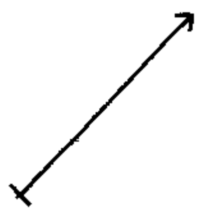
Vectores



dirección, sentido, magnitud

- → suma
- → ponderación
- → norma

$$\|\vec{r}\| = \sqrt{x^2+y^2+z^2} \quad , \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \vee \quad x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

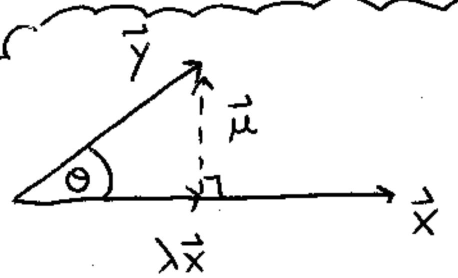
$$\hat{\imath} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ; \quad \hat{\jmath} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ; \quad \hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

→ Producto Punto

$$\vec{A} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} ; \quad \vec{B} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$$

$$\vec{A} \cdot \vec{B} = a_x \cdot b_x + a_y \cdot b_y + a_z \cdot b_z \quad \text{(escalar)}$$

$$\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cdot \cos \measuredangle (\vec{A}, \vec{B})$$



$$\vec{y} = \lambda\vec{x} + \vec{u} \quad / \langle \ , \vec{x} \rangle$$

$$\langle \vec{y}, \vec{x} \rangle = \langle \lambda\vec{x}, \vec{x} \rangle + \cancel{\langle \vec{u}, \vec{x} \rangle} \; 0$$

$$\langle \vec{y}, \vec{x} \rangle = \lambda \|\vec{x}\|^2$$

$$\Rightarrow \quad \lambda = \frac{\langle \vec{y}, \vec{x} \rangle}{\|\vec{x}\|^2}$$

$$\cos\theta = \frac{\|\lambda\vec{x}\|}{\|\vec{y}\|} = \frac{|\lambda| \|\vec{x}\|}{\|\vec{y}\|}$$

$$= \frac{|\langle \vec{y}, \vec{x} \rangle|}{\|\vec{x}\|^2} \cdot \frac{\|\vec{x}\|}{\|\vec{y}\|} = \frac{|\langle \vec{y}, \vec{x} \rangle|}{\|\vec{x}\| \cdot \|\vec{y}\|}$$

$\Rightarrow$ Producto Cruz

$$\vec{A} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} ; \quad \vec{B} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y \cdot b_z - a_z b_y)\hat{i} + (a_z b_x - a_x b_z)\hat{j} + (a_x b_y - a_y b_x)\hat{k}$$

$\vec{A} \times \vec{B} = \vec{C}$ (vector) $\perp$ a $\vec{A}$ y a $\vec{B}$ (mano derecha)

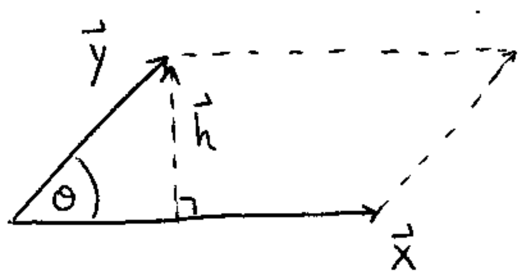
Prop.

- $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $\lambda \vec{a} \times \vec{b} = \vec{a} \times \lambda \vec{b} = \lambda (\vec{a} \times \vec{b})$
- $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{b} \cdot (\vec{a} \cdot \vec{c}) - (\vec{b} \cdot \vec{c}) \cdot \vec{a}$
- $\hat{i} \times \hat{j} = \hat{k}$

  $\hat{j} \times \hat{k} = \hat{i}$

  $\hat{k} \times \hat{i} = \hat{j}$

$$\| \vec{A} \times \vec{B} \| = \| \vec{A} \| \cdot \| \vec{B} \| \, | \mathrm{sen} \measuredangle (\vec{A}, \vec{B}) |$$

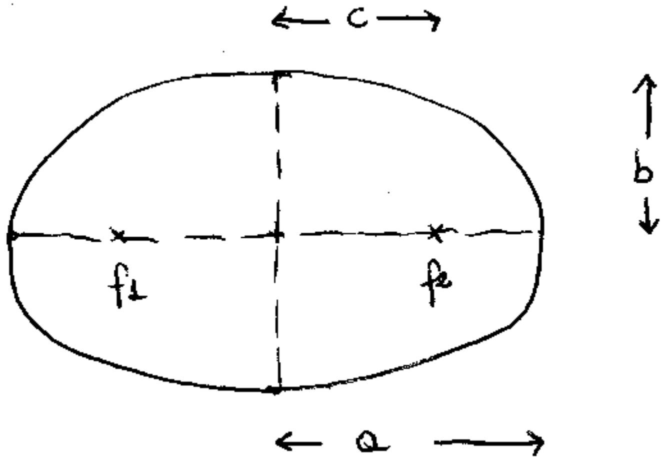


$$\text{Area} = \| \vec{x} \| \cdot \| \vec{h} \|$$

$$\mathrm{sen}\,\theta = \frac{\| \vec{h} \|}{\| \vec{y} \|}$$

$$\text{Area} = \| \vec{x} \| \cdot \| \vec{y} \| \cdot \mathrm{sen}\,\theta$$

Elipse



a: semi-eje mayor
b: semi-eje menor
c: distancia del centro a un foco
e: excentricidad
r_p: radio perihelio (distancia mín.)
r_a: radio afelio (distancia máx.)

$$a^2 = b^2 + c^2$$

$$e = \frac{c}{a} \quad \text{(definición de } e\text{)}$$

$$r_p = a - c = a(1-e)$$

$$r_a = a + c = a(1+e)$$

Problema 1

$r_p^L = 363.000$ [Km]
$r_a^L = 405.500$ [Km]

$T = 27{,}322$ días

Radio $\oplus$ = $6{,}37 \times 10^6$ [m] = $6{,}37 \times 10^3$ [Km] = 6370 [Km]

r_p^s = 225 [Km] + 6370 [Km] = 6595 [Km]
r_a^s = 710 [Km] + 6370 [Km] = 7080 [Km]

Entonces:

$$a^L = \frac{r_p^L + r_a^L}{2} = 384250 \text{ [Km]} =$$

$$a^s = \frac{r_p^s + r_a^s}{2} = 6838 \text{ [Km]}$$

$$\Rightarrow \frac{T^{L^2}}{a^{L^3}} = \frac{T^{s^2}}{a^{s^3}}$$

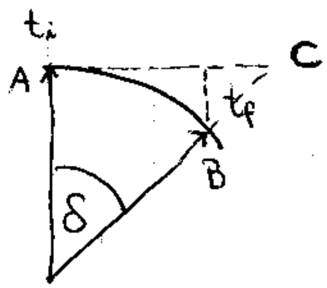
$$\Rightarrow \frac{746{,}5}{(384250)^3} \cdot (6838)^3 = T^{s^2}$$

$$\Rightarrow T^s = 0{,}064 \text{ días}$$

$$\Rightarrow T = 1 \text{ hr. } 32 \text{ min } 9 \text{ seg.}$$

Problema 2

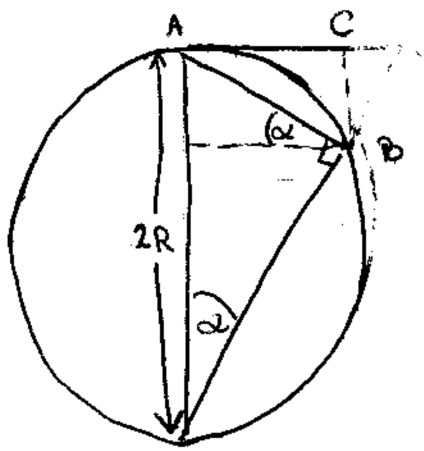
a).



Consideramos $\Delta t = t_f - t_i$ muy pequeño

$\Rightarrow \widehat{AB} \approx \overline{AC}$ y además $\overline{BC} \approx$ recta vertical

$\widetilde{AB} = v_0 \cdot \Delta t$ y $\overline{BC} = \frac{1}{2} a (\Delta t)^2$ (lo q' cae)



$$\Rightarrow \tan\alpha = \frac{v_0 \Delta t}{2R - s}$$

$$\tan\alpha = \frac{s}{v_0 \cdot \Delta t}$$

$$\Rightarrow v_0^2 \cdot (\Delta t)^2 = (2R - s) \cdot s$$

$$\approx 2R \cdot s$$

$$\Rightarrow \cancel{2} \cdot \frac{1}{\cancel{2}} a \cancel{(\Delta t^2)} R = v_0^2 \cancel{(\Delta t)^2}$$

$$\Rightarrow \boxed{a = \frac{v_0^2}{R}}$$

b)

$$\cancel{m}a = \frac{GM\cancel{m}}{r^2}$$

$$\Rightarrow a = \frac{GM}{r^2}$$

$$\Rightarrow \frac{v_0^2}{r} = \frac{GM}{r^2}$$

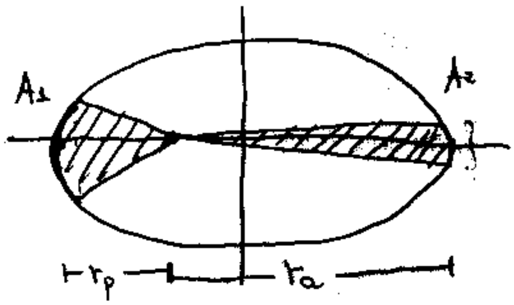
pero $v_0 \cdot T = 2\pi r$

$$\Rightarrow \frac{4\pi^2 r^{\cancel{2}}}{T^2 \cdot \cancel{r}} = \frac{GM}{r^2}$$

$$\Rightarrow \boxed{\frac{T^2}{r^3} = \left(\frac{GM}{4\pi^2}\right)^{-1}}$$

Pauta Ejercicio 4

4-a Debido a la segunda Ley de Kepler, que dice: "El radio vector desde el sol al planeta barre áreas iguales en intervalos iguales de tiempo", podemos deducir que:



$$A_1 = A_2$$

Podemos tomar Δt pequeño tq $A_1 = A_2 \approx$ Área de un triángulo

$$\Rightarrow A_1 = \frac{r_p \cdot \text{arco } 1}{2}, \quad \text{arco } 1 \approx v_p \cdot \Delta t$$

$$\Rightarrow A_1 = \frac{r_p \cdot v_p \cdot \Delta t}{2}$$

y $\quad A_2 = \frac{r_a \cdot \text{arco } 2}{2}, \quad \text{arco } 2 \approx v_a \cdot \Delta t$

$$\Rightarrow A_2 = \frac{r_a \cdot v_a \cdot \Delta t}{2}$$

$$\Rightarrow A_1 = A_2 \Rightarrow \frac{r_a \cdot v_a \cdot \Delta t}{\cancel{2}} = \frac{r_p \cdot v_p \cdot \Delta t}{\cancel{2}}$$

$$\Rightarrow \frac{v_p}{v_a} = \frac{r_a}{r_p}$$

pero $v_{perihelio} = v_{máx}$
$v_{afelio} = v_{min}$

$$\Rightarrow \frac{v_{máx}}{v_{min}} = \frac{r_a}{r_p}$$

$$e = \frac{c}{a}$$

$r_a: a + c = a + a \cdot e = a(1 + e) = a(1 + 0{,}0167)$
$r_p: a - c = a - a \cdot e = a(1 - e) = a(1 - 0{,}0167)$

$$\Rightarrow \frac{v_{máx}}{v_{min}} = \frac{\cancel{a}(1 + 0{,}0167)}{\cancel{a}(1 - 0{,}0167)} = \frac{1{,}0167}{0{,}9833} \left(\approx 1{,}03\right)$$

4-5

$$\vec{F} = \frac{K\cdot m\cdot m'}{r^{\alpha}}\,\vec{u} \qquad K\left[\frac{m^{1+\alpha}}{s^2\cdot Kg}\right]$$

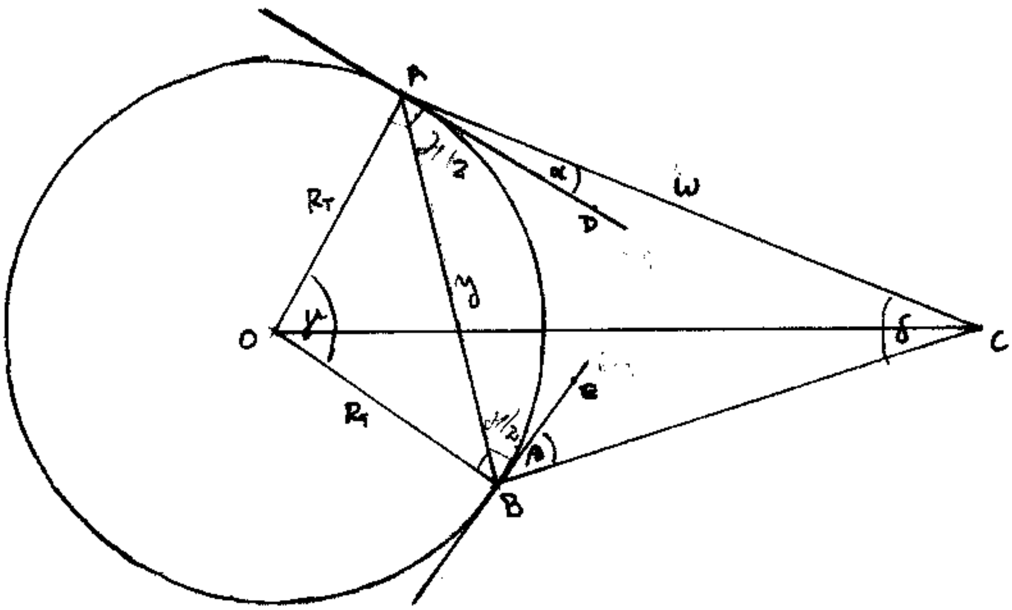
Sabemos que la tercera Ley de Kepler relaciona el periodo con el semi-eje mayor de las órbitas, En este caso de órbita circular; $a = R$. Sabemos que esta relación debe ser una cte adimensional, entonces por análisis dimensional

$$K\cdot\left[\frac{m^{\cancel{1+\alpha}}}{\cancel{s^2}\cdot \cancel{Kg}}\right]\cdot \underbrace{T^2\,[\cancel{s^2}]}_{\text{necesitamos cancelar } [s^2]}\cdot \underbrace{m\,[\cancel{Kg}]}_{\text{necesitamos cancelar } [Kg]}\cdot \underbrace{\frac{1}{R^{\alpha+1}}\left[\frac{1}{\cancel{m^{\alpha+1}}}\right]}_{\text{necesitamos cancelar } [m^{\alpha+1}]} = cte.$$

$$\Rightarrow \quad \frac{K\cdot T^2\cdot m}{R^{\alpha+1}} = cte$$

$$\Rightarrow \boxed{\frac{T^2}{R^{\alpha+1}} = \widetilde{cte}}$$

Ejercicio N°3



Datos:
R_T, α, β, μ

Pregunta: ¿$\overline{OC}$?

→ Por el teorema del coseno: $y^2 = R_T^2 + R_T^2 - 2R_T \cdot R_T \cdot \cos\mu = 2R_T^2(1-\cos\mu) = y^2$

ΔAOB isósceles $\Rightarrow \measuredangle OAB = \measuredangle OBA$, $\mu + 2\measuredangle OAB = \pi \Rightarrow \measuredangle OAB = \frac{\pi}{2} - \frac{\mu}{2}$

EB tangente $\Rightarrow \overline{OB} \perp \overline{BE} \Rightarrow \measuredangle OBE = \pi/2$

$\measuredangle OBE = \measuredangle OBA + \measuredangle ABE \Rightarrow \measuredangle ABE = \frac{\pi}{2} - \left(\frac{\pi}{2} - \frac{\mu}{2}\right) = \frac{\mu}{2}$

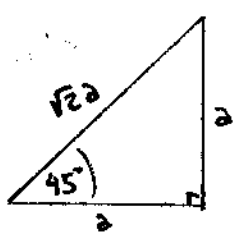
Por teorema del seno: $\frac{\omega}{\operatorname{sen}(\beta+\mu/2)} = \frac{y}{\operatorname{sen}\delta} \Rightarrow \omega = y\,\frac{\operatorname{sen}(\beta+\mu/2)}{\operatorname{sen}\delta}$

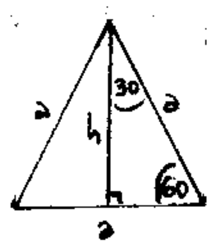
AD tangente $\Rightarrow \overline{AD} \perp \overline{OA} \Rightarrow \measuredangle OAD = \pi/2 \Rightarrow \measuredangle OAC = \pi/2 + \alpha$

Por teo del coseno: $\overline{OC}^2 = R_T^2 + \omega^2 - 2R_T\omega\cos(\pi/2+\alpha)$

pero ¿δ? $\Delta ABC \Rightarrow \left(\frac{\mu}{2}+\alpha\right) + \left(\frac{\mu}{2}+\beta\right) + \delta = \pi \Rightarrow \delta = \pi - (\alpha+\beta+\mu)$

→ $\mu = 60°$
$\alpha = 45°$
$\beta = 30°$





$h = \frac{\sqrt{3}}{2}a \quad \left(\frac{a^2}{4} + h^2 = a^2\right)$

$\Rightarrow \operatorname{sen}45° = \cos 45° = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\operatorname{sen}30° = \cos 60° = \frac{a/2}{a} = \frac{1}{2}$

$\operatorname{sen}60° = \cos 30° = \frac{\sqrt{3}/2\,a}{a} = \frac{\sqrt{3}}{2}$

$$\operatorname{sen}(\pi-(\alpha+\beta+\gamma)) = \overset{0}{\cancel{\operatorname{sen}\pi}}\cdot\cos(\alpha+\beta+\gamma) - \cos\pi\,\operatorname{sen}(\alpha+\beta+\gamma) = \operatorname{sen}(\alpha+\beta+\gamma)$$

$$\boxed{(\overline{OC})^2 = R_T^2 + 2R_T^2(1-\cos\gamma)\,\frac{\operatorname{sen}^2(\beta+\frac{\gamma}{2})}{\operatorname{sen}^2(\alpha+\beta+\gamma)} + 2R_T\sqrt{2R_T^2(1-\cos\gamma)}\,\frac{\operatorname{sen}(\beta+\gamma/2)\,\operatorname{sen}(\alpha)}{\operatorname{sen}(\alpha+\beta+\gamma)}}$$

ya que $\cos(\pi/2+\alpha) = \overset{0}{\cancel{\cos\pi/2}}\cos\alpha - \overset{1}{\cancel{\operatorname{sen}\pi/2}}\operatorname{sen}\alpha = -\operatorname{sen}\alpha$

$\gamma = 60°$
$\alpha = 45°$
$\beta = 30°$

$$\Rightarrow\quad 2(1-\cos\gamma) = 2(1-1/2) = 1$$

$$\operatorname{sen}(\beta+\gamma/2) = \operatorname{sen}(30°+30°) = \operatorname{sen}60° = \frac{\sqrt{3}}{2}$$

$$\operatorname{sen}(\alpha+\beta+\gamma) = \operatorname{sen}(45°+30°+60°) = \operatorname{sen}(45°+90°) = \operatorname{sen}45°\overset{0}{\cancel{\cos 90°}} + \overset{1}{\cancel{\operatorname{sen}90°}}\cdot\cos 45° = \frac{\sqrt{2}}{2}$$

$$\operatorname{sen}(\alpha) = \operatorname{sen}45° = \frac{\sqrt{2}}{2}$$

$$\Rightarrow (\overline{OC})^2 = R_T^2 + R_T^2\frac{\left(\frac{\sqrt{3}}{2}\right)^2}{\left(\frac{\sqrt{2}}{2}\right)^2} + 2R_T\cdot R_T\frac{\left(\frac{\sqrt{3}}{2}\right)}{\cancel{\left(\frac{\sqrt{2}}{2}\right)}}\cancel{\frac{\sqrt{2}}{2}} = R_T^2 + \frac{3}{2}R_T^2 + \sqrt{3}R_T^2 = \left(\frac{5}{2}+\sqrt{3}\right)R_T^2$$