CÁLCULO - PRIMAVERA 2004

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CLASE AUXILIAR #15

Problema 13.1. Resuelva las siguientes integrales:

1. $I = \displaystyle\int_a^b \operatorname{cosec} x dx, a \leq b$

2. $I = \displaystyle\int_0^1 (a^{2x} + 3a^x - 7) dx$

Solución 13.1. .

1. $I = \displaystyle\int_a^b \operatorname{cosec} x dx, a \leq b$

   Se sabe que $\operatorname{sen} x = 2\operatorname{sen}(\frac{x}{2})\cos(\frac{x}{2})$, luego:

$$\begin{aligned} I &= \int_a^b \operatorname{cosec} x dx \\ &= \int_a^b \frac{1}{2\operatorname{sen}(\frac{x}{2})\cos(\frac{x}{2})} dx \\ &= \int_a^b \frac{\frac{1}{\cos^2(\frac{x}{2})}}{\frac{2\operatorname{sen}(\frac{x}{2})\cos(\frac{x}{2})}{\cos^2(\frac{x}{2})}} dx \\ &= \int_a^b \frac{\frac{1}{2\cos^2(\frac{x}{2})}}{\tan(\frac{x}{2})} dx \end{aligned}$$

   Haciendo el cambio de variable $\tan(\frac{x}{2}) = y \rightarrow \frac{1}{2}\frac{1}{\cos^2(\frac{x}{2})}dx = dy$, luego:

$$\begin{aligned} I &= \int_{y(a)}^{y(b)} \frac{dy}{y} \\ &= \ln|y(b)| - \ln|y(a)| \\ &= \ln|\tan(\frac{b}{2})| - \ln|\tan(\frac{a}{2})| \\ &= \ln\left(\left|\frac{\tan(\frac{b}{2})}{\tan(\frac{a}{2})}\right|\right) \end{aligned}$$

2. $I = \int_0^1 (a^{2x} + 3a^x - 7)dx$
Haciendo el cambio de variable $t = a^x \to dt = \ln a a^x dx$, se tiene:

$$
\begin{aligned}
I &= \int_0^1 (a^{2x} + 3a^x - 7)dx \\
&= \int_1^a \frac{t^2 + 3t - 7}{t \ln a} dt \\
&= \frac{1}{\ln a} \int_1^a (t + 3 - 7t^{-1})dt \\
&= \frac{1}{\ln a} \left(\frac{a^2}{2} + 3a - 7\ln a - (\frac{1}{2} + 3 - 7\ln 1) \right) \\
&= \frac{1}{\ln a} \left(\frac{a^2}{2} + 3a - 7\ln a - \frac{7}{2} \right)
\end{aligned}
$$

Teorema 13.1. Sea $R(x) = \frac{P(x)}{Q(x)}$ una funcion racional con P y Q polinomios a coeficientes en $\mathbb{R}$, tal que Q puede descomponerse de la forma:

$$
Q(x) = (x - r_1)^{n_1} \dots (x - r_j)^{n_j} (a_1x^2 + b_1x + c_1)^{m_1} \dots (a_kx^2 + b_kx + c_k)^{m_k}
$$

si P y Q no tienen factores comunes y $gr(P) < gr(Q)$, entonces

$$
\begin{aligned}
R(x) = &\frac{A_1^1}{x - r_1} + \dots + \frac{A_{n_1}^1}{(x - r_1)^{n_1}} + \\
&\vdots \\
&\frac{A_1^j}{x - r_j} + \dots + \frac{A_{n_j}^j}{(x - r_j)^{n_j}} + \\
&\frac{C_1^1 x + B_1^1}{a_1x^2 + b_1x + c_1} + \dots + \frac{C_{m_1}^1 x + B_{m_1}^1}{(a_1x^2 + b_1x + c_1)^{m_1}} + \\
&\vdots \\
&\frac{C_1^k x + B_1^k}{a_kx^2 + b_kx + c_k} + \dots + \frac{C_{m_k}^k x + B_{m_k}^k}{(a_kx^2 + b_kx + c_k)^{m_k}}
\end{aligned}
$$

para todo $x \in \mathbb{R}$ tal que $Q(x) \neq 0$, con $A_1^1, \dots, B_1^1, \dots, C_1^1, \dots, C_{m_k}^k$ constantes.

Sabemos que:

- $\int \frac{A}{x - a} dx = A \ln |x - a| + C$
- $\int \frac{A}{(x - a)^n} dx = \frac{-A}{(n - 1)(x - a)^{n-1}} + C, n \neq 1$

- $\displaystyle\int \frac{Ax+B}{(ax^2+bx+c)^n}dx = C\underbrace{\int \frac{z}{(z^2+1)^n}dx}_{U} + D\underbrace{\int \frac{1}{(z^2+1)^n}dx}_{V}$

  $U = \begin{cases} \dfrac{1}{2(n-1)(z^2+1)^{n-1}} + E & n \neq 1 \\ \dfrac{\ln|z^2+1|}{2} + E & n = 1 \end{cases}$

  $V = I_n = I_{n-1} + \dfrac{x}{2(n-1)(1+z^2)^{n-1}} - \dfrac{1}{2(n-1)}I_{n-1} + E$

  $ax^2+bx+c = a\left[\left(x+\dfrac{b}{2a}\right)^2 - \dfrac{b^2}{4a^2} + \dfrac{c}{a}\right]$

  Cambio de variable: $a\left(x+\dfrac{b}{2a}\right)^2 = \left(\dfrac{4ac-b^2}{4a^2}\right)z^2$

Problema 13.2. Calcule $I = \displaystyle\int \frac{3x-2}{x^2+x+1}dx$.

Solución 13.2. Reescribiendo el denominador:

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

Y haciendo el cambio de variable:

$$\begin{aligned} \left(x+\frac{1}{2}\right)^2 &= \frac{3}{4}z^2 \\ x &= \frac{\sqrt{3}}{2}z - \frac{1}{2} \\ dx &= \frac{\sqrt{3}}{2}dz \end{aligned}$$

Con esto, $(x^2+x+1) = \dfrac{3}{4}(z^2+1)$:

$$\begin{aligned} I &= \int \frac{\left(\frac{3\sqrt{3}}{2}z - \frac{3}{2} - 2\right)}{\frac{3}{4}(z^2+1)}\frac{\sqrt{3}}{2}dz \\ &= \int \frac{9z-7\sqrt{3}}{3(z^2+1)}dz \\ &= 3\int \frac{z}{z^2+1}dz - \frac{7\sqrt{3}}{3}\int \frac{1}{z^2+1}dz \\ &= \frac{3}{2}\ln(z^2+1) - \frac{7\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C \end{aligned}$$