda x} + A_4 e^{-(-1+i)\lambda x}$$

$$w = e^{\lambda x}\left(A_1 e^{i\lambda x} + A_4 e^{-i\lambda x}\right) + e^{-\lambda x}\left(A_2 e^{-i\lambda x} + A_3 e^{i\lambda x}\right)$$

Sabiendo que

$$e^{i\lambda x} = \cos\lambda x + i \operatorname{sen}\lambda x$$
$$e^{-i\lambda x} = \cos\lambda x - i \operatorname{sen}\lambda x$$

Reemplazando

$$W = e^{\lambda x}\left(\cos\lambda x\,(A_1+A_4) + i\operatorname{sen}\lambda x\,(A_1-A_4)\right)$$
$$+\, e^{-\lambda x}\left(\cos\lambda x\,(A_2+A_3) + i\operatorname{sen}\lambda x\,(A_3-A_2)\right)$$

Se define

$$C_1 = A_1 + A_4$$
$$C_2 = i\,(A_1 - A_4)$$
$$C_3 = A_2 + A_3$$
$$C_4 = i\,(A_3 - A_2)$$

La solución

$$w = e^{\lambda x}\left(C_1\cos\lambda x + C_2\operatorname{sen}\lambda x\right) + e^{-\lambda x}\left(C_3\cos\lambda x + C_4\operatorname{sen}\lambda x\right)$$

1ª cond de Borde

Para $x \to \infty$ $\quad W = 0 \Rightarrow C_1 = C_2 = 0$

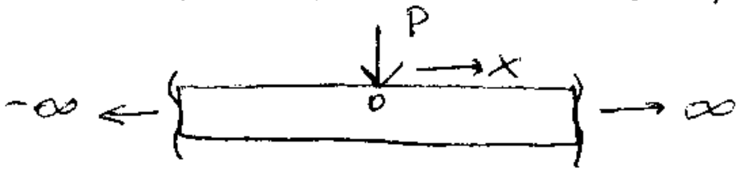
$$\Rightarrow\quad W = e^{-\lambda x}\left(C_3\cos\lambda x + C_4\operatorname{sen}\lambda x\right)$$

sabemos que

$$y = w \qquad M = -EI\,\frac{d^2w}{dx^2}$$
$$\theta = \frac{dw}{dx} \qquad V = \frac{dM}{dx}$$

Caso a

Viga infinita con carga puntual centrada



1ª Cond. Borde

$$\theta = \left.\frac{dw}{dx}\right|_{x=0} = 0$$

$$\frac{dw}{dx} = -\lambda e^{-\lambda x} \cdot (C_3 \cos\lambda x + C_4 \operatorname{sen}\lambda x) + e^{-\lambda x}(-\lambda C_3 \operatorname{sen}\lambda x + \lambda C_4 \cos\lambda x)$$

en $x=0$

$$\Rightarrow \left.\frac{dw}{dx}\right|_{x=0} = -\lambda C_3 + \lambda C_4 \Rightarrow C_3 = C_4 = C$$

luego $\boxed{w = C e^{-\lambda x}(\cos\lambda x + \operatorname{sen}\lambda x)}$

2ª Cond. Borde

$$\Sigma F_v = 0 \qquad P = 2\int_0^\infty k' w \, dx$$

$$P = 2k'C\int_0^\infty e^{-\lambda x}(\cos\lambda x + \operatorname{sen}\lambda x)\,dx$$

integrando

$$\Rightarrow P = \frac{2k'C}{\lambda} \Rightarrow C = \frac{P\lambda}{2k'}$$

$$\Rightarrow w = \frac{P\lambda}{2k'} e^{-\lambda x}(\cos\lambda x + \operatorname{sen}\lambda x)$$

$$\theta = \frac{-P\lambda^2}{2k'} \operatorname{sen}(\lambda x)\, e^{-\lambda x}$$

$$M = \frac{P}{4\lambda} \left(-\sin \lambda x + \cos \lambda x\right) e^{-\lambda x}$$

$$V = \frac{-P}{2} \cos(\lambda x)\, e^{-\lambda x}$$

Llamamos $A\lambda x = e^{-\lambda x}(\cos \lambda x + \sin \lambda x)$

$$B\lambda x = e^{-\lambda x}(\sin \lambda x)$$

$$C\lambda x = e^{-\lambda x}(\cos \lambda x - \sin \lambda x)\, e^{-\lambda x}$$

$$D\lambda x = \cos \lambda x \cdot e^{-\lambda x}$$

$\Rightarrow$ $\quad w = \dfrac{P\lambda}{2k'} A\lambda(x) \qquad M = \dfrac{P}{4\lambda} C\lambda x$

$\qquad \theta = \dfrac{-P\lambda}{2k'} B\lambda x \qquad V = \dfrac{-P}{2} D\lambda x$

* $\Rightarrow$ Se define los siguientes casos

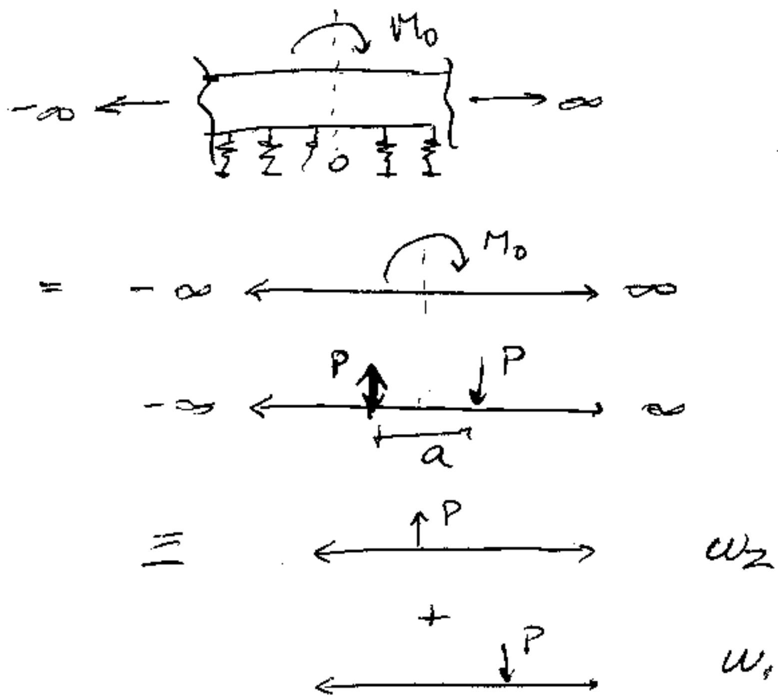
$\lambda \cdot L > \pi$ viga infinita o semi finita

$\dfrac{\pi}{4} < \lambda L < \pi$ viga finita

$\lambda \cdot L < \pi/4$ viga rígida o corta ()

Caso 6

Viga infinita con momento



Luego $w_T = w_1 + w_2$

$$= \frac{P\lambda}{2k'} A_{\lambda x} - \frac{P\lambda}{2k'} A_{\lambda (x+a)}$$

$$= \frac{\lambda P a}{2k'} \left[\frac{A_{\lambda x} - A_{\lambda(x+a)}}{a} \right]$$

$$= -\frac{\lambda M_0}{2k'} \left[\frac{A_{\lambda(x+a)} - A_{\lambda x}}{a} \right]$$

haciendo lim $a \to 0$

$$w = -\frac{\lambda M_0}{2k'} A'_{\lambda x}$$

$$A_{\lambda x} = e^{-\lambda x} (\cos \lambda x + \sin \lambda x)$$
$$A'_{\lambda x} = (-\lambda \sin \lambda + \lambda \cos \lambda x) e^{-\lambda x} - \lambda (\cos \lambda x + \sin \lambda x) e^{-\lambda x}$$
$$A'_{\lambda x} = -2\lambda \sin \lambda x \, e^{-\lambda x} = -2\lambda B_{\lambda x}$$

$$\Rightarrow \quad w = \frac{\lambda^2 M_0}{k'} \quad B\lambda x$$

de forma similar obtenemos

$$\theta = \frac{M_0 \lambda^3}{k'} \quad B\lambda x$$

$$M = \frac{M_0}{2} \quad C\lambda x$$

$$V = -\frac{\lambda M_0}{2} \quad A\lambda x$$

INFINITE BEAM

CONCENTRATED LOAD

DEFLECTION: $y = \frac{P\lambda}{2K} A_{\lambda x}$

MOMENT: $M = \frac{P}{4\lambda} C_{\lambda x}$

SHEAR: $Q = -\frac{P}{2} D_{\lambda x}$

APPLIED MOMENT

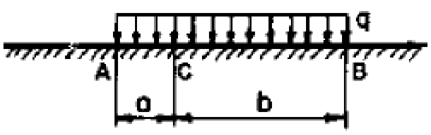
DEFLECTION: $y = \frac{M_o \lambda^2}{K} B_{\lambda x}$

MOMENT: $M = \frac{M_o}{2} D_{\lambda x}$

SHEAR: $Q = -\frac{M_o \lambda}{2} A_{\lambda x}$

UNIFORMLY DISTRIBUTED LOAD

POINT C IS UNDER LOAD

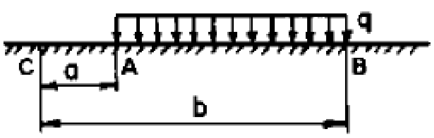


DEFLECTION: $y_C = \frac{q}{2K}(2 - D_{\lambda a} - D_{\lambda b})$

MOMENT: $M_C = \frac{q}{4\lambda^2}(B_{\lambda a} + B_{\lambda b})$

SHEAR: $Q_C = \frac{q}{4\lambda}(C_{\lambda a} - C_{\lambda b})$

POINT C IS LEFT OF LOAD

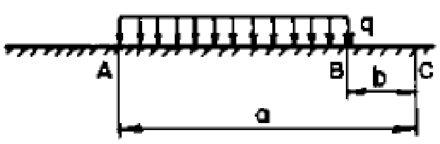


DEFLECTION: $y_C = \frac{q}{2K}(D_{\lambda a} - D_{\lambda b})$

MOMENT: $M_C = -\frac{q}{4\lambda^2}(B_{\lambda a} - B_{\lambda b})$

SHEAR: $Q_C = \frac{q}{4\lambda}(C_{\lambda a} - C_{\lambda b})$

POINT C IS RIGHT OF LOAD



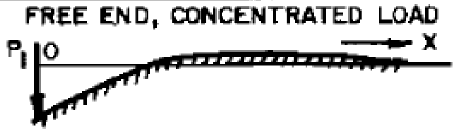
DEFLECTION: $y_C = -\frac{q}{2K}(D_{\lambda a} - D_{\lambda b})$

MOMENT: $M_C = \frac{q}{4\lambda^2}(B_{\lambda a} - B_{\lambda b})$

SHEAR: $Q_C = \frac{q}{4\lambda}(C_{\lambda a} - C_{\lambda b})$

SEMI-INFINITE BEAM

FREE END, CONCENTRATED LOAD

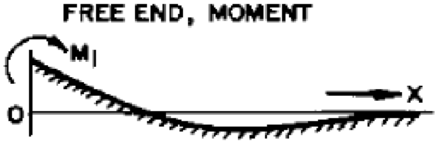


DEFLECTION: $y = \frac{2P_1\lambda}{K} D_{\lambda x}$

MOMENT: $M = -\frac{P_1}{\lambda} B_{\lambda x}$

SHEAR: $Q = -P_1 C_{\lambda x}$

FREE END, MOMENT

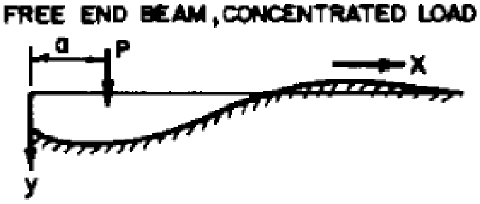


DEFLECTION: $y = -\frac{2M_1\lambda^2}{K} C_{\lambda x}$

MOMENT: $M = M_1 A_{\lambda x}$

SHEAR: $Q = -2M_1\lambda B_{\lambda x}$

FREE END BEAM, CONCENTRATED LOAD NEAR END



DEFLECTION: $y = \frac{P\lambda}{2K}\left[(C_{\lambda a} + 2D_{\lambda a})A_{\lambda x} - 2(C_{\lambda a} + D_{\lambda a})B_{\lambda x} + A_{\lambda(a+x)}\right]$

IF NOTATION $(C_{\lambda a} + 2D_{\lambda a}) = \alpha$
AND $(C_{\lambda a} + D_{\lambda a}) = \beta$ IS USED

MOMENT: $M = \frac{P}{4\lambda}\left\{\alpha C_{\lambda x} - 2\beta D_{\lambda x} + C_{\lambda(a-x)}\right\}$

SHEAR: $Q = -\frac{P}{2}\left\{\alpha D_{\lambda x} - \beta A_{\lambda x} \pm D_{\lambda(a-x)}\right\}$

FIGURE 10
Computation of Shear, Moment, and Deflection, Beams on Elastic Foundation

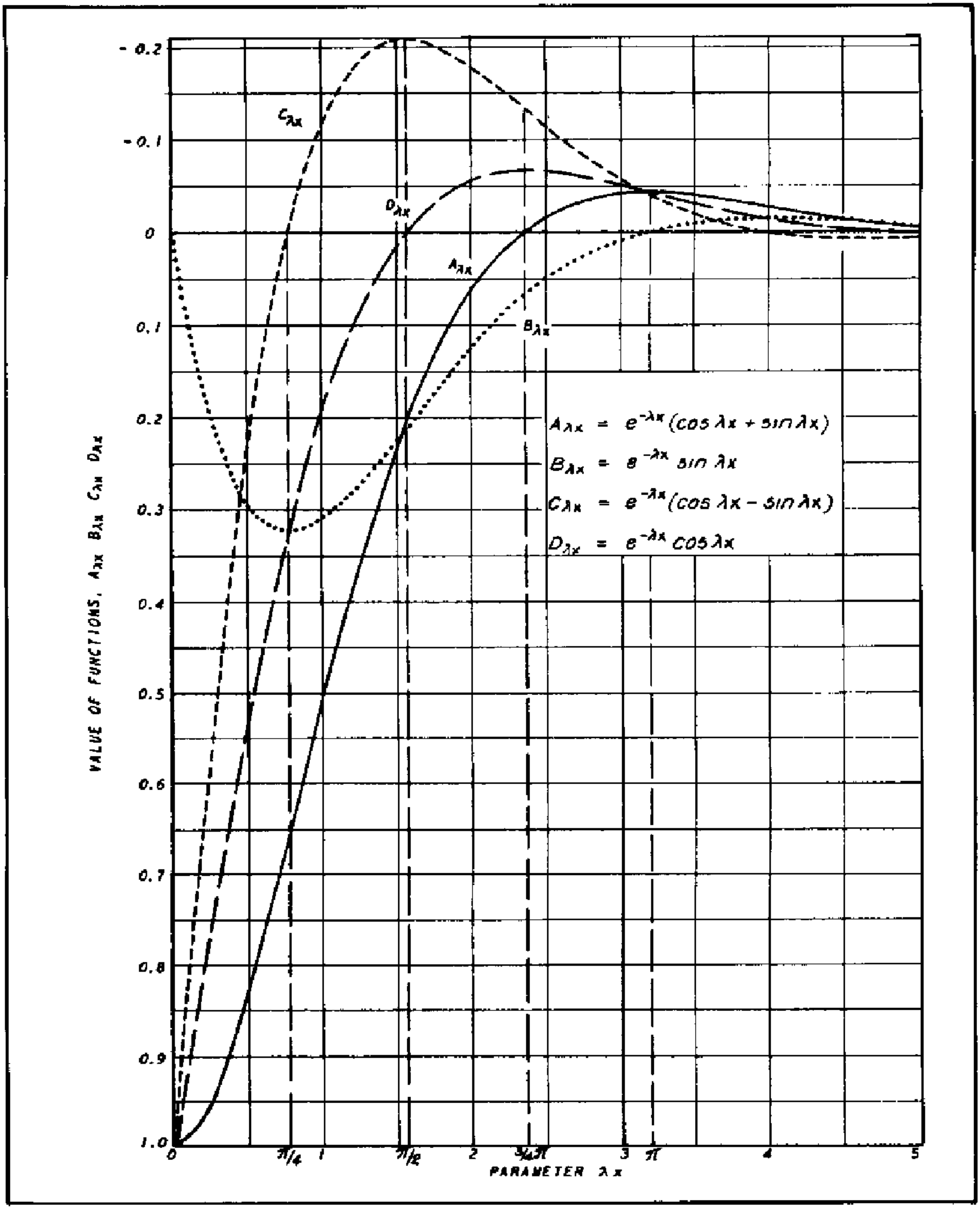


FIGURE 11

Functions for Shear, Moment, and Deflection, Beams on Elastic Foundations