# CÁLCULO - PRIMAVERA 2004

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## CLASE AUXILIAR #1

**Resumen 1.** (Algebra de derivadas) Si $f, g$ son funciones derivables, con $g'(x) \neq 0$ para todo $x \in Domg$ y $\lambda \in \mathbb{R}$:

1. $(f+g)' = f' + g'$
2. $(\lambda f)' = \lambda f'$
3. $(fg)' = f'g + fg'$
4. $\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$
5. $\left(\frac{1}{g}\right) = \frac{-g'}{g^2}$

(Regla de la cadena) Si $f, g$ son funciones derivables y $Recf \subseteq Domg$, entonces $(g \circ f)'(x) = g'(f(x))f'(x)$.
Las derivadas mas comunes estan en la siguiente tabla:

| | |
|---|---|
| $\dfrac{d}{dx}a = 0$ | $\dfrac{d}{dx}x^k = kx^{k-1}$ |
| $\dfrac{d}{dx}e^x = e^x$ | $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$ |
| $\dfrac{d}{dx}\operatorname{sen} x = \cos x$ | $\dfrac{d}{dx}\cos x = -\operatorname{sen} x$ |
| $\dfrac{d}{dx}\tan x = \sec^2 x$ | $\dfrac{d}{dx}x = -\operatorname{cosec}^2 x$ |
| $\dfrac{d}{dx}\sec x = \sec x \tan x$ | $\dfrac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \tan x$ |

**Problema 1.1.** Calcular la derivada de las siguientes funciones:

1. $f(x) = \ln \sqrt{\dfrac{1 + \operatorname{sen} x}{1 - \operatorname{sen} x}}$

2. $f(x) = x^{\ln x}$

**Solución 1.1.** 1.

$$\begin{aligned}
f'(x) &= \left(\ln\sqrt{\frac{1+\operatorname{sen}x}{1-\operatorname{sen}x}}\right)' \\
&= \frac{1}{\sqrt{\frac{1+\operatorname{sen}x}{1-\operatorname{sen}x}}}\left(\sqrt{\frac{1+\operatorname{sen}x}{1-\operatorname{sen}x}}\right)' \\
&= \sqrt{\frac{1-\operatorname{sen}x}{1+\operatorname{sen}x}}\frac{1}{2\sqrt{\frac{1+\operatorname{sen}x}{1-\operatorname{sen}x}}}\left(\frac{1+\operatorname{sen}x}{1-\operatorname{sen}x}\right)' \\
&= \sqrt{\frac{1-\operatorname{sen}x}{1+\operatorname{sen}x}}\frac{1}{2\sqrt{\frac{1+\operatorname{sen}x}{1-\operatorname{sen}x}}}\left(\frac{(1+\operatorname{sen}x)'(1-\operatorname{sen}x)-(1+\operatorname{sen}x)(1-\operatorname{sen}x)'}{(1-\operatorname{sen}x)^2}\right) \\
&= \sqrt{\frac{1-\operatorname{sen}x}{1+\operatorname{sen}x}}\frac{1}{2\sqrt{\frac{1+\operatorname{sen}x}{1-\operatorname{sen}x}}}\left(\frac{\cos x(1-\operatorname{sen}x)+(1+\operatorname{sen}x)\cos x}{(1-\operatorname{sen}x)^2}\right) \\
&= \sqrt{\frac{1-\operatorname{sen}x}{1+\operatorname{sen}x}}\frac{1}{2\sqrt{\frac{1+\operatorname{sen}x}{1-\operatorname{sen}x}}}\left(\frac{2\cos x}{(1-\operatorname{sen}x)^2}\right)
\end{aligned}$$

2.

$$\begin{aligned}
f'(x) &= \left(x^{\ln x}\right)' \\
&= \left(e^{\ln x\ln x}\right)' \\
&= \left(e^{\ln^2 x}\right)' \\
&= e^{\ln^2 x}\left(\ln^2 x\right)' \\
&= e^{\ln^2 x}\left((\ln x)'\ln x+\ln x(\ln x)'\right) \\
&= e^{\ln^2 x}\frac{2\ln x}{x}
\end{aligned}$$

**Problema 1.2.** Dado $f(x) = (1+x)^n$, deduzca que $\displaystyle\sum_{k=1}^{n}\binom{n}{k}k = n2^{n-1}$.

Para ello:

1. Muestre que $f'(x) = n(1+x)^{n-1}$
2. Muestre que si $g(x) = \displaystyle\sum_{k=0}^{n}\alpha_k x^k$, entonces $g'(x) = \displaystyle\sum_{k=1}^{n}\alpha_k k x^{k-1}$
3. Pruebe que $\displaystyle\sum_{k=1}^{n}\binom{n}{k}k = f'(1) = n2^{n-1}$

**Solución 1.2.** 1. Directo.

2. La derivada de una suma es la suma de las derivadas (algebra de derivadas).

$$\begin{aligned} g'(x) &= \left(\sum_{k=0}^{n} \alpha_k x^k\right)' \\ &= \sum_{k=0}^{n} \alpha_k k x^{k-1} \\ &= \sum_{k=1}^{n} \alpha_k k x^{k-1} \end{aligned}$$

3. Del curso de 'Algebra sabemos que $(1+x)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} x^k$, por lo tanto:

$$\begin{aligned} f'(x) &= \left(\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} x^k\right)' \\ &= \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} k x^{k-1} \\ &= \sum_{k=1}^{n} \begin{pmatrix} n \\ k \end{pmatrix} k x^{k-1} \\ \underbrace{=}_{(1)} & n(1+x)^{n-1} \end{aligned}$$

Evaluando en $x = 1$, se tiene que:

$$\sum_{k=1}^{n} \begin{pmatrix} n \\ k \end{pmatrix} k = n2^{n-1}$$

**Problema 1.3.** Sea $g : \mathbb{R} \to \mathbb{R}$ dos veces derivable con $g'(x) \neq 0$ para todo $x \in \mathbb{R}$. Sea $f : \mathbb{R} \to \mathbb{R}, f(x) = \cos(kg(x)), k \in \mathbb{R}$:

1. Muestre que $f''(x) - \dfrac{f'(x)g''(x)}{g'(x)} + (kg'(x))^2 f(x) = 0$.

2. Calcule $f^{(n)}(0)$ para $g(x) = x$.

**Solución 1.3.** 1.

$$\begin{aligned} f'(x) &= -\operatorname{sen}(kg(x))kg'(x) \\ f''(x) &= -(\cos(kg(x))kg'(x)kg'(x) + \operatorname{sen}(kg(x))kg''(x)) \\ &= -\cos(kg(x))(kg'(x))^2 - \operatorname{sen}(kg(x))kg''(x) \end{aligned}$$

Reemplazando:

$$\begin{aligned} -\cos(kg(x))(kg'(x))^2 - \operatorname{sen}(kg(x))kg''(x) + \\ \frac{\operatorname{sen}(kg(x))kg'(x)g''(x)}{g'(x)} + (kg'(x))^2\cos(kg(x)) &= 0 \end{aligned}$$

2.

$$\begin{aligned} f(x) &= \cos(kx) \\ f'(x) &= -\operatorname{sen}(kx)k \\ &= \cos(kx - \frac{\pi}{2})k \\ f''(x) &= -\cos(kx)k^2 \\ &= \cos(kx-\pi)k^2 \\ f'''(x) &= \operatorname{sen}(kx)k^3 \\ &= \cos(kx - \frac{3\pi}{2})k^3 \\ &\vdots \\ f^{(n)}(x) &= \cos(kx - \frac{n\pi}{2})k^n \end{aligned}$$

**Problema 1.4.** Si $f(x) = (\tan x + \sec x)^m$, pruebe que $f'(x) = mf(x)\sec x$.

**Solución 1.4.** Se sabe que $(\tan x)' = \sec^2 x$ y $(\sec x)' = \sec x \tan x$, por lo tanto:

$$\begin{aligned} ((\tan x + \sec x)^m)' &= m(\tan x + \sec x)^{m-1}((\tan x)' + (\sec x)') \\ &= m(\tan x + \sec x)^{m-1}(\sec^2 x + \sec x \tan x) \\ &= m(\tan x + \sec x)^{m-1}(\tan x + \sec x)\sec x \\ &= mf(x)\sec x \end{aligned}$$