Lagrange Multipliers
(Adams, 4th edition, section 13.3, p.786)

A **constrained extremum value problem** for a function $f(x, y)$ is one in which the variables *x* and *y* are not independent, but are related by some other condition. In the problem "maximize $f(x, y)$ subject to the condition $h(x, y) = c$", or equivalently, $g(x, y) = h(x, y) - c = 0$, the condition $h(x, y) = c$ or $g(x, y) = 0$ is called the **constraint**.

In the previous section, the problem of finding the maximum of a two dimensional function restricted to the boundary of a region can be considered as a problem of finding an extremum subject to a constraint. Problems of this kind were solved previously by reducing them to a one dimensional problem. This was done by expressing *y* as a function of *x* on the boundary and substituting in $f(x, y)$, or else by finding a parametric representation $x(t)$, $y(t)$ for the boundary curve and substituting in the function *f*. In the general case these solution methods may not be possible.

Example.

A problem previously considered is to maximize $f(x, y) = x + y$ subject to the constraint $g(x, y) = x^2 + y^2 - 1 = 0$. This can be solved by expressing $y = \sqrt{1 - x^2}$ and maximising the function $q(x) = x + y = x + \sqrt{1 - x^2}$ which is a function of the one variable *x*. It can also be solved by expressing *x* and *y* in terms of the parameter *t* as $x = \cos(t)$, $y = \sin(t)$ and maximizing the function $s(t) = x + y = \cos(t) + \sin(t)$.

In the general case, it may not be possible to express the boundary curve in terms of a single parameter *t*. In this case we can introduce an extra variable λ (known as a **Lagrange multiplier**) and a new function $L(x, y, \lambda)$ (known as a **Lagrange function**).

Theorem.
Suppose that $f(x, y)$ and $g(x, y)$ have continuous first partial derivatives near the point $P_0 = (x_0, y_0)$ on the curve *C* defined by the equation $g(x, y) = 0$. Suppose that, **restricted to points on the curve** *C*, the function $f(x, y)$ has a local maximum or minimum value at P_0. Also suppose that

(i) P_0 is not an endpoint of the curve *C*

(ii) $\nabla g(P_0) \neq 0$.

Then there exists a number λ_0 such that (x_0, y_0, λ_0) is a critical point (in three dimensions) of the **Lagrangian function** $L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$.

Comment.
The three conditions for a critical point of L (a function of three variables) are

(i) $\dfrac{\partial L}{\partial x} = \dfrac{\partial f}{\partial x} + \lambda \dfrac{\partial g}{\partial x} = 0$

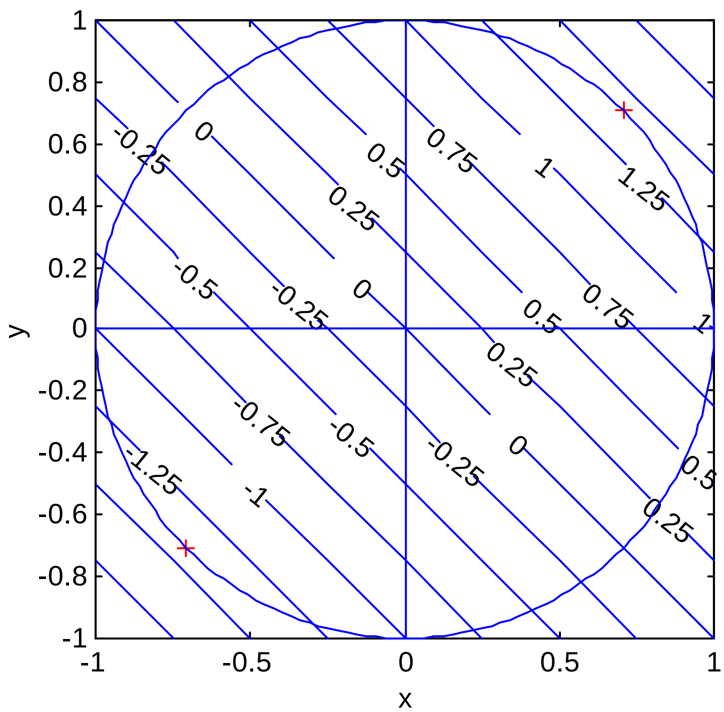
(ii) $\dfrac{\partial L}{\partial y} = \dfrac{\partial f}{\partial y} + \lambda \dfrac{\partial g}{\partial y} = 0$

(iii) $\dfrac{\partial L}{\partial \lambda} = g(x, y) = 0$, which is just the constraint condition.

At the point P_0, $\nabla L = \nabla(f + \lambda g) = 0$, so that in two dimensions $\nabla f = -\lambda_0 \nabla g$ and $\mathrm{grad}(f)$ is parallel to $\mathrm{grad}(g)$.

Example 1.

Use the method of Lagrange multipliers to maximize $f(x, y) = x + y$ subject to the constraint $g(x, y) = x^2 + y^2 - 1 = 0$.



Level curves of the function $f(x, y) = x + y$, with the curve C defined by the constraint $g(x, y) = x^2 + y^2 - 1 = 0$.

Form the Lagrange function
$L(x,y,\lambda) = f(x,y) + \lambda g(x,y) = x + y + \lambda\left(x^2 + y^2 - 1\right)$, with λ the Lagrange multiplier.

Look for a critical point satisfying

$$\frac{\partial L}{\partial x} = 1 + 2\lambda x = 0$$

$$\frac{\partial L}{\partial y} = 1 + 2\lambda y = 0$$

$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1 = 0 .$$

From the first two equations,

$x = -\dfrac{1}{2\lambda}$ and $y = -\dfrac{1}{2\lambda}$.

Substituting in the third (constraint) equation,

$\dfrac{1}{4\lambda^2} + \dfrac{1}{4\lambda^2} = 1$, $\lambda^2 = \frac{1}{2}$, so $\lambda = \pm\frac{\sqrt{2}}{2}$.

In case 1, $\lambda = \frac{\sqrt{2}}{2}$, so $x = -\frac{\sqrt{2}}{2}$ and $y = -\frac{\sqrt{2}}{2}$, and $f(x,y) = -\sqrt{2}$.

In case 2, $\lambda = -\frac{\sqrt{2}}{2}$, so $x = \frac{\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$, and $f(x,y) = \sqrt{2}$.

The function *f* restricted to the curve *g* has a maximum value of $f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \sqrt{2}$.

Example 2.

Find the shortest distance from the origin $(0,0)$ to the curve $x^2 y = 16$.

Note. Let $d(x,y) = \sqrt{x^2 + y^2}$ be the distance from $(0,0)$ to a point (x,y) on the curve $x^2 y = 16$. Then write Q for the square of the distance, $Q = d^2 = x^2 + y^2$. The distance *d* is not differentiable at the origin, but Q is differentiable. Also, $\dfrac{\partial Q}{\partial x} = 2d\dfrac{\partial d}{\partial x}$ for example, so as long as $d \neq 0$, any critical point of *d* is also a critical point of Q. It is much easier to look for critical points of Q.

Solution.
The problem is equivalent to finding the minimum of $Q = d^2 = x^2 + y^2$ subject to the constraint $g(x,y) = x^2 y - 16 = 0$.

Form the Lagrange function
$L(x,y,\lambda) = f(x,y) + \lambda g(x,y) = x^2 + y^2 + \lambda\left(x^2 y - 16\right)$, with λ the Lagrange multiplier.

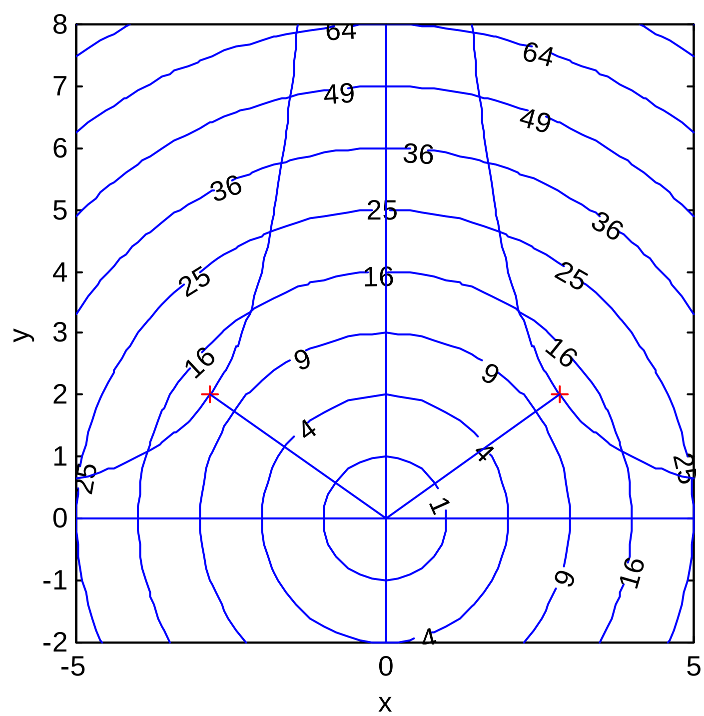
Look for a critical point satisfying

$$\frac{\partial L}{\partial x} = 2x + 2\lambda xy = 2x(1+\lambda y) = 0 \qquad \text{(A)}$$

$$\frac{\partial L}{\partial y} = 2y + \lambda x^2 = 0 \qquad \text{(B)}$$

$$\frac{\partial L}{\partial \lambda} = x^2 y - 16 = 0. \qquad \text{(C)}$$

Equation (A) requires that
$x = 0$ or $\lambda y = -1$. The condition $x = 0$ does not satisfy equation (C). Multiplying equation (B) by y and substituting for λ gives $2y^2 + \lambda y x^2 = 0$ or $2y^2 - x^2 = 0$, or $x = \pm\sqrt{2}y$. Substituting in equation (C) gives $2y^3 = 16$, or $y = 2$. The two critical points are $(x, y) = \left(-2\sqrt{2}, 2\right)$ and $(x, y) = \left(2\sqrt{2}, 2\right)$. The distance from each point to the origin is $d(x, y) = \sqrt{8+4} = 2\sqrt{3}$.



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Level curves of the function $d^2 = Q(x, y) = x^2 + y^2$, with
the curve C defined by the constraint $g(x, y) = x^2 y - 16 = 0$.
The shortest distances from the origin to the curve are shown.

Extension to 3 dimensions.

Suppose that we wish to find a maximum or a minimum of a function $f(x,y,z)$ of three variables on the surface S defined by the constraint $g(x,y,z)=0$.

As in the two dimensional case, we can use λ a Lagrange multiplier and define the Lagrangian function

$$L(x,y,z,\lambda)= f(x,y,z)+\lambda g(x,y,z).$$

And look for a critical points of L satisfying

$\frac{\partial L}{\partial x}=0$, $\frac{\partial L}{\partial y}=0$, $\frac{\partial L}{\partial z}=0$ and
$\frac{\partial L}{\partial \lambda}=g(x,y,z)=0$.

Example 3.

Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x,y,z)=x+y+z$ on the surface of the sphere $x^2+y^2+z^2=1$.

Solution. We wish to find the maximum and minimum values of $f(x,y,z)=x+y+z$ subject to the constraint $g(x,y,z)=x^2+y^2+z^2-1=0$.

Form the Lagrange function
$L(x,y,z,\lambda)= f(x,y,z)+\lambda g(x,y,z)=x+y+z+\lambda\left(x^2+y^2+z^2-1\right)$, with λ the Lagrange multiplier.

Look for a critical point satisfying

$$\frac{\partial L}{\partial x}=1+2\lambda x=0$$
$$\frac{\partial L}{\partial y}=1+2\lambda y=0$$
$$\frac{\partial L}{\partial z}=1+2\lambda z=0$$
$$\frac{\partial L}{\partial \lambda}=x^2+y^2+z^2-1=0.$$

From the first three equations,

$$x=-\frac{1}{2\lambda}\ ,\ y=-\frac{1}{2\lambda} \text{ and } z=-\frac{1}{2\lambda}.$$

Substituting in the third (constraint) equation,

$\frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}=1$, $\lambda^2=\frac{3}{4}$, so $\lambda=\pm\frac{\sqrt{3}}{2}$.

In case 1, $\lambda=\frac{\sqrt{3}}{2}$, so $x=-\frac{\sqrt{3}}{3}$, $y=-\frac{\sqrt{3}}{3}$, $z=-\frac{\sqrt{3}}{3}$, and $f\left(x,y\right)=-\sqrt{3}$.

In case 2, $\lambda=-\frac{\sqrt{3}}{2}$, so $x=\frac{\sqrt{3}}{3}$, $y=\frac{\sqrt{3}}{3}$, $z=\frac{\sqrt{3}}{3}$, and $f\left(x,y\right)=\sqrt{3}$.

The function *f* restricted to the surface *g* has a maximum value of $f\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)=\sqrt{3}$, and a minimum value of $f\left(-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right)=-\sqrt{3}$.