EXAMINATION PAPERS

Math2011 Several Variable Calculus — June 2000.

1 **a.** Consider the surface defined by

$$y^2e^{3z} - \cos(xy) + z = 5.$$

Find the equation of the tangent plane at the point $(\frac{\pi}{2}, 2, 0)$.

b. For the function

$$f(u,v) = \int_1^u \frac{e^{vt}}{t}\,dt$$

calculate $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ and hence find

$$\frac{d}{dx}\int_1^{x^2} \frac{e^{xt}}{t}\,dt.$$

c. Suppose $f(x,y) = e^{x^2-y^2}$ and let $\mathbf{u} = (1,3)$. Find the directional derivative of f at the point $(-1,1)$ in the direction of $\mathbf{u}$.

d. Calculate the Taylor series expansion of $z = \sin(xy)$ about the point $(1,\pi)$. Your expansion should include all second order terms.

e. Consider the iterated integral

$$I = \int_{-1}^{1}\int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} f(x,y)\,dydx.$$

i) Sketch the region of integration for I in the $x-y$ plane.

ii) Rewrite I as an iterated integral with the order of integration reversed.

2 **a.** Consider the function

$$f(x,y) = 3y^2 + 2y^3 + 3x^2y - 3x^2, \quad (x,y) \in \mathbb{R}^2.$$

Find and classify the critical points of f.

b. Use the method of Lagrange multipliers to minimize

$$f(x,y,z) = x^2 + 2y^2 - 2z^2$$

subject to $x+y+z=1$ and $x+z=2$.

c. Use polar coordinates to evaluate the integral

$$\iint_\Omega x^3\,dA,$$

where $\Omega = \{(x,y) \mid x^2+y^2 \le 4,\ x \ge 1\}$.

3 **a.** Let S be the region which is bounded below by the cone $z = \sqrt{x^2+y^2}$ and bounded above by the hemisphere $x^2+y^2+(z-1)^2 = 1$, $(z \ge 1)$.

i) Sketch the region S and express the volume V of S as an iterated integral using:

α) Cartesian coordinates
β) cylindrical coordinates
γ) spherical coordinates.

ii) Use one of the above integrals to find the volume V.

b. Find the area of the region, lying in the first quadrant (i.e. $x \ge 0$, $y \ge 0$) bounded by the ellipses $x^2+2y^2 = 2$ and $x^2+2y^2=4$ and the lines $y = x/\sqrt{6}$ and $y = x/\sqrt{2}$ by using the change of variables $u = x^2+2y^2$ and $v = y/x$.

4 **a.** Find the point(s) at which the twisted cubic

$$\vec{r}(t) = (t, t^2, t^3)$$

intersects the plane $4x + 2y + z = 24$. What is the angle(s) of intersection between the curve and the plane?

b. A particle moves so that

$$\vec{r}(t) = 2\cos 2t\,\mathbf{i} + 3\cos t\,\mathbf{j}, \quad t \geq 0.$$

i) Show that the particle oscillates on an arc of the parabola $4y^2 - 9x = 18$. Draw the path.
ii) What are the acceleration vectors at the points of zero velocities?

c. Calculate the work done by the force $\vec{F} = x\mathbf{i} + xy\mathbf{j} + xyz\mathbf{k}$ applied to an object that moves in a straight line from $(0, 1, 4)$ to $(1, 0, -4)$.

5 **a.** Find the surface area of the plane $2x + y + 2z = 16$ bounded by $x = 0$, $y = 0$ and $x^2 + y^2 = 64$.

b. Let ϕ be a scalar field, $\vec{a}$ be a constant vector, and $\vec{r} = (x, y, z)$. Show that:
i) $\operatorname{curl} \vec{r} = 0 \qquad \operatorname{div} \vec{r} = 3$
ii) $\operatorname{curl}(\vec{a} \times \vec{r}) = 2\vec{a}$
iii) $\operatorname{grad} \dfrac{1}{\|\vec{r}\|^3} = -\dfrac{3}{\|\vec{r}\|^5}\vec{r}$.

c. Show that $\vec{F} = (y^2, 2xy - e^y)$ is a conservative field, i.e. $\vec{F} = \nabla f$ for some scalar field f.

d. Let C be the unit circle

$$\vec{r}(t) = (\cos t, \sin t), \quad t \in [0, 2\pi).$$

Let $\vec{G} = (y^2 + y, 2xy - e^y)$.
Evaluate

$$\int_C \vec{G} \cdot d\vec{r}.$$

[Hint: write $\vec{G} = \vec{F} + (y, 0)$]

6 **a.** Write down a statement of Stokes' theorem.

b. Verify Stokes' theorem for

$$\vec{F}(x, y, z) = (z - y, x + z, -(x + y))$$

when S is the surface

$$z = 4 - x^2 - y^2 \quad \text{above} \quad z = 0$$

and $d\vec{S}$ is an upward normal.

c. State the Divergence theorem (Gauss's theorem). Use the divergence theorem to calculate the flux of $\vec{v} = (x, 2y^2, 3z^2)$ out of the solid $0 \leq x^2 + y^2 \leq 9$, $0 \leq z \leq 1$.

Solutions to Math2011 June 2000 exam

1 **a.** The surface is $F(x, y, z) = 0$ where $F(x, y, z) = y^2e^{3z} - \cos(xy) + z - 5$.
Hence the tangent plane at the point $(\pi/2, 2, 0)$ in point–normal form is

$$\nabla F(x, y, z) \cdot (x - \pi/2, y - 2, z) = 0.$$

$$\text{Now} \quad \nabla F(x, y, z) = (y\sin(xy),\ 2ye^{3z} + x\sin(xy),\ 3y^2e^{3z} + 1)$$

$$\therefore \quad \nabla F(\pi/2, 2, 0) = \left(2\sin(\pi),\ 4e^0 + \frac{\pi}{2}\sin(\pi),\ 3 \cdot 2^2 \cdot e^0 + 1\right) = (0, 4, 13)$$

and so the tangent plane has equation

$$\begin{aligned}(0,4,13)\cdot(x,y,z) &= (0,4,13)\cdot(\pi/2,2,0)\\ \text{i.e.}\quad 4y+13z &= 8\end{aligned}$$

b.

$$\frac{\partial f}{\partial u} = \left.\frac{e^{vt}}{t}\right|_{t=u} = \frac{e^{uv}}{u} \quad \text{(by Fund. Thm. of Cal.)}$$

$$\begin{aligned}\text{and}\quad \frac{\partial f}{\partial v} &= \int_1^u \frac{\partial}{\partial v}\left(\frac{e^{vt}}{t}\right)dt = \int_1^u e^{vt}\,dt\\ &= \left.\frac{e^{vt}}{v}\right|_{t=1}^{t=u} = \frac{1}{v}\left(e^{uv}-e^{v}\right)\end{aligned}$$

$$\begin{aligned}\therefore\quad \frac{d}{dx}\int_1^{x^2}\frac{e^{xt}}{t}\,dt &= \frac{d}{dx}f(x^2,x)\\ &= \frac{\partial f}{\partial u}\cdot\frac{du}{dx}+\frac{\partial f}{\partial v}\cdot\frac{dv}{dx} \quad (\text{with } u=x^2,\ v=x)\\ &= \frac{e^{x^3}}{x^2}\cdot 2x+\frac{1}{x}\left(e^{x^3}-e^{x}\right)\cdot 1 = \frac{3e^{x^3}-e^{x}}{x}\end{aligned}$$

c. For $\underset{\sim}{\mathbf{u}}=(1,3)$, let $\underset{\sim}{\mathbf{v}}$ be the unit vector which is the normalisation of $\underset{\sim}{\mathbf{u}}$,

$$\underset{\sim}{\mathbf{v}} = \frac{\underset{\sim}{\mathbf{u}}}{|\underset{\sim}{\mathbf{u}}|} = \frac{1}{\sqrt{1^2+3^2}}(1,3) = \frac{1}{2}(1,3).$$

$$\begin{aligned}\therefore\quad \frac{\partial f}{\partial \underset{\sim}{\mathbf{v}}}(-1,1) &= \text{directional derivative in direction } \underset{\sim}{\mathbf{u}} \text{ at } (-1,1)\\ &= \text{rate of change in } f \text{ per unit increase}\\ &\quad\ \text{in direction of } \underset{\sim}{\mathbf{u}} \text{ at } (-1,1)\\ &= \nabla f(-1,1)\cdot\underset{\sim}{\mathbf{v}}.\end{aligned}$$

$$\text{Now } \nabla f(x,y) = \left(2xe^{x^2-y^2},\ -2ye^{x^2-y^2}\right) \quad \therefore\ \nabla f(-1,1)=(-2,-2)$$

$$\text{hence}\quad \frac{\partial f}{\partial \underset{\sim}{\mathbf{v}}}(-1,1) = (-2,-2)\cdot\frac{1}{2}(1,3) = \frac{1}{2}(-2-6) = -4.$$

d. Let $u=x-1$ and $v=y-\pi$, hence

$$\begin{aligned}xy &= (u+1)(v+\pi) = uv+u\pi+v+\pi\\ \therefore\quad \sin(xy) &= \sin(uv+u\pi+v+\pi) = -\sin(uv+u\pi+v)\\ &= -(uv+u\pi+v)+\frac{(uv+u\pi+v)^3}{3!}-\frac{(uv+u\pi+v)^5}{5!}+\cdots\\ &\quad \text{(using the Maclaurin series for sin)}\\ &= -\pi u - v - uv + \text{higher order terms}\\ &= -\pi(x-1)-(y-\pi)-(x-1)(y-\pi)+\text{higher order terms}\end{aligned}$$

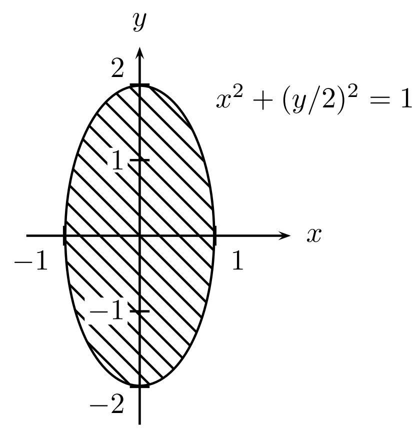
OR directly from the definition of the Taylor series,

$$\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{\partial^n f}{\partial x^k\partial y^{n-k}}(1,\pi)\,(x-1)^k(y-\pi)^{n-k},$$

we calculate

$$
\begin{aligned}
f(1,\pi) &= \sin\pi = 0 \\
f_x = \frac{\partial f}{\partial x} &= y\cos(xy), \quad \therefore \quad f_x(1,\pi) = \pi\cos\pi = -\pi, \\
f_y = \frac{\partial f}{\partial y} &= x\cos(xy), \quad \therefore \quad f_x(1,\pi) = 1\cdot\cos\pi = -1, \\
f_{xx} = \frac{\partial^2 f}{\partial x^2} &= -y^2\sin(xy), \quad \therefore \quad f_{xx}(1,\pi) = -\pi^2\cdot\sin\pi = 0, \\
f_{xy} = \frac{\partial^2 f}{\partial y\partial x} &= \cos(xy) - xy\sin(xy), \quad \therefore \quad f_{xy}(1,\pi) = \cos\pi - \pi\sin\pi = -1, \\
f_{yy} = \frac{\partial^2 f}{\partial y^2} &= -x^2\sin(xy), \quad \therefore \quad f_{yy}(1,\pi) = -1^2\cdot\sin\pi = 0, \\
\therefore \quad \text{T. series} &= 0 + \frac{1}{1!}\left[-\pi(x-1) - 1(y-\pi)\right] \\
&\quad + \frac{1}{2!}\left[0(x-1)^2 + 2(-1)(x-1)(y-\pi) + 0(y-\pi)^2\right] + \cdots \\
&= -\pi(x-1) - (y-\pi) - (x-1)(y-\pi) + \text{higher order terms}
\end{aligned}
$$

e. i)



ii) $I = \int_{-2}^{2}\int_{-\sqrt{1-(y/2)^2}}^{\sqrt{1-(y/2)^2}} f(x,y)\,dx\,dy$

2 a.

$$\nabla f(x,y) = (f_x, f_y) = (6xy - 6x,\ 6y + 6y^2 + 3x^2) = (0,0)$$

$$\text{gives} \quad 6x(y-1) = 0 \tag{1}$$

$$\text{and} \quad 6y + 6y^2 + 3x^2 = 0 \tag{2}$$

Hence from (1), we have $x = 0$ or $y = 1$. Substituting $x = 0$ into (2) gives $6y(1+y) = 0$ so $y = 0$ or $y = -1$, hence we have two critical points $(0,0)$ and $(0,-1)$.
Substituting $y = 1$ into (2) gives $12 + 3x^2 = 0$ which has no real solutions. Hence there are only two critical points.
Letting $A = f_{xx} = 6y - 6,\ B = f_{xy} = 6x,\ C = f_{yy} = 6 + 12y$, we have the following table:

Critical Point	A	B	C	$B^2 - AC$	Type
$(0,0)$	-6	0	6	$36 > 0$	Saddle Point
$(0,-1)$	-12	0	-6	$-72 < 0$	Local Max.

b. Note: It was announced at the exam that this question has an **error**, "minimum" should be replaced by "maximum".
Define the functions $g_1(x,y,z) = x+y+z-1$ and $g_2(x,y,z) = x+z-2$, then local maxima or minima for f on the intersection of the two surfaces (planes) $g_1 = 0$ and $g_2 = 0$, by the Lagrange Multiplier Theorem, occur at critical points of $f - \alpha g_1 - \beta g_2$ for some real constants α, β.

$$\therefore\ \nabla f = \alpha \nabla g_1 + \beta \nabla g_2 \quad \text{at such points.} \qquad (*)$$

Now $\nabla f(x,y) = (2x, 4y, -4z)$, $\nabla g_1(x,y,z) = (1,1,1)$, $\nabla g_2(x,y,z) = (1,0,1)$, so $(*)$ plus the surface conditions $g_1 = 0$ and $g_2 = 0$ gives five equations:

$$\begin{aligned} 2x &= \alpha + \beta & (1)\\ 4y &= \alpha & (2)\\ -4z &= \alpha + \beta & (3)\\ x+y+z &= 1 & (4)\\ x+z &= 2 & (5) \end{aligned}$$

Hence $x = \dfrac{\alpha+\beta}{2}$, $y = \dfrac{\alpha}{4}$, $z = -\left(\dfrac{\alpha+\beta}{4}\right)$, which when substituted into (4) and (5) give

$$\frac{\alpha}{2} + \frac{\beta}{4} = 1 \quad \text{or} \quad 2\alpha + \beta = 4, \quad \text{and} \quad \frac{\alpha}{4} + \frac{\beta}{4} = 2 \quad \text{or} \quad \alpha + \beta = 8$$

Solving these equations, we see $\alpha = -4$ and $\beta = 12$, hence the only possible critical point is

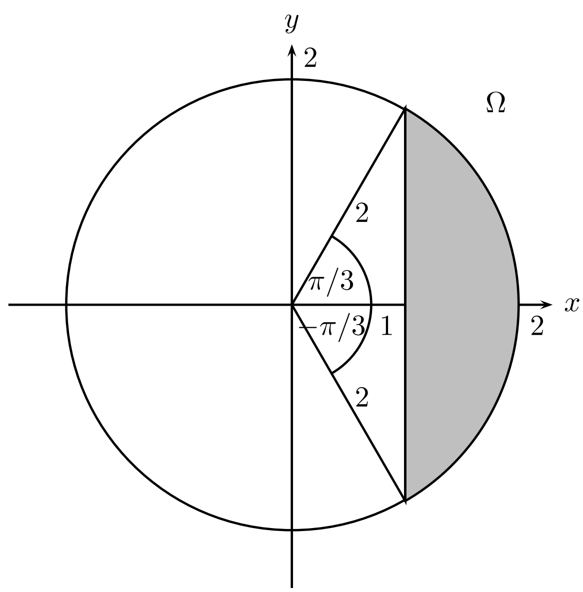
$$(x,y,z) = (4,-1,-2).$$

Which has an f - value of $f(4,-1,-2) = 16 + 2 - 8 = 10$.
This point is in fact a global or absolute **maximum** point for f on the intersection of the two planes. To see this, the intersection of the two planes is a line, given parametrically by

$$(x,y,z) = (2-t,-1,t).$$

Now $f(2-t,-1,t) = 6 - 4t - t^2 = 10 - (t+2)^2$, which has a global maximum when $t = -2$ i.e. at the point $(4,-1,-2)$, and it has no global minimum.

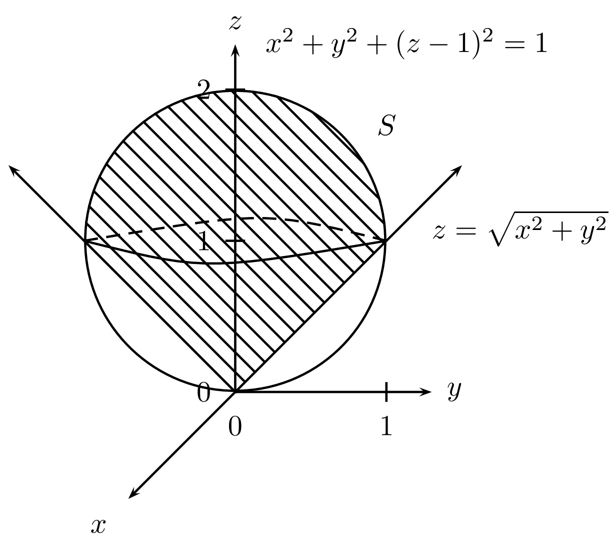
c.



In Ω we see that θ varies from $-\cos^{-1}(1/2) = -\pi/3$ to $\cos^{-1}(1/2) = \pi/3$. Given any θ in this range, r varies along the ray determined by θ from the r solution to $x = 1$ to $r = 2$, i.e. as $x = r\cos\theta$, from $r = 1/\cos\theta$ to $r = 2$.

$$\begin{aligned}
\therefore \quad \iint_\Omega x^3\,dA &= \int_{-\pi/3}^{\pi/3}\int_{1/\cos\theta}^{2} (r\cos\theta)^3 \cdot r\,dr\,d\theta = \int_{-\pi/3}^{\pi/3} \frac{r^5}{5}\bigg|_{r=1/\cos\theta}^{r=2} \cdot \cos^3\theta\,d\theta \\
&= \frac{1}{5}\int_{-\pi/3}^{\pi/3}\left(32 - \frac{1}{\cos^5\theta}\right)\cos^3\theta\,d\theta \\
&= \frac{2}{5}\int_0^{\pi/3} 32\cos^3\theta - \sec^2\theta\,d\theta \qquad \text{(by symmetry)} \\
&= \frac{2}{5}\left[32\left(\sin\theta - \frac{\sin^3\theta}{3}\right) - \tan\theta\right]\bigg|_0^{\pi/3} \\
&= \frac{2}{5}\left[32\left(\frac{\sqrt{3}}{2} - \frac{1}{3}\cdot\left(\frac{\sqrt{3}}{2}\right)^3\right) - \sqrt{3}\right] \\
&= \frac{2}{5}\left[16\sqrt{3}\left(1 - \frac{1}{4}\right) - \sqrt{3}\right] = \frac{2}{5}\left[12\sqrt{3} - \sqrt{3}\right] = \frac{22\sqrt{3}}{5}
\end{aligned}$$

3 **a.** i)



Note that the projection of S onto the xy–plane is $\mathrm{Proj}_{xy}(S) = \{(x,y) \mid x^2 + y^2 \le 1\}$

α) $\quad \mathrm{Vol}(S) = \displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1+\sqrt{1-x^2-y^2}} 1\,dz\,dy\,dx$

β) $\quad \mathrm{Vol}(S) = \displaystyle\int_0^{2\pi}\int_0^1\int_r^{1+\sqrt{1-r^2}} 1\cdot r\,dz\,dr\,d\theta$

γ) $\quad$ All θ–cross-sections are congruent. In the $\theta = \pi/2$ cross-section, we see ϕ varies from 0 to $\pi/4$. Given any θ in $[0, 2\pi]$ and ϕ in $[0, \pi/4]$, then ρ varies from 0 to the ρ solution to the sphere equation:

$$\begin{aligned} x^2+y^2+(z-1)^2 &= 1 \\ \text{so} \quad x^2+y^2+z^2 &= 2z \\ \therefore \quad \rho^2 &= 2\rho\cos\phi \\ \text{hence} \quad \rho &= 2\cos\phi \end{aligned}$$

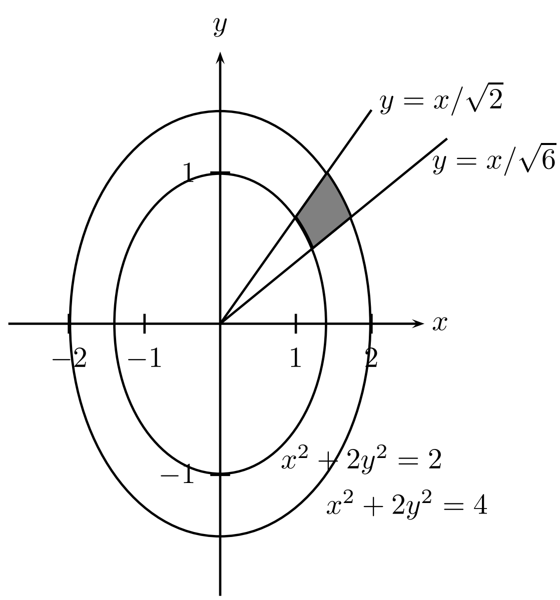
Hence

$$\mathrm{Vol}(S) = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi} 1\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

ii) Using γ) above,

$$\begin{aligned} \mathrm{Vol}(S) &= 2\pi\int_0^{\pi/4} \left.\frac{\rho^3}{3}\right|_{\rho=0}^{\rho=2\cos\phi}\cdot\sin\phi\,d\phi \\ &= 2\pi\cdot\frac{8}{3}\int_0^{\pi/4}\cos^3\phi\sin\phi\,d\phi \\ &= \frac{16\pi}{3}\left[-\frac{\cos^4\phi}{4}\right]\Bigg|_0^{\pi/4} \\ &= \frac{4\pi}{3}\left[1-\left(\frac{1}{\sqrt{2}}\right)^4\right] = \frac{4\pi}{3}\left[1-\frac{1}{4}\right] = \frac{4\pi}{3}\cdot\frac{3}{4} \\ &= \pi. \end{aligned}$$

b.



Call the region R, then with the change of variables

$$u = x^2+2y^2, \qquad v = \frac{y}{x},$$

we have

$$\begin{aligned}
\text{Area}(R) &= \iint_R 1 \, dx\, dy = \iint_{R'} 1 \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv \\
&\text{with } R = \{(x,y) \mid 2 \le x^2 + 2y^2 \le 4,\ x/\sqrt{6} \le y \le x/\sqrt{2},\ x, y > 0\} \\
&\therefore \quad R' = \{(u,v) \mid 2 \le u \le 4,\ 1/\sqrt{6} \le v \le 1/\sqrt{2}\}
\end{aligned}$$

$$\text{Now} \quad \frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)} \right)^{-1} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}^{-1} = \begin{vmatrix} 2x & 4y \\ -y/x^2 & 1/x \end{vmatrix}^{-1}$$

$$= \frac{1}{2 + 4y^2/x^2} = \frac{1}{2+4v^2}$$

$$\begin{aligned}
\therefore \quad \text{Area}(R) &= \iint_{R'} \frac{1}{2+4v^2} \, du\, dv = \frac{1}{4} \int_2^4 \int_{1/\sqrt{6}}^{1/\sqrt{2}} \frac{1}{1/2 + v^2} \, dv\, du \\
&= \frac{1}{4} \cdot \sqrt{2} \tan^{-1}(\sqrt{2}v) \Big|_{v=1/\sqrt{6}}^{v=1/\sqrt{2}} \cdot (4-2) \\
&= \frac{\sqrt{2}}{2} \left[\tan^{-1}(1) - \tan^{-1}(1/\sqrt{3}) \right] \\
&= \frac{\sqrt{2}}{2} \left[\frac{\pi}{4} - \frac{\pi}{6} \right] \\
&= \frac{\pi\sqrt{2}}{24}
\end{aligned}$$

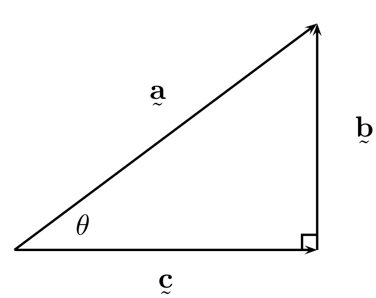
4 **a.** $\underset{\sim}{\mathbf{r}}(t) = (t, t^2, t^3)$ meets the plane $4x + 2y + z = 24$ when $4t + 2t^2 + t^3 = 24$, i.e. $(t-2)(t^2 + 4t + 12) = 0$.

Now since the quadratic factor has discriminant $4^2 - 4 \cdot 1 \cdot 12 = 16 - 48 < 0$, there are no other real roots, so the only point of intersection occurs when $t = 2$, i.e. at the point $(2, 4, 8)$

At this point, the curve $\underset{\sim}{\mathbf{r}}(t)$ has a tangent vector

$$\dot{\underset{\sim}{\mathbf{r}}}(2) = (1, 2t, 3t^2)\big|_{t=2} = (1, 4, 12) = \underset{\sim}{\mathbf{a}} \quad \text{(say)}$$

The angle θ the curve makes with the plane at the point $(2, 4, 8)$, is determined by the right-angled vector triangle:



with $\underset{\sim}{\mathbf{b}} \parallel \underset{\sim}{\mathbf{n}} = (4, 2, 1)$ normal to the plane and $\underset{\sim}{\mathbf{c}}$ lying in the plane.

$$\text{Hence}\quad \underset{\sim}{\mathbf{b}} = \text{proj}_{\underset{\sim}{\mathbf{n}}}(\underset{\sim}{\mathbf{a}}) = \left(\frac{\underset{\sim}{\mathbf{a}} \cdot \underset{\sim}{\mathbf{n}}}{\underset{\sim}{\mathbf{n}} \cdot \underset{\sim}{\mathbf{n}}}\right) \underset{\sim}{\mathbf{n}}$$
$$= \left(\frac{(1,4,12) \cdot (4,2,1)}{(4,2,1) \cdot (4,2,1)}\right)(4,2,1) = \left(\frac{4+8+12}{16+4+1}\right)(4,2,1) = \frac{8}{7}(4,2,1)$$
$$\therefore \quad \sin\theta = \frac{|\underset{\sim}{\mathbf{b}}|}{|\underset{\sim}{\mathbf{a}}|} = \frac{8}{7}\sqrt{\frac{21}{161}} = \frac{8}{7}\sqrt{\frac{3}{23}} \approx 0.41275$$
$$\therefore \quad \theta \approx 24.377^\circ \quad \text{or} \quad 0.42547 \text{ rads}$$

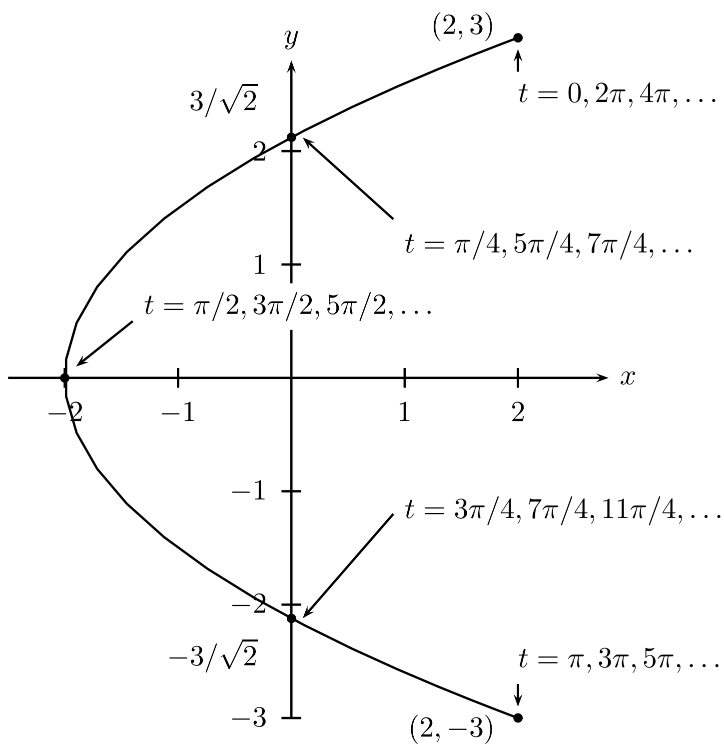
Alternatively, $\theta = \pi/2 - \phi$ where ϕ is the angle (in radians) between $\underset{\sim}{\mathbf{a}}$ and that normal vector to the plane pointing to the same side of the plane as $\underset{\sim}{\mathbf{a}}$. Let ω be the angle between $\underset{\sim}{\mathbf{a}}$ and $\underset{\sim}{\mathbf{n}}$, where $0 \le \omega \le \pi$, then

$$\phi = \begin{cases} \omega, & \text{if } 0 \le \omega < \pi/2, \\ \pi - \omega, & \text{if } \omega > \pi/2 \end{cases}$$

$$\therefore \quad \cos\omega = \frac{\underset{\sim}{\mathbf{a}} \cdot \underset{\sim}{\mathbf{n}}}{|\underset{\sim}{\mathbf{a}}|\,|\underset{\sim}{\mathbf{n}}|} = \frac{(1,4,12) \cdot (4,2,1)}{\sqrt{1+16+144}\,\sqrt{16+4+1}}$$
$$= \frac{4+8+12}{\sqrt{161}\,\sqrt{21}} = \frac{24}{\sqrt{161}\,\sqrt{21}}$$
$$> 0 \qquad \therefore \quad 0 < \omega < \pi/2 \quad \text{and so} \quad \omega = \phi,$$
$$\therefore \quad \cos\phi \approx 0.41275$$
$$\text{so} \quad \phi \approx 1.14533 \text{ rads}$$
$$\therefore \quad \theta = \pi/2 - \phi \approx 0.42547 \text{ rads.}$$

b. i) For $\underset{\sim}{\mathbf{r}}(t) = (x, y) = (2\cos 2t,\, 3\cos t)$,

$$\frac{x}{2} = \cos 2t = 2\cos^2 t - 1 = 2\left(\frac{y}{3}\right)^2 - 1 = \frac{2y^2}{9} - 1$$
$$\therefore \quad 9x = 4y^2 - 18$$
$$\text{or} \quad 4y^2 - 9x = 18$$



ii) Velocity vector is $\dot{\underset{\sim}{\mathbf{r}}}(t) = (-4\sin 2t,\ -3\sin t)$ and the acceleration vector is $\ddot{\underset{\sim}{\mathbf{r}}}(t) = (-8\cos 4t,\ -3\cos t)$.
Now $\dot{\underset{\sim}{\mathbf{r}}}(t) = \underset{\sim}{\mathbf{0}} = (0,0)$ when $\sin 2t = 2\sin t\cos t = 0$ and $\sin t = 0$, i.e. when $\sin t = 0$ so when $t = n\pi$ for n a nonnegative integer (as $t \geq 0$). For $t = n\pi$,

$$\ddot{\underset{\sim}{\mathbf{r}}}(t) = \ddot{\underset{\sim}{\mathbf{r}}}(n\pi) = (-8,\ -3(-1)^n)$$

c. The work done is

$$W = \int_C \underset{\sim}{\mathbf{F}} \cdot d\underset{\sim}{\mathbf{r}}$$

where the straight line interval is parameterised as

$$\begin{aligned}
C:\quad \underset{\sim}{\mathbf{r}}(t) &= (x,y,z) = (0,1,4) + t\,((1,0,4) - (0,1,4)) = (0,1,4) + t(1,-1,8)\\
&= (t, 1-t, 4-8t) \quad \text{for } t = 0 \text{ to } t = 1\\
\therefore\ \dot{\underset{\sim}{\mathbf{r}}}(t) &= (1,-1,-8)\\
\text{hence}\quad W &= \int_0^1 \underset{\sim}{\mathbf{F}}(\underset{\sim}{\mathbf{r}}(t)) \cdot \dot{\underset{\sim}{\mathbf{r}}}(t)\ dt\\
&= \int_0^1 \underset{\sim}{\mathbf{F}}(t, 1-t, 4-8t) \cdot (1,-1,-8)\ dt\\
&= \int_0^1 (t, t(1-t), t(1-t)(4-8t)) \cdot (1,-1,-8)\ dt\\
&= \int_0^1 -32t + 97t^2 - 64t^3\ dt = \left[-16t^2 + \frac{97t^3}{3} - 16t^4\right]\Bigg|_0^1\\
&= -16 + \frac{97}{3} - 16 = -32 + \frac{97}{3} = \frac{-96+97}{3} = \frac{1}{3}
\end{aligned}$$

5 **a.** Call the surface $\mathcal{S}$. Using x and y co-ordinates for the surface parameters, we have the parameterisation of $\mathcal{S}$:

$$\mathcal{S}:\quad \underset{\sim}{\mathbf{r}}(x,y) = (x,\ y,\ z) = (x,\ y,\ 8 - x - y/2) \quad \text{for } (x,y) \in \Omega$$

where Ω is the parameter domain $\{(x,y) \mid x^2 + y^2 \leq 64,\ x \geq 0,\ y \geq 0\}$

$$\therefore\quad \frac{\partial \underset{\sim}{\mathbf{r}}}{\partial x} = \left(1,\ 0,\ \frac{\partial z}{\partial x}\right) = (1,\ 0,\ -1), \quad \frac{\partial \underset{\sim}{\mathbf{r}}}{\partial y} = \left(0,\ 1,\ \frac{\partial z}{\partial y}\right) = (0,\ 1,\ -1/2)$$

$$\therefore\quad \frac{\partial \underset{\sim}{\mathbf{r}}}{\partial x} \times \frac{\partial \underset{\sim}{\mathbf{r}}}{\partial y} = \begin{vmatrix} \underset{\sim}{\mathbf{i}} & \underset{\sim}{\mathbf{j}} & \underset{\sim}{\mathbf{k}} \\ 1 & 0 & \frac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{vmatrix} = \left(-\frac{\partial z}{\partial x},\ -\frac{\partial z}{\partial y},\ 1\right)$$

$$\begin{aligned}
\therefore\quad dS &= \left|\frac{\partial \underset{\sim}{\mathbf{r}}}{\partial x} \times \frac{\partial \underset{\sim}{\mathbf{r}}}{\partial y}\right| = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dx\,dy\\
&= \sqrt{1 + (-1)^2 + (-1/2)^2}\ dx\,dy = \sqrt{\frac{9}{4}}\ dx\,dy = \frac{3}{2}\ dx\,dy
\end{aligned}$$

where dS is the scalar surface area element

$$\text{hence}\quad \text{Area}(S) = \int_{\mathcal{S}} 1\ dS = \iint_\Omega \frac{3}{2}\ dx\,dy = \frac{3}{2} \cdot \text{Area}(\Omega) = \frac{3}{2} \cdot \frac{1}{4} \cdot \pi \cdot 8^2 = 24\pi.$$

b. i)

$$\operatorname{curl} \underset{\sim}{\mathbf{r}} = \underset{\sim}{\nabla} \times \underset{\sim}{\mathbf{r}} = \begin{vmatrix} \underset{\sim}{\mathbf{i}} & \underset{\sim}{\mathbf{j}} & \underset{\sim}{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0-0,\ -(0-0),\ 0-0) = (0,0,0) = \underset{\sim}{\mathbf{0}}$$

$$\operatorname{div} \underset{\sim}{\mathbf{r}} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

ii)

$$\underset{\sim}{\mathbf{a}} \times \underset{\sim}{\mathbf{r}} = \begin{vmatrix} \underset{\sim}{\mathbf{i}} & \underset{\sim}{\mathbf{j}} & \underset{\sim}{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y,\ a_3 x - a_1 z,\ a_1 y - a_2 x)$$

$$\begin{aligned} \therefore \quad \operatorname{curl}(\underset{\sim}{\mathbf{a}} \times \underset{\sim}{\mathbf{r}}) &= \underset{\sim}{\nabla} \times (\underset{\sim}{\mathbf{a}} \times \underset{\sim}{\mathbf{r}}) \\ &= \begin{vmatrix} \underset{\sim}{\mathbf{i}} & \underset{\sim}{\mathbf{j}} & \underset{\sim}{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix} \\ &= (a_1 + a_1,\ -(-a_2 - a_2),\ a_3 + a_3) = (2a_1,\ 2a_2,\ 2a_3) = 2\underset{\sim}{\mathbf{a}} \end{aligned}$$

iii)

$$\begin{aligned} \operatorname{grad}\left(\frac{1}{||\underset{\sim}{\mathbf{r}}||^3}\right) &= \underset{\sim}{\nabla}\left(\frac{1}{r^3}\right) \qquad \text{where } r = ||\underset{\sim}{\mathbf{r}}|| = \sqrt{x^2+y^2+z^2} \\ &= \left(\frac{\partial}{\partial x}\left(r^{-3}\right), \frac{\partial}{\partial y}\left(r^{-3}\right), \frac{\partial}{\partial z}\left(r^{-3}\right)\right) \end{aligned}$$

$$\text{Now} \quad \frac{\partial}{\partial x}\left(r^{-3}\right) = -3r^{-4} \cdot \frac{\partial r}{\partial x} = -3r^{-4} \cdot \frac{x}{r} = -\frac{3x}{r^5}$$

$$\text{as} \quad \frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2x = \frac{x}{r}$$

$$\text{Similarly} \quad \frac{\partial}{\partial y}\left(r^{-3}\right) = -\frac{3y}{r^5}, \qquad \frac{\partial}{\partial z}\left(r^{-3}\right) = -\frac{3z}{r^5}$$

$$\begin{aligned} \therefore \quad \underset{\sim}{\nabla}\left(\frac{1}{r^3}\right) &= \left(-\frac{3x}{r^5}, -\frac{3y}{r^5}, -\frac{3z}{r^5}\right) \\ &= -\frac{3}{r^5}(x,y,z) = -\frac{3}{||\underset{\sim}{\mathbf{r}}||^5}\underset{\sim}{\mathbf{r}} \end{aligned}$$

c. Method 1. Directly solve

$$\underset{\sim}{\mathbf{F}} = (y^2,\ 2xy - e^y) = \underset{\sim}{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\text{so} \quad \frac{\partial f}{\partial x} = y^2 \tag{1}$$

$$\text{and} \quad \frac{\partial f}{\partial y} = 2xy - e^y \tag{2}$$

Integrating (1) w.r.t. y, we have $f(x,y) = xy^2 + h(y)$. Substituting this into (2) we have

$$\begin{aligned} 2xy + h'(y) &= 2xy - e^y \\ \therefore \quad h'(y) &= -e^y \quad \text{so} \quad h(y) = -e^y + \text{const.} \end{aligned}$$

In other words we have solutions $f(x,y) = xy^2 - e^y + \text{const.}$

Method 2. Extend $\underset{\sim}{\mathbf{F}}$ to a vector field in $\mathbb{R}^3$ by

$$\underset{\sim}{\mathbf{G}}(x,y,z) = (\underset{\sim}{\mathbf{F}}(x,y),\, 0) = (y^2,\, 2xy - e^y,\, 0)$$

then calculate

$$\operatorname{curl}\underset{\sim}{\mathbf{G}} = \begin{vmatrix} \underset{\sim}{\mathbf{i}} & \underset{\sim}{\mathbf{j}} & \underset{\sim}{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy - e^y & 0 \end{vmatrix} = (0,\, -(0-0),\, 2y - 2y) = (0,0,0) = \underset{\sim}{\mathbf{0}}$$

Hence $\underset{\sim}{\mathbf{G}}$ is a conservative vector field in $\mathbb{R}^3$ so there is a scalar field ϕ on $\mathbb{R}^3$ with $\underset{\sim}{\nabla}\phi = \underset{\sim}{\mathbf{G}}$. Since then $\dfrac{\partial\phi}{\partial z} = 0$, ϕ is independent of z and we may write $\phi(x,y,z) = f(x,y)$, hence $\underset{\sim}{\nabla} f = \underset{\sim}{\mathbf{F}}$.

d. Using the hint, let $\underset{\sim}{\mathbf{G}} = \underset{\sim}{\mathbf{F}} + \underset{\sim}{\mathbf{H}}$ where $\underset{\sim}{\mathbf{F}}$ is the conservative vector field in part c) and $\underset{\sim}{\mathbf{H}}(x,y) = (y,0)$.

$$\begin{aligned}
\therefore \quad \int_C \underset{\sim}{\mathbf{G}} \cdot d\underset{\sim}{\mathbf{r}} &= \int_C \underset{\sim}{\mathbf{F}} \cdot d\underset{\sim}{\mathbf{r}} + \int_C \underset{\sim}{\mathbf{H}} \cdot d\underset{\sim}{\mathbf{r}} = 0 + \int_C \underset{\sim}{\mathbf{H}} \cdot d\underset{\sim}{\mathbf{r}} \\
&\qquad \text{(as } F \text{ is conservative in } \mathbb{R}^2 \text{ and } C \text{ is a closed path)} \\
&= \int_0^{2\pi} \underset{\sim}{\mathbf{H}}(\underset{\sim}{\mathbf{r}}(t)) \cdot \dot{\underset{\sim}{\mathbf{r}}}(t)\, dt \\
&= \int_0^{2\pi} (\sin t,\, 0) \cdot (-\sin t,\, \cos t)\, dt \\
&= -4 \int_0^{\pi/2} \sin^2 t\, dt \qquad \text{(by symmetry)} \\
&= -4I_{2,0} = -4 \cdot \left(\frac{2-1}{2+0}\right) \cdot I_{0,0} = -4 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = -\pi
\end{aligned}$$

(where a reduction formula for a trig. integral has been used.)

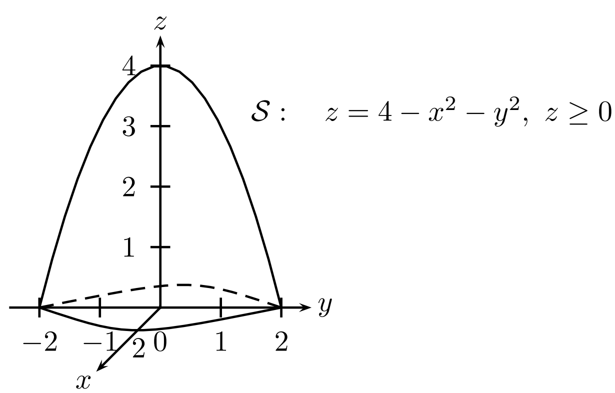
6 a.

Stokes' Theorem. *If $\underset{\sim}{\mathbf{F}}$ is a continuously differentiable vector field on an open subset of $\mathbb{R}^3$ containing an open smooth (orientable) connected surface $\mathcal{S}$ with a boundary curve C made up of finitely many smooth curves, then*

$$\int_C \underset{\sim}{\mathbf{F}} \cdot d\underset{\sim}{\mathbf{r}} = \int_{\mathcal{S}} \operatorname{curl}\underset{\sim}{\mathbf{F}} \cdot d\underset{\sim}{\mathbf{S}}$$

(up to $\pm$ with a "consistent" choice of normal to the surface $\mathcal{S}$)

b.



Parameterising the surface $\mathcal{S}$ by x and y co-ordinates,

$$\mathcal{S}: \quad \underset{\sim}{\mathbf{r}}(x,y) = (x,y,z) = (x,y,4-x^2-y^2) \quad \text{for} \quad (x,y) \in \Omega$$
$$\text{(for parameter domain } \Omega = \{(x,y) \mid x^2+y^2 \le 4\}),$$

$$\therefore \quad \frac{\partial \underset{\sim}{\mathbf{r}}}{\partial x} = (1,0,-2x), \quad \frac{\partial \underset{\sim}{\mathbf{r}}}{\partial y} = (0,1,-2y)$$

$$\text{so} \quad \frac{\partial \underset{\sim}{\mathbf{r}}}{\partial x} \times \frac{\partial \underset{\sim}{\mathbf{r}}}{\partial y} = \begin{vmatrix} \underset{\sim}{\mathbf{i}} & \underset{\sim}{\mathbf{j}} & \underset{\sim}{\mathbf{k}} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = (2x,2y,1)$$

$$\therefore \quad d\underset{\sim}{\mathbf{S}} = (2x,2y,1)\ dx\,dy \qquad \text{(outward normal)}$$

$$\text{and} \quad \operatorname{curl} \underset{\sim}{\mathbf{F}} = \begin{vmatrix} \underset{\sim}{\mathbf{i}} & \underset{\sim}{\mathbf{j}} & \underset{\sim}{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y & x+z & -x-y \end{vmatrix}$$
$$= (-1-1,\ -(-1-1),\ 1-(-1)) = (-2,2,2)$$

$$\begin{aligned} \therefore \quad \int_{\mathcal{S}} \operatorname{curl} \underset{\sim}{\mathbf{F}} \cdot d\underset{\sim}{\mathbf{S}} &= \iint_{\Omega} (-2,2,2) \cdot (2x,2y,1)\ dx\,dy \\ &= \iint_{\Omega} -4x+4y+2\ dx\,dy \\ &= -4 \iint_{\Omega} x\ dx\,dy + 4 \iint_{\Omega} y\ dx\,dy + 2 \iint_{\Omega} 1\ dx\,dy \\ &= -4 \cdot 0 + 4 \cdot 0 + 2 \cdot \text{Area}(\Omega) \quad \text{(by symmetry)} \\ &= 0 + 0 + 2 \cdot \pi \cdot 2^2 \\ &= 8\pi. \end{aligned}$$

The boundary curve C of the open surface $\mathcal{S}$ is the circle $x^2+y^2=4$ in the xy–plane, so parameterising C by

$$C: \quad \underset{\sim}{\mathbf{r}} = (x,y,z) = (2\cos\theta, 2\sin\theta, 0), \quad \theta = 0 \quad \text{to} \quad \theta = 2\pi,$$

$$\begin{aligned} \therefore \quad \int_C \underset{\sim}{\mathbf{F}} \cdot d\underset{\sim}{\mathbf{r}} &= \int_0^{2\pi} (0-2\sin\theta, 2\cos\theta+0, -2\cos\theta-2\sin\theta) \cdot \dot{\underset{\sim}{\mathbf{r}}}(\theta)\ d\theta \\ &= \int_0^{2\pi} (-2\sin\theta, 2\cos\theta, -2\cos\theta-2\sin\theta) \cdot (-2\sin\theta, 2\cos\theta, 0)\ d\theta \\ &= \int_0^{2\pi} 4\sin^2\theta + 4\cos^2\theta\ d\theta = \int_0^{2\pi} 4\ d\theta = 4 \cdot 2\pi = 8\pi. \end{aligned}$$

c.

Divergence or Gauss's Theorem. *If a vector field $\underset{\sim}{\mathbf{F}}$ is continuously differentiable in an open subset of $\mathbb{R}^3$ containing a solid Ω bounded by a closed surface $\mathcal{S}$, made up of finitely many smooth surfaces, then*

$$\int_{\Omega} \operatorname{div} \underset{\sim}{\mathbf{F}}\ dV = \int_{\mathcal{S}} \underset{\sim}{\mathbf{F}} \cdot d\underset{\sim}{\mathbf{S}}$$

By definition, the flux of the vector field $\underset{\sim}{\mathbf{v}} = (x, 2y^2, 3z^2)$ out of the cylindrical solid $\Omega = \{(x,y,x) \mid 0 \le x^2+y^2 \le 9,\ 0 \le z \le 1\}$ is the surface integral

$$\text{Flux}(\underset{\sim}{\mathbf{v}}) = \int_{\mathcal{S}} \underset{\sim}{\mathbf{v}} \cdot d\underset{\sim}{\mathbf{S}}$$

where $\mathcal{S}$ is the closed surface made up of

$$\begin{gathered}\text{the curved cylindrical surface } \{(x,y,z) \mid x^2+y^2=9,\ 0 \le z \le 1\} \\ \text{and the two (flat) circular disks } x^2+y^2 \le 9 \text{ at } z=0 \text{ and } z=1,\end{gathered}$$

and the vector surface area element $d\underset{\sim}{\mathbf{S}}$ is always taken outward from Ω.

$$\text{Now} \quad \operatorname{div} \underset{\sim}{\mathbf{v}} = \frac{\partial}{\partial x}(x)\frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}(3z^2) = 1 + 4y + 6z$$

Hence by the Divergence Theorem,

$$\begin{aligned}
\text{Flux}(\underset{\sim}{\mathbf{v}}) &= \int_\Omega 1 + 4y + 6z \ dV = \iiint_\Omega 1 + 4y + 6z \ dx\,dy\,dz \\
&= \text{Vol}(\Omega) + 4\iiint_\Omega y \ dx\,dy\,dz + 6\iiint_\Omega z \ dx\,dy\,dz \\
&= \pi \cdot 3^2 \cdot 1 + 0 + 6\int_0^{2\pi}\int_0^3\int_0^1 z \cdot r \ dz\,dr\,d\theta \\
&\qquad \text{(by symmetery for 2nd integral and using cylindricals in 3rd)} \\
&= 9\pi + 6 \cdot 2\pi \left(\int_0^3 r\,dr\right)\left(\int_0^1 z\,dz\right) \\
&= 9\pi + 12\pi \cdot \left(\left.\frac{r^2}{2}\right|_0^3\right)\left(\left.\frac{z^2}{2}\right|_0^1\right) = 9\pi + 12\pi \cdot \frac{9}{2} \cdot \frac{1}{2} = 9\pi + 27\pi = 36\pi
\end{aligned}$$

Math2011 Several Variable Calculus — June 2001.

1 **a.** Determine the equation of the tangent plane to the surface

$$z = f(x, y) = e^x \ln y \quad \text{at} \quad (4, 1).$$

b. If $z = f(t)$ and $t = \dfrac{1}{x} + \dfrac{1}{y}$, show that

$$x^2 \frac{\partial z}{\partial x} = y^2 \frac{\partial z}{\partial y}.$$

c. Use differentials to approximate the value of $\sqrt{(3.03)^2 + (1.98)^2 + (5.98)^2}$.

d. Your engineering company has won the contract to build a ski slope for the 2006 Winter Olympics. The mountain side is approximated by the function

$$z = f(x, y) = 5000 - \frac{x^2}{1000} - \frac{y^2}{40}.$$

The company decides to start the ski slope at the point $(200, 40)$.

i) In which direction initially should the ski slope point to ensure the fastest possible ski run?

ii) What is the slope of the mountain side in this direction at $(200, 40)$?

e. Consider the following double integral

$$\int_0^3 \int_{y^2}^9 y \cos(x^2)\, dx\, dy.$$

i) Sketch the region of integration.

ii) Hence determine an equivalent double integral with the order of integration reversed. [**DO NOT EVALUATE THIS DOUBLE INTEGRAL.**]

2 **a.** A function $f : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x^2 - y^2 \\ y \end{pmatrix}$$

and the function $g : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$g\begin{pmatrix} u \\ v \\ w \end{pmatrix} = (uvw).$$

Let $h = g \circ f$ be the composite function.

i) Calculate $J_{f(\mathbf{x}_0)} g \cdot J_{\mathbf{x}_0} f$ where $\mathbf{x}_0 = (1, 1)$.

ii) Hence determine the gradient of h at $\mathbf{x}_0 = (1, 1)$.

b. i) Using the method of Lagrange Multipliers determine the maximum value of $x^2y^2z^2$ on a sphere of radius r centred at the origin.

ii) Using your result from i), show that

$$(x^2y^2z^2)^{\frac{1}{3}} \le \frac{x^2 + y^2 + z^2}{3}.$$

3 **a.** Consider the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 2x$.

i) Sketch the solid.

ii) Hence write down a triple integral to determine the required volume in

- cartesian co-ordinates, and
- cylindrical polar co-ordinates.

Clearly indicate the limits of integration in both cases.

iii) Determine the volume of the solid.

b. Consider the double integral

$$\iint_{\mathcal{R}} (x^2 - xy + y^2)\, dA,$$

where $\mathcal{R}$ is the region bounded by the ellipse $x^2 - xy + y^2 = 2$ in the xy-plane.

i) Using the transformation T:

$$T : x = \sqrt{2}\, u - \sqrt{\frac{2}{3}}\, v, \quad y = \sqrt{2}\, u + \sqrt{\frac{2}{3}}\, v,$$

sketch the region $T(\mathcal{R})$ in the uv-plane.

ii) Hence evaluate the double integral.

4 **a.** Find the length of the curve

$$\underset{\sim}{\mathbf{r}}(t) = 2 \ln t\, \mathbf{i} + 2t\, \mathbf{j} + \frac{1}{2} t^2\, \mathbf{k}$$

from $t = 1$ to $t = 2$.

b. Consider the two curves given in parametric form as

$$\begin{array}{lll} C_1 : & \underset{\sim}{\mathbf{r}}_1(t) = (t, t, t), & t \in \mathbb{R} \\ C_2 : & \underset{\sim}{\mathbf{r}}_2(s) = (s, s^2, s^3), & s \in \mathbb{R} \end{array}$$

i) Find the angle between the two curves at each point of intersection.

ii) Find all points on C_2 where the tangent is parallel to the vector $(1, 0, 0)$, and find the parametric vector equations of the tangent lines at these points.

c. Let $\underset{\sim}{\mathbf{F}}(x, y, z) = \alpha \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}, \quad \alpha \in \mathbb{R}.$

i) Show that $\underset{\sim}{\mathbf{F}}$ is a gradient vector field by finding φ such that $\underset{\sim}{\mathbf{F}} = \nabla \varphi$.

ii) Show that the work done by $\underset{\sim}{\mathbf{F}}$ in moving an object along any curve C joining any two point on the unit sphere $x^2 + y^2 + z^2 = 1$ is 0.

d. Use Green's theorem to show that

$$\oint_C e^x \sin y\, dx + e^x \cos y\, dy = 0$$

for any smooth simple closed curve in $\mathbb{R}^2$.

5 **a.** Let $\underset{\sim}{\mathbf{r}} = (x, y, z)$ and $r = \|\underset{\sim}{\mathbf{r}}\| = \sqrt{x^2 + y^2 + z^2}$.

i) Show that $\nabla(\frac{1}{r}) = \frac{-\underset{\sim}{\mathbf{r}}}{r^3}$, $\nabla \cdot \underset{\sim}{\mathbf{r}} = 3$, and $\nabla \times \underset{\sim}{\mathbf{r}} = \underset{\sim}{\mathbf{0}}$.

ii) If f is a differentiable function of one variable, show that

$$\nabla \cdot (f(r)\underset{\sim}{\mathbf{r}}) = r f'(r) + 3 f(r).$$

Find $f(r)$ if $f(r)\underset{\sim}{\mathbf{r}}$ is solenoidal, i.e., $\nabla\cdot(f(r)\underset{\sim}{\mathbf{r}})=0$, for $r\neq 0$.

b. Let G be the region in $\mathbb{R}^3$ bounded by the paraboloid $z=1-(x^2+y^2)$ and the x-y plane, and let its bounding surface be denoted by S. Let $\underset{\sim}{\mathbf{v}}=(x,y,z)$.

i) Evaluate $\displaystyle\iiint_G \nabla\cdot\underset{\sim}{\mathbf{v}}\,dx\,dy\,dz$ directly.

ii) Give a parametrization for each of the two separate surfaces of S, the top and bottom.

iii) Use ii) to compute the flux of $\underset{\sim}{\mathbf{v}}$ out of S, directly.

iv) What theorem asserts the equality of the results of i) and iii). f i) and iii).

6 **a.** Let $\underset{\sim}{\mathbf{F}}(x,y,z)=(-3y,3x,z^4)$. Let S_1 be the portion of the ellipsoid $z=\sqrt{1-2(x^2+y^2)}$ that lies above the plane $z=1/\sqrt{2}$. Let C be the bounding curve, and S_2 be the portion of the plane $z=1/\sqrt{2}$ bounded by C.

i) Compute $\nabla\times\underset{\sim}{\mathbf{F}}$.

ii) Give a precise statement of Stokes' theorem applied to S_1 and C.

iii) Give a precise statement of Stokes' theorem applied to S_2 and C.

iv) Deduce from ii) and iii) or by direct calculations that

$$\iint_{S_1}[(\nabla\times\underset{\sim}{\mathbf{F}})\cdot\underset{\sim}{\mathbf{n}}]\,dS=\frac{3}{2}\pi.$$

b. Find the surface area of the portion of the plane $x+y+z=8$ that lies within the cylinder $x^2+y^2=1$.

Solutions to Math2011 June 2001 exam

1 **a.** Let $\phi(x,y,z)=z-e^x\ln y$. So the surface is $\phi(x,y,z)=0$. The normal $\underset{\sim}{\mathbf{n}}$ to the surface at $(4,1,1)$ is given by

$$\begin{aligned}\underset{\sim}{\mathbf{n}}=\pm\nabla\phi(4,1,1)&=\pm\left(\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y},\frac{\partial\varphi}{\partial z}\right)\Bigg|_{(4,1,1)}\\&=\pm\left(-e^x\ln y,\ -\frac{e^x}{y},\ 1\right)\Bigg|_{(4,1,1)}\\&=\pm\left(-e^4\ln 1,\ -\frac{e^4}{1},\ 1\right)\\&=\pm(0,\ -e^4,\ 1)\end{aligned}$$

The tangent plane is given by $\nabla\phi(4,1,1)\bullet\big(\underset{\sim}{\mathbf{x}}-(4,1,1)\big)=0$. Hence

$$(x-4)\cdot 0+(y-1)\cdot(-e^4)+(z-0)\cdot 1=0\quad\Rightarrow\quad e^4y-z=e^4.$$

b.

$$\begin{aligned}\frac{\partial z}{\partial x}&=f'(t)\frac{\partial t}{\partial x}=f'(t)\left(-\frac{1}{x^2}\right)\\\frac{\partial z}{\partial y}&=f'(t)\frac{\partial t}{\partial y}=f'(t)\left(-\frac{1}{y^2}\right)\\\therefore\quad x^2\frac{\partial z}{\partial x}&=f'(t)=y^2\frac{\partial z}{\partial y}\end{aligned}$$

c. Let $f(\underset{\sim}{\mathbf{x}}) = \sqrt{x^2+y^2+z^2}$. Thus using differentials we have

$$\begin{aligned}
&f(x+\Delta x,\ y+\Delta y,\ z+\Delta z)\\
&\approx f(x,y,z) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z\\
&= f(x,y,z) + \frac{x}{\sqrt{x^2+y^2+z^2}}\Delta x + \frac{y}{\sqrt{x^2+y^2+z^2}}\Delta y + \frac{z}{\sqrt{x^2+y^2+z^2}}\Delta z\\
&= f(x,y,z) + \frac{x}{f(x,y,z)}\Delta x + \frac{y}{f(x,y,z)}\Delta y + \frac{z}{f(x,y,z)}\Delta z
\end{aligned}$$

Hence we can approximate $f(3.03,\ 1.98,\ 5.98)$ with $x = 3$, $y = 2$, $z = 6$, and $\Delta x = 0.03$, $\Delta y = -0.02$, $\Delta z = -0.02$. Thus

$$\begin{aligned}
&f(3.03,\ 1.98,\ 5.98)\\
&\approx f(3,2,6) + \frac{3}{f(3,2,6)}(0.03) + \frac{2}{f(3,2,6)}(-0.02) + \frac{6}{f(3,2,6)}(-0.02)\\
&= 7 + \frac{0.09}{7} - \frac{0.04}{7} - \frac{0.12}{7}\\
&= 7 - \frac{0.07}{7} = 6.99
\end{aligned}$$

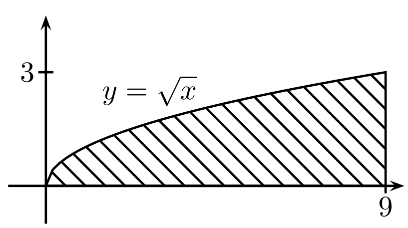
d. The gradient vector is given by $\nabla f(x,y) = (-x/500,\ -y/20)$. At the point $(200,\ 40)$ the gradient is $\nabla f(200,\ 40) = (-0.4,\ -2)$.

i) Direction of fastest ski run is direction of the vector $\underset{\sim}{\mathbf{v}} = -\nabla f(200,40) = (0.4,\ 2)$.

ii) The slope of the mountain at $(200,\ 40)$ in this direction is

$$\begin{aligned}
\frac{\partial f}{\partial \hat{\underset{\sim}{\mathbf{v}}}}(200,\ 40) &= \nabla f(200,\ 40) \bullet \hat{\underset{\sim}{\mathbf{v}}}\\
&= -\underset{\sim}{\mathbf{v}} \bullet \frac{\underset{\sim}{\mathbf{v}}}{||\underset{\sim}{\mathbf{v}}||}\\
&= -||\underset{\sim}{\mathbf{v}}|| = -\sqrt{(0.04)^2+2^2} = -\sqrt{4.16}.
\end{aligned}$$

e.



ii) $\displaystyle\int_0^9 \int_0^{\sqrt{x}} y\cos(x^2)\,dy\,dx$

2 a. i)

$$\begin{aligned}
J_{f(\mathbf{x}_0)}g &= (vw \quad uw \quad uv)_{u=x_0,\ v=x_0^2-y_0^2,\ w=y_0}\\
&= \left(y_0(x_0^2-y_0^2)\ x_0y_0\ x_0(x_0^2-y_0^2)\right)\\
&= (0\ 1\ 0) \quad \text{for} \quad \mathbf{x}_0 = (x_0,y_0) = (1,1)
\end{aligned}$$

$$J_{\mathbf{x}_0}f = \begin{pmatrix} 1 & 0\\ 2x_0 & -2y_0\\ 0 & 1\end{pmatrix} = \begin{pmatrix} 1 & 0\\ 2 & -2\\ 0 & 1\end{pmatrix}$$

$$\therefore \quad J_{f\mathbf{x}_0}g \cdot J_{\mathbf{x}_0}f = (0\ 1\ 0)\begin{pmatrix} 1 & 0\\ 2 & -2\\ 0 & 1\end{pmatrix} = (2\ -2)$$

ii) Hence $\nabla h(\mathbf{x}_0) = \nabla(g \circ f)(\mathbf{x}_0) = (2,\,-2)$.

b. i) Let $f(x,y,z) = x^2y^2z^2$ and $g(x,y,z) = x^2+y^2+z^2-r^2$. Hence we wish to maximise f subject to the constraint $g=0$. By the Lagrange Multiplier theorem, local extrema (i.e. local maxima and minima) for f on $g=0$ occur at critical points of $f-\lambda g$ for some constant $\lambda \in \mathbb{R}$. This leads to

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \lambda\frac{\partial g}{\partial x} & \therefore\quad 2xy^2z^2 &= \lambda 2x \\
\frac{\partial f}{\partial y} &= \lambda\frac{\partial g}{\partial y} & \therefore\quad yx^2z^2 &= \lambda 2y \\
\frac{\partial f}{\partial z} &= \lambda\frac{\partial y}{\partial z} & \therefore\quad 2zx^2y^2 &= \lambda 2z \\
&\text{and } x^2+y^2+z^2 = r^2
\end{aligned}$$

From the first three equations, $y^2z^2 = x^2z^2 = x^2y^2 = \lambda$ provided $x,\ y,\ z, \neq 0$. Hence, (with the same assumption) it follows

$$x = \pm y, \qquad z = \pm y$$

(with independent choice of signs). Substituting into $g=0$ gives $3y^2 = r^2$, hence

$$x = \pm\frac{r}{\sqrt{3}}, \qquad y = \pm\frac{r}{\sqrt{3}}, \qquad z = \pm\frac{r}{\sqrt{3}} \qquad\qquad (*)$$

(with independent choice of signs). They all have the same f value of

$$\left(\frac{r^2}{3}\right)^3 = \frac{r^6}{27}.$$

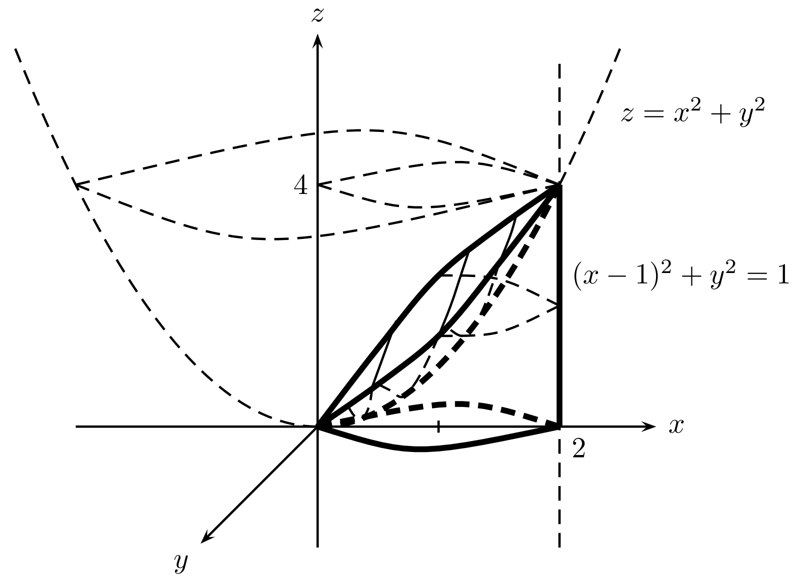
If any of x, y, z are 0 then f has the value 0 at such points. Hence the points $(*)$ are clearly (absolute) maxima points for f on $g=0$. It follows the absolute maxima points are

$$\left(\pm\frac{r}{\sqrt{3}}, \pm\frac{r}{\sqrt{3}}, \pm\frac{r}{\sqrt{3}}\right) \quad \text{with maximum value} \quad \frac{r^6}{27}.$$

ii) Given (x,y,z), let $r = \sqrt{x^2+y^2+z^2}$, then part i) shows,

$$\begin{aligned}
f(x,y,z) &\le \frac{r^6}{27} = \frac{(x^2+y^2+z^2)^3}{27} \\
\text{i.e.}\quad x^2y^2z^2 &\le \frac{(x^2+y^2+z^2)^3}{27} \\
\therefore\quad (x^2y^2z^2)^{1/3} &\le \frac{x^2+y^2+z^2}{3}
\end{aligned}$$

3 **a.** i)



ii)

- Cartesian: $\text{Vol} = \int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \int_0^{x^2+y^2} dz\,dy\,dx.$
- Cylindrical polar: $\text{Vol} = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_0^{r^2} r\,dz\,dr\,d\theta.$

  Since the paraboloid is $z = r^2$, and the cylinder is $r^2 = 2r\cos\theta$ or $r = 2\cos\theta$.

iii) Using cylindrical polar co–ordinates,

$$\begin{aligned}
\text{Vol} &= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_0^{r^2} r\,dz\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} rz\Big|_{z=0}^{z=r^2} dr\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^3\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{r^4}{4}\Big|_{r=0}^{r=2\cos\theta} d\theta \\
&= \frac{2^4}{4} \int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta = 4 \cdot 2 \int_0^{\pi/2} \cos^4\theta\,d\theta \\
&= 8 \cdot \left(\frac{4-1}{4+0}\right)\left(\frac{2-1}{2+0}\right) \cdot \frac{\pi}{2} = 8 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{2}.
\end{aligned}$$

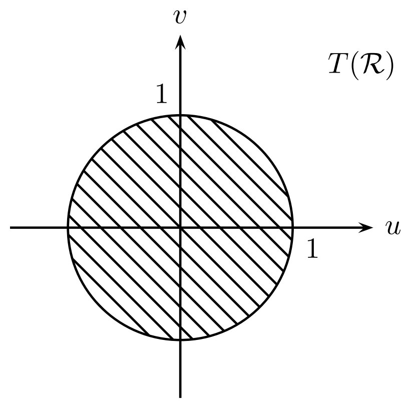
b. i) The transformation here $T : \mathbb{R}^2 \to \mathbb{R}^2$ is regarded as going from the xy–plane to the uv–plane, i.e. $(u,v) = T(x,y)$ so

$$(x,y) = T^{-1}(u,v) = (\sqrt{2}u - \sqrt{2/3}v,\ \sqrt{2}u + \sqrt{2/3}v).$$

Hence the region $\mathcal{R}$ in the xy–plane which is the region bounded by the ellipse $x^2 - xy + y^2 = 2$, corresponds to the region $T(\mathcal{R})$ in the uv–plane which is bounded by the curve:

$$\begin{aligned}
\left(\sqrt{2}u - \sqrt{\frac{2}{3}}v\right)^2 - \left(\sqrt{2}u - \sqrt{\frac{2}{3}}v\right)\left(\sqrt{2}u + \sqrt{\frac{2}{3}}v\right) + \left(\sqrt{2}u + \sqrt{\frac{2}{3}}v\right)^2 &= 2 \\
\therefore\ 2\left(2u^2 + \frac{2}{3}v^2\right) - \left(2u^2 - \frac{2}{3}v^2\right) &= 2 \\
\text{i.e.}\quad 2u^2 + 2v^2 &= 2 \\
\text{or}\quad u^2 + v^2 &= 1
\end{aligned}$$

So $T(\mathcal{R})$ is the unit disc in the uv–plane:



ii)

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \sqrt{2} & -\sqrt{\dfrac{2}{3}} \\ \sqrt{2} & \sqrt{\dfrac{2}{3}} \end{vmatrix} = \frac{4}{\sqrt{3}}$$

$$\begin{aligned} \therefore \qquad \iint_{\mathcal{R}} (x^2 - xy + y^2)\ dx\,dy &= \iint_{T(\mathcal{R})} (2u^2 + 2v^2) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \ du\,dv \\ &= 2 \cdot \frac{4}{\sqrt{3}} \iint_{T(\mathcal{R})} (u^2 + v^2)\ du\,dv \\ &= \frac{8}{\sqrt{3}} \int_0^{2\pi} \int_0^1 r^2 \cdot r\ dr\,d\theta \quad \text{(by polars in the } uv\text{–plane)} \\ &= \frac{8}{\sqrt{3}} \cdot 2\pi \cdot \frac{1}{4} = \frac{4\pi}{\sqrt{3}}\ . \end{aligned}$$

4 a.

$$\underset{\sim}{\mathbf{r}}(t) = \left(2\ln t,\ 2t,\ \frac{1}{2}t^2\right) \qquad \text{for } t = 1 \text{ to } t = 2.$$

$$\therefore \quad \underset{\sim}{\mathbf{r}}'(t) = \left(\frac{2}{t},\ 2,\ t\right)$$

$$\begin{aligned} \text{so} \quad \text{length} &= \int_1^2 \|\underset{\sim}{\mathbf{r}}'(t)\|\ dt = \int_1^2 \sqrt{\frac{4}{t^2} + 4 + t^2}\ dt \\ &= \int_1^2 \sqrt{\left(\frac{2}{t} + t\right)^2}\ dt = \int_1^2 \frac{2}{t} + t\ dt \\ &= 2\ln t + \frac{t^2}{2}\bigg|_1^2 = 2\ln 2 + 2 - 0 - \frac{1}{2} = \frac{3}{2} + 2\ln 2. \end{aligned}$$

b. i) To determine the point(s) of intersection, we solve $\underset{\sim}{\mathbf{r}}_1(t) = \underset{\sim}{\mathbf{r}}_2(s)$.

$$\therefore \qquad \left.\begin{aligned} t &= s \\ t &= s^2 \\ t &= s^3 \end{aligned}\right\} \implies \begin{array}{ccc} t = 0 & \text{and} & s = 0 \\ & \text{or} & \\ t = 1 & \text{and} & s = 1 \end{array}$$

Hence the two points of intersection are

1. $\underset{\sim}{\mathbf{r}}_1(0) = \underset{\sim}{\mathbf{r}}_2(0) = (0,\, 0,\, 0)$
2. $\underset{\sim}{\mathbf{r}}_1(1) = \underset{\sim}{\mathbf{r}}_2(1) = (1,\, 1,\, 1)$

Let the angles of intersection be θ_1 and θ_2 respectively. Note that

$$\underset{\sim}{\mathbf{r}}_1'(t) = (1,\, 1,\, 1) \quad \text{and} \quad \underset{\sim}{\mathbf{r}}_2'(s) = (1,\, 2s,\, 3s^2) \quad \text{for all} \quad s,\, t.$$

1. At $(0,\, 0,\, 0)$:

$$\cos\theta_1 = \frac{\underset{\sim}{\mathbf{r}}_1'(0) \bullet \underset{\sim}{\mathbf{r}}_2'(0)}{||\underset{\sim}{\mathbf{r}}_1'(0)||\, ||\underset{\sim}{\mathbf{r}}_2'(0)||} = \frac{(1,1,1) \bullet (1,0,0)}{||(1,1,1)||\, ||(1,0,0)||} = \frac{1}{\sqrt{3}\cdot 1} = \frac{1}{\sqrt{3}}$$

$$\text{so} \quad \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.9553 \text{ rads} \approx 54.735^\circ$$

2. At $(1,\, 1,\, 1)$:

$$\cos\theta_2 = \frac{\underset{\sim}{\mathbf{r}}_1'(1) \bullet \underset{\sim}{\mathbf{r}}_2'(1)}{||\underset{\sim}{\mathbf{r}}_1'(1)||\, ||\underset{\sim}{\mathbf{r}}_2'(1)||} = \frac{(1,1,1) \bullet (1,2,3)}{||(1,1,1)||\, ||(1,2,3)||} = \frac{6}{\sqrt{3}\,\sqrt{14}}$$

$$\text{so} \quad \theta_2 = \cos^{-1}\left(\frac{6}{\sqrt{3}\,\sqrt{14}}\right) \approx 0.3876 \text{ rads} \approx 22.208^\circ$$

ii) The condition is $\underset{\sim}{\mathbf{r}}_2'(s) = (1, 2s, 3s^2) = \alpha(1,0,0)$ for some $\alpha \in \mathbb{R}$. Hence $s = 0$, so the only such point is at $\underset{\sim}{\mathbf{r}}_2(0) = (0,0,0)$. The tangent line at this point is given parametrically by

$$\underset{\sim}{\mathbf{r}}(u) = (0,0,0) + u(1,0,0) = u(1,0,0).$$

c. i) Solving $\underset{\sim}{\mathbf{F}} = \nabla\phi$, gives

$$\frac{\partial\phi}{\partial x} = \frac{\alpha x}{(x^2+y^2+z^2)^{3/2}} \implies \phi = \frac{-\alpha}{(x^2+y^2+z^2)^{1/2}} + h_1(y,z) \qquad (1)$$

$$\frac{\partial\phi}{\partial y} = \frac{\alpha y}{(x^2+y^2+z^2)^{3/2}} \implies \phi = \frac{-\alpha}{(x^2+y^2+z^2)^{1/2}} + h_2(z,x) \qquad (2)$$

$$\frac{\partial\phi}{\partial z} = \frac{\alpha z}{(x^2+y^2+z^2)^{3/2}} \implies \phi = \frac{-\alpha}{(x^2+y^2+z^2)^{1/2}} + h_3(x,y) \qquad (3)$$

Equating these expressions for ϕ, it follows that

$$h_1(y,z) = h_2(z,x) = h_3(x,y) \qquad \text{for all } x,\ y,\ z \in \mathbb{R}.$$

The relation between h_1 and h_2 shows they depend on z alone, i.e.

$$h_1(y,z) = h_2(z,x) = g(z) \qquad \text{for all } z \in \mathbb{R}$$

for some function $g : \mathbb{R} \to \mathbb{R}$, but then

$$g(z) = h_3(x,y) \qquad \text{for all } x,\ y,\ z \in \mathbb{R}$$

shows h_3 and g, hence h_1 and h_2 are constant. Hence there is a solution to $\underset{\sim}{\mathbf{F}} = \nabla\phi$ which is unique up to a constant:

$$\phi = \frac{-\alpha}{(x^2+y^2+z^2)^{3/2}} + \text{constant}$$

Alternatively, substitute the solution for ϕ in (1) into the equation for $\dfrac{\partial\phi}{\partial y}$ to see that

$$\frac{\partial h_1}{\partial y}(y,z)=0 \quad \text{hence} \quad h_1(y,z)=g(z) \quad \text{for all } y,\ z\in\mathbb{R}.$$

Now substituting the revised expression for ϕ into the 3rd equation, shows that

$$g'(z)=0 \quad \text{for all } z\in\mathbb{R}, \text{ hence } g(z)=\text{constant}.$$

ii) Let the constant in ϕ be c. Let C be any curve on $x^2+y^2+z^2=1$ from one point $A(\underset{\sim}{\mathbf{a}})$ to another point $B(\underset{\sim}{\mathbf{b}})$. On the unit sphere $x^2+y^2+z^2=1$, $\phi=-\alpha+c$,

$$\begin{aligned}\therefore\quad \int_C \underset{\sim}{\mathbf{F}}\bullet d\underset{\sim}{\mathbf{r}} &= \int_C \nabla\phi\bullet d\underset{\sim}{\mathbf{r}}\\ &= \phi(\underset{\sim}{\mathbf{a}})-\phi(\underset{\sim}{\mathbf{b}})\\ &= (-\alpha+c)-(-\alpha+c)=0.\end{aligned}$$

d. By Green's Theorem, if R is the region bounded by the smooth simple closed curve C, and C is traversed in the positive sense,

$$\oint_C F_1\,dx+F_2\,dy=\iint_R\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\,dx\,dy.$$

Hence with $F_1(x,y)=e^x\sin y$ and $F_2(x,y)=e^x\cos y$,

$$\begin{aligned}\oint_C e^x\sin y\,dx+e^x\cos y\,dy &= \iint_R\left(\frac{\partial}{\partial x}(e^x\cos y)-\frac{\partial}{\partial y}(e^x\sin y)\right)\,dx\,dy\\ &= \iint_R (e^x\cos y-e^x\cos y)\,dx\,dy=\iint_R 0\,dx\,dy=0\end{aligned}$$

5 a. i)

$$\begin{aligned}\nabla\left(\frac{1}{r}\right) &= \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\underset{\sim}{\mathbf{i}}+\frac{\partial}{\partial y}\left(\frac{1}{r}\right)\underset{\sim}{\mathbf{j}}+\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\underset{\sim}{\mathbf{k}}\\ &= -\frac{x}{(x^2+y^2+z^2)^{3/2}}\underset{\sim}{\mathbf{i}}-\frac{y}{(x^2+y^2+z^2)^{3/2}}\underset{\sim}{\mathbf{j}}-\frac{z}{(x^2+y^2+z^2)^{3/2}}\underset{\sim}{\mathbf{k}}\\ &= -\frac{\underset{\sim}{\mathbf{r}}}{r^3}\end{aligned}$$

$$\nabla\bullet\underset{\sim}{\mathbf{r}}=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)=1+1+1=3$$

$$\begin{aligned}\nabla\times\underset{\sim}{\mathbf{r}} &= \begin{vmatrix}\underset{\sim}{\mathbf{i}} & \underset{\sim}{\mathbf{j}} & \underset{\sim}{\mathbf{k}}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x & y & z\end{vmatrix}\\ &= \left(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\right)\underset{\sim}{\mathbf{i}}-\left(\frac{\partial z}{\partial x}-\frac{\partial x}{\partial z}\right)\underset{\sim}{\mathbf{j}}+\left(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\right)\underset{\sim}{\mathbf{k}}=(0,\,0,\,0)=\underset{\sim}{\mathbf{0}}.\end{aligned}$$

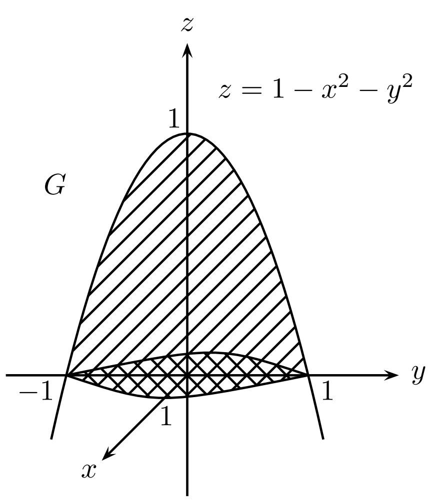
ii)

$$\begin{aligned}
\nabla \bullet (f(r)\underset{\sim}{\mathbf{r}}) &= \frac{\partial}{\partial x}(f(r)x) + \frac{\partial}{\partial y}(f(r)y) + \frac{\partial}{\partial z}(f(r)z) \\
&= f'(r) \cdot \frac{\partial r}{\partial x} \cdot x + f(r) \cdot 1 + f'(r) \cdot \frac{\partial r}{\partial y} \cdot y + f(r) \cdot 1 + f'(r) \cdot \frac{\partial r}{\partial z} \cdot z + f(r) \cdot 1 \\
&= f'(r) \cdot \frac{x}{r} \cdot x + f'(r) \cdot \frac{y}{r} \cdot y + f'(r) \cdot \frac{z}{r} \cdot z + 3f(r) \\
&= f'(r) \cdot \frac{x^2+y^2+z^2}{r} + 3f(r) = f'(r) \cdot \frac{r^2}{r} + 3f(r) \\
&= rf'(r) + 3f(r).
\end{aligned}$$

If $\nabla(f(r)\underset{\sim}{\mathbf{r}}) = 0$ for $r \neq 0$, then

$$\begin{aligned}
& rf'(r) + 3f(r) = 0 \\
\therefore \quad & \frac{f'(r)}{f(r)} = -\frac{3}{r} \\
\text{so} \quad & \ln f(r) = -3\ln r + c \qquad (c \text{ a constant}) \\
\text{or} \quad & \ln f(r) = \ln(r^{-3}) + c = \ln\left(\frac{A}{r^3}\right) \qquad (A \text{ a constant}) \\
\therefore \quad & f(r) = \frac{A}{r^3}.
\end{aligned}$$

b.



G is the solid region bounded by the surface S where

$$\begin{aligned}
S &= S_{\text{top}} \cup S_{\text{bot}} \\
S_{\text{top}} &= \{(x, y, 1-x^2-y^2) \mid x^2+y^2 \leq 1\} \\
S_{\text{bot}} &= \{(x, y, 0) \mid x^2+y^2 \leq 1\}
\end{aligned}$$

b. i)

$$\begin{aligned}
\iiint_G \nabla \bullet \underset{\sim}{\mathbf{v}}\; dx\,dy\,dz &= \iiint_G \operatorname{div}(\underset{\sim}{\mathbf{v}})\; dx\,dy\,dz \quad \text{with } \underset{\sim}{\mathbf{v}} = (x, y, z) \\
&= \iiint_G 3\; dx\,dy\,dz \quad \text{as } \operatorname{div}(\underset{\sim}{\mathbf{v}}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \\
&= 3\operatorname{vol}(G) = 3\int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\; dz\,dr\,d\theta \\
&= 3 \cdot 2\pi \int_0^1 r(1-r^2)\; dr = 6\pi\int_0^1 r - r^3\; dr = 6\pi\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^1 \\
&= \frac{6\pi}{4} = \frac{3\pi}{2}.
\end{aligned}$$

ii) We will use polar/cylindrical co-ords to parameterise S_{top} and S_{bot}:

$$\begin{aligned} S_{\text{top}}: &\quad \underset{\sim}{\mathbf{r}}(r,\theta) = (r\cos\theta, r\sin\theta, 1-r^2), &\quad 0 \le \theta \le 2\pi,\ 0 \le r \le 1 \\ S_{\text{bot}}: &\quad \underset{\sim}{\mathbf{r}}(r,\theta) = (r\cos\theta, r\sin\theta, 0), &\quad 0 \le \theta \le 2\pi,\ 0 \le r \le 1 \end{aligned}$$

We could use the cartesian co-ordinates x and y as the parameters, but one of the the later integrals will simplify with a polar/cylindrical change of variables.

iii) The direct calculation of the flux is

$$\iint_S \underset{\sim}{\mathbf{v}} \bullet d\underset{\sim}{\mathbf{S}} = \iint_{S_{\text{top}}} \underset{\sim}{\mathbf{v}} \bullet d\underset{\sim}{\mathbf{S}} + \iint_{S_{\text{bot}}} \underset{\sim}{\mathbf{v}} \bullet d\underset{\sim}{\mathbf{S}}$$

(using outward normals).
For the surface integral on S_{bot}, $d\underset{\sim}{\mathbf{S}} = -\underset{\sim}{\mathbf{k}}\, r\ dr\, d\theta$, hence

$$\begin{aligned} \iint_{S_{\text{bot}}} \underset{\sim}{\mathbf{v}} \bullet d\underset{\sim}{\mathbf{S}} &= -\int_0^{2\pi}\int_0^1 zr\ dr\, d\theta \qquad \text{where } z = 0 \\ &= 0 \end{aligned}$$

For the surface integral on S_{top},

$$\begin{aligned} d\underset{\sim}{\mathbf{S}} &= \pm\left(\frac{\partial \underset{\sim}{\mathbf{r}}}{\partial r} \times \frac{\partial \underset{\sim}{\mathbf{r}}}{\partial \theta}\right) dr\, d\theta \\ &= \pm(\cos\theta, \sin\theta, -2r) \times (-r\sin\theta, r\cos\theta, 0)\ dr\, d\theta \\ &= \pm\begin{vmatrix} \underset{\sim}{\mathbf{i}} & \underset{\sim}{\mathbf{j}} & \underset{\sim}{\mathbf{k}} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} dr\, d\theta = \pm(2r^2\cos\theta, 2r^2\sin\theta, r)\ dr\, d\theta \end{aligned}$$

and we choose the + sign for the outward normal, hence

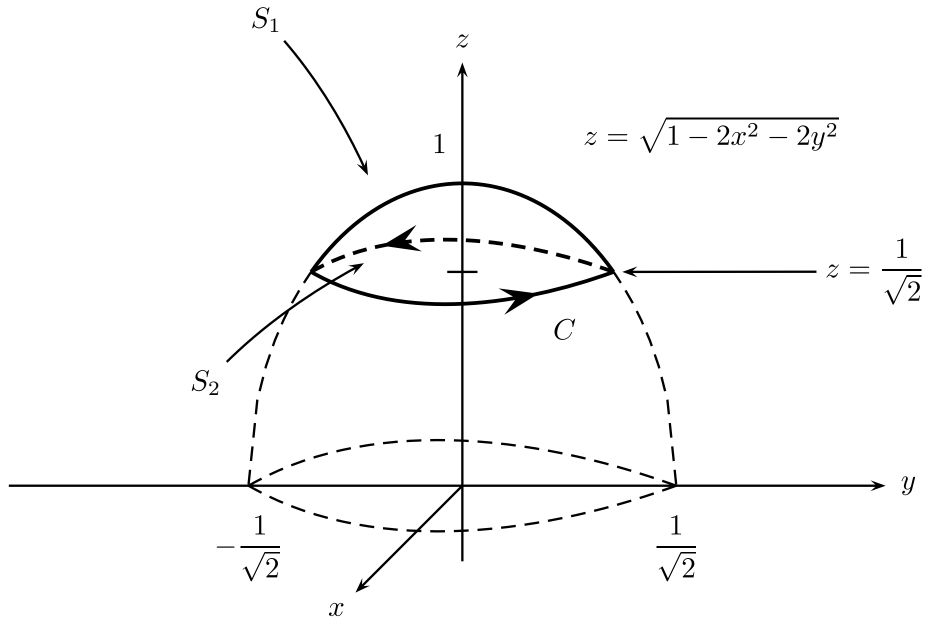
$$\begin{aligned} \iint_{S_{\text{top}}} \underset{\sim}{\mathbf{v}} \bullet d\underset{\sim}{\mathbf{S}} &= \int_0^{2\pi}\int_0^1 (r\cos\theta, r\sin\theta, 1-r^2) \bullet (2r^2\cos\theta, 2r^2\sin\theta, r)\ dr\, d\theta \\ &= \int_0^{2\pi}\int_0^1 2r^3\cos^2\theta + 2r^3\sin^2\theta + r - r^3\ dr\, d\theta \\ &= \int_0^{2\pi}\int_0^1 2r^3 + r - r^3\ dr\, d\theta = \int_0^{2\pi}\int_0^1 r^3 + r\ dr\, d\theta \\ &= 2\pi\left(\frac{r^4}{4} + \frac{r^2}{2}\right)\Big|_0^1 = 2\pi \cdot \frac{3}{4} = \frac{3\pi}{2} \end{aligned}$$

Hence

$$\iint_S \underset{\sim}{\mathbf{v}} \bullet d\underset{\sim}{\mathbf{S}} = \frac{3\pi}{2} + 0 = \frac{3\pi}{2}.$$

iv) Gauss's Divergence Theorem.

6 a.



$$S_1 = \left\{ (x, y, z) \mid x^2 + y^2 \le \frac{1}{4},\ \frac{1}{\sqrt{2}} \le z \le \sqrt{1 - 2x^2 - 2y^2} \right\}$$

$$S_2 = \left\{ \left(x, y, \frac{1}{\sqrt{2}} \right) \mid x^2 + y^2 \le \frac{1}{4} \right\}$$

$$C: \quad \underset{\sim}{\mathbf{r}}(t) = \left(\frac{\cos t}{2}, \frac{\sin t}{2}, \frac{1}{\sqrt{2}} \right) \qquad t = 0 \quad \text{to} \quad t = 2\pi.$$

Note: C is positively oriented with respect to the orientation for S_1 where the normal vectors to S_1 have z–component ≥ 0.
C is also positively oriented with respect to the orientation for S_2 where the normal vectors to S_2 are up (indeed in the direction of $\underset{\sim}{\mathbf{k}}$).

i)

$$\nabla \times \underset{\sim}{\mathbf{F}} = \begin{vmatrix} \underset{\sim}{\mathbf{i}} & \underset{\sim}{\mathbf{j}} & \underset{\sim}{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -3y & 3x & z^4 \end{vmatrix} = (0,\ 0,\ 6) = 6\underset{\sim}{\mathbf{k}}$$

ii) Stokes' Theorem applied to S_1 and C

$$\iint_{S_1} (\nabla \times \underset{\sim}{\mathbf{F}}) \bullet \underset{\sim}{\mathbf{n}}\, dS = \oint_C \underset{\sim}{\mathbf{F}} \bullet d\underset{\sim}{\mathbf{r}}$$

with unit normal vectors $\underset{\sim}{\mathbf{n}}$ to the surface having non-negative z–component so that the direction of C (shown in the diagram) is positively oriented w.r.t. S_1.

iii) Stokes' Theorem applied to S_2 and C

$$\iint_{S_2} (\nabla \times \underset{\sim}{\mathbf{F}}) \bullet \underset{\sim}{\mathbf{n}}\, dS = \oint_C \underset{\sim}{\mathbf{F}} \bullet d\underset{\sim}{\mathbf{r}}$$

with unit normal vectors $\underset{\sim}{\mathbf{n}} = \underset{\sim}{\mathbf{k}}$ at all points of S_2 so that the direction of C (shown in the diagram) is positively oriented w.r.t. S_2.

iv) Hence, by ii) and iii) the harder surface integral over S_2 can be evaluated as the easier surface integral over S_1:

$$\begin{aligned}\iint_{S_1} (\nabla \times \underset{\sim}{\mathbf{F}}) \bullet \underset{\sim}{\mathbf{n}}\, dS &= \iint_{S_2} (\nabla \times \underset{\sim}{\mathbf{F}}) \bullet \underset{\sim}{\mathbf{n}}\, dS \\ &= \iint_{S_2} (\nabla \times \underset{\sim}{\mathbf{F}}) \bullet \underset{\sim}{\mathbf{k}}\, dS \\ &= \iint_{S_2} 6\underset{\sim}{\mathbf{k}} \bullet \underset{\sim}{\mathbf{k}}\, dS \\ &= 6 \iint_{S_2} dS = 6 \times \text{ area of } S_2\end{aligned}$$

At $z = \dfrac{1}{\sqrt{2}} = \sqrt{1 - 2x^2 - 2y^2}$. Therefore

$$\frac{1}{2} = 1 - 2x^2 - 2y^2 \Longrightarrow x^2 + y^2 = \frac{1}{4}.$$

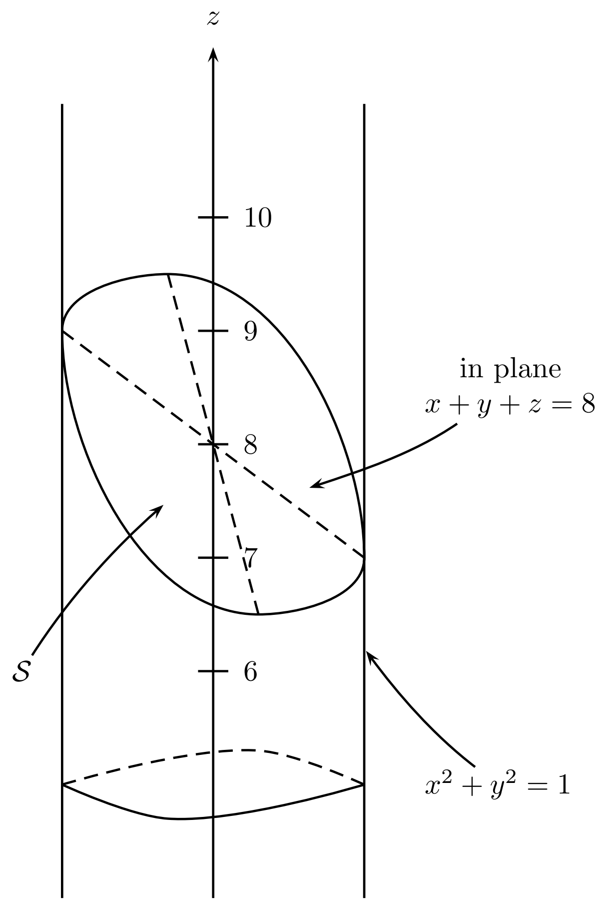
Hence S_2 is a disc of radius $\dfrac{1}{2}$ and so has area $\dfrac{\pi}{4}$. Thus

$$\iint_{S_1} (\nabla \times \underset{\sim}{\mathbf{F}}) \bullet \underset{\sim}{\mathbf{n}}\, dS = 6 \cdot \frac{\pi}{4} = \frac{3\pi}{2}.$$

Less simple is a direct calculation of the line integral in Stokes' theorem – see the parameterisation of C above under the diagram.

$$\begin{aligned}\therefore \qquad \iint_{S_1} (\nabla \times \underset{\sim}{\mathbf{F}}) \bullet \underset{\sim}{\mathbf{n}}\, dS &= \oint_C \underset{\sim}{\mathbf{F}} \bullet d\underset{\sim}{\mathbf{r}} \\ &= \int_0^{2\pi} \left(-\frac{3\sin t}{2}, \frac{\cos t}{2}, \frac{1}{4}\right) \bullet \left(-\frac{\sin t}{2}, \frac{\cos t}{2}, 0\right)\, dt \\ &= \int_0^{2\pi} -\frac{3\sin^2 t}{4} + \frac{\cos^2 t}{4}\, dt \\ &= \int_0^{2\pi} -\frac{3}{4} + \cos^2 t\, dt \\ &= \int_0^{2\pi} -\frac{3}{4} + \cos^2 t\, dt \\ &= -\frac{3}{4} \cdot 2\pi + 4 \cdot \left(\frac{2-1}{2+0}\right) \cdot \frac{\pi}{2} \\ &= -\frac{3\pi}{2} + 4\pi = \frac{3\pi}{2}.\end{aligned}$$

b.



Call the surface region $\mathcal{S}$. A parameterisation of $\mathcal{S}$, using the x and y co–ordinates as parameters is:

$$\mathcal{S}: \qquad \underset{\sim}{\mathbf{r}}(x, y) = (x,\ y,\ 8 - x - y) \qquad \text{for} \quad x^2 + y^2 \leq 1$$

$$\text{Hence} \quad dS = \left|\left| \frac{\partial \underset{\sim}{\mathbf{r}}}{\partial x} \times \frac{\partial \underset{\sim}{\mathbf{r}}}{\partial y} \right|\right| \ dx\, dy$$

$$= \left|\left| \begin{vmatrix} \underset{\sim}{\mathbf{i}} & \underset{\sim}{\mathbf{j}} & \underset{\sim}{\mathbf{k}} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} \right|\right| \ dx\, dy \qquad \text{where } f(x, y) = 8 - x - y$$

$$= ||(-f_x, f_y, 1)|| \ dx\, dy$$

$$= \sqrt{1 + f_x^2 + f_y^2} \ dx\, dy = \sqrt{1 + 1 + 1} \ dx\, dy$$

$$= \sqrt{3}\, dx\, dy$$

$$\therefore \quad \text{area}(\mathcal{S}) = \iint_{\mathcal{S}} dS = \iint_D \sqrt{3} \ dx\, dy \qquad \text{where } D = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

$$= \sqrt{3} \cdot \text{area}(D) = \sqrt{3} \cdot \pi \cdot 1^2 = \pi\sqrt{3}.$$

Math2011 Several Variable Calculus — November 2001.

1 **a.** Consider the surface defined by

$$\ln(2xy) - z^3 = 8.$$

i) Show that the point $(-1, -\frac{1}{2}, -2)$ lies on this surface.

ii) Find a unit normal vector to the surface at the point $(-1, -\frac{1}{2}, -2)$.

b. Consider the function

$$f(x, y) = (2y - 1)^5 + (x + 4y)^2 - x + 2.$$

i) Find the Taylor series expansion of $f(x, y)$ about the point $(1, \frac{1}{2})$ up to and including all second order terms.

ii) From the expansion in part (i), find the linear function that gives the best approximation of f near $(1, \frac{1}{2})$.

c. Suppose $f(x, y) = \ln(x - y)$ and let $\mathbf{u} = (\cos\theta, \sin\theta)$.

i) Find the directional derivative $f'_{\mathbf{u}}$ at the point $(2, 1)$.

ii) At the point $(2, 1)$ in what direction is f increasing most rapidly?

d. Evaluate the double integral

$$\int_0^1 \int_{2y}^2 \cosh(x^2)\, dx\, dy$$

by first reversing the order of integration.

2 **a.** Locate and classify the critical points of the function

$$f(x, y) = x^2 + y^3 + 2xy - x - y + 1.$$

b. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = x^2 - y^2$$

and the function $g : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$g\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} uv\cos w \\ uv\sin w \end{pmatrix}.$$

Let $h = f \circ g$ be the composite function.

i) Calculate $J_{g(\mathbf{u}_0)}f.J_{\mathbf{u}_0}g$ when $\mathbf{u}_0 = \begin{pmatrix} 1 \\ 1 \\ \pi \end{pmatrix}$.

ii) Hence determine ∇h at $(1, 1, \pi)$.

c. A rectangle lying in the xy-plane has vertices at $(0, 0)$, $(x, 0)$, $(0, y)$ and (x, y) where $x, y > 0$. Use the method of Lagrange multipliers to find the maximum area of the rectangle given that the vertex (x, y) lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are positive constants.

3 **a.** The solid S is bounded below by the cone $z = \sqrt{3x^2 + 3y^2}$ and above by the sphere $x^2+y^2+z^2 = 4$.

i) Sketch the solid S and express the volume V of S as an iterated triple integral using:

α) Cartesian coordinates
β) cylindrical coordinates
γ) spherical coordinates

iii) Use any one of the above integrals to evaluate the volume V.

b. Evaluate the area between the curves

$$(x+y)^2 = 4(x-2y) \quad \text{and} \quad x - 2y = 1$$

by making the substitutions $u = x + y$ and $v = x - 2y$.

4 **a.** Let $\underset{\sim}{\mathbf{F}} = (2xy^3 + 1, 3x^2y^2 - 2y)$. Find a scalar potential φ such that $\underset{\sim}{\mathbf{F}} = \operatorname{grad}\varphi$. Hence find the work done by $\underset{\sim}{\mathbf{F}}$ in moving an object from $(1, -2)$ to $(3, 1)$.

b. Use Green's theorem to compute the line integral

$$\oint_C (x+y)dx + x^2 dy,$$

where C is the counterclockwise-oriented triangle with vertices $(0,0)$, $(2,0)$ and $(0,1)$.

c. A particle moves in a plane such that its position at time t is given by $\underset{\sim}{\mathbf{r}}(t) = (3t^2, t^3 - 9t)$. Show that there are no positions where the velocity of the particle is parallel to its acceleration.

d. Show that the three vectors $\underset{\sim}{\mathbf{u}} = (1,1,1)$, $\underset{\sim}{\mathbf{v}} = (2,3,1)$ and $\underset{\sim}{\mathbf{w}} = (-1,5,-1)$ define a parallelepiped. Find its volume.

5 **a.** Find the area of the portion of the surface $z = 1 - \frac{2}{3}y^{3/2}$ that lies above the triangle in the xy-plane with vertices $(0,0)$, $(1,0)$ and $(0,1)$.

b. Let

$$\underset{\sim}{\mathbf{F}} = (x^2 + yz, -2y(x+z), xy + z^2)$$

$$\underset{\sim}{\mathbf{G}} = (-yz^2 - \frac{1}{2}xy^2, 0, x^2y + \frac{1}{2}y^2z)$$

i) Show that $\underset{\sim}{\mathbf{F}}$ is solenoidal (i.e., div $\underset{\sim}{\mathbf{F}} = 0$).

ii) Show that curl $\underset{\sim}{\mathbf{G}} = \underset{\sim}{\mathbf{F}}$.

c. Find the flux $\iint_S \underset{\sim}{\mathbf{F}}.\underset{\sim}{\mathbf{n}}\, ds$ out of the unit box $S = [0,1] \times [0,1] \times [0,1]$ of the vector field $\underset{\sim}{\mathbf{F}} = (y - x, x - xz, xy - z)$.

6 **a.** Let $\underset{\sim}{\mathbf{F}}(x,y,z) = (yz, -xz, xy)$ and let C be the triangle with vertices $(1,0,0)$, $(0,2,0)$ and $(0,0,3)$ oriented by the order in which the points are given.

i) Compute $\nabla \times \underset{\sim}{\mathbf{F}}$.

ii) Let S be the plane of the triangle C. Show that S is given by

$$z = 3 - 3x - \frac{3}{2}y.$$

iii) Parametrize the surface S.

iv) Parametrize C.

v) Verify Stokes' theorem in this case by calculating $\iint_S \nabla \times \underset{\sim}{\mathbf{F}} \cdot \underset{\sim}{\mathbf{n}}\, ds$ in two different ways.

b. State the Divergence theorem.

Solutions to Math2011 November 2001 exam

1 **a.** i) Point $(-1,\ -\frac{1}{2},\ -2)$ lies on the surface $\ln(2xy) - z^3 = 8$ since

$$\ln\left(2\cdot(-1)\cdot\left(-\frac{1}{2}\right)\right) - (-2)^3 = \ln(1) + 8 = 8.$$

ii) Let $F(x, y, z) = \ln(2xy) - z^3$, then

$$\therefore \quad \nabla F(x, y, z) = \left(\frac{2y}{2xy},\ \frac{2x}{2xy},\ 3z^2\right) = \left(\frac{1}{x},\ \frac{1}{y},\ -3z^2\right).$$

Hence a normal vector to the surface at the point $(-1,\ -\frac{1}{2},\ -2)$ is

$$\begin{aligned} \nabla F(-1,\ -\tfrac{1}{2},\ -2) &= (-1,\ -2,\ -12) \\ \therefore \quad \text{a unit normal is} \quad &= \frac{1}{\sqrt{149}}(-1,\ -2,\ -12). \end{aligned}$$

b. i) Method 1:

$$\begin{aligned} & & f(1, \tfrac{1}{2}) &= 0 + (1+2)^2 - 1 + 2 = 10 \\ f_x(x, y) &= 2(x + 4y)\cdot 1 - 1 & \therefore\ f_x(1, \tfrac{1}{2}) &= 2\cdot(1+2) - 1 = 5 \\ f_y(x, y) &= 5\cdot(x+4y)^4\cdot 2 + 2\cdot(x+4y)\cdot 4 & \therefore\ f_y(1, \tfrac{1}{2}) &= 10\cdot 0 + 8(1+2) = 24 \\ f_{xx}(x, y) &= 2 & \therefore\ f_{xx}(1, \tfrac{1}{2}) &= 2 \\ f_{xy}(x, y) &= 8 & \therefore\ f_{xx}(1, \tfrac{1}{2}) &= 8 \\ f_{yy}(x, y) &= 32 & \therefore\ f_{xx}(1, \tfrac{1}{2}) &= 32 \end{aligned}$$

$$\begin{aligned} \therefore \quad f(x, y) &= 10 + \frac{5(x-1) + 24(y - \frac{1}{2})}{1!} + \\ &\qquad \frac{1}{2!}\left[2(x-1)^2 + 2\cdot 8(x-1)(y-\tfrac{1}{2}) + 32(y-\tfrac{1}{2})^2\right] + \cdots \\ &= 10 + 5(x-1) + 24(y-\tfrac{1}{2}) + (x-1)^2 + 8(x-1)(y-\tfrac{1}{2}) + 16(y-\tfrac{1}{2})^2 + \cdots \end{aligned}$$

Method 2:

$$\begin{aligned} &\text{Let} \quad u = x - 1, \quad v = y - \tfrac{1}{2} \\ \therefore \quad & 2y - 1 = 2v, \quad \text{and} \quad x + 4y = (u+1) + 4(v + \tfrac{1}{2}) = u + 4v + 3 \\ \text{and} \quad & -x + 2 = -(u+1) + 2 = -u + 1 \\ \therefore \quad & f(x, y) = (2v)^5 + (u + 4v + 3)^2 - u + 1 \\ &\qquad = 1 - u + 9 + 6u + 24v + u^2 + 8uv + 16v^2 + 32v^5 \\ &\qquad = 10 + 5u + 24v + u^2 + 8uv + 16v^2 + 32v^5 \\ &\qquad = 10 + 5(x-1) + 24(y-\tfrac{1}{2}) + (x-1)^2 \\ &\qquad\qquad + 8(x-1)(y-\tfrac{1}{2}) + 16(y-\tfrac{1}{2})^2 + \cdots \quad . \end{aligned}$$

ii) The best "linear" (strictly speaking "affine") approximation of f near $(1, \frac{1}{2})$ is given by the function

$$A(x, y) = 10 + 5(x-1) + 24(y - \tfrac{1}{2}).$$

c. i)

$$f'_{\mathbf{u}} = \frac{\partial f}{\partial \mathbf{u}}(2,1) = \nabla f(2,1) \bullet \mathbf{u}$$

$$\text{Now} \quad \nabla f(x,y) = \left(\frac{1}{x-y}, \frac{-1}{x-y}\right) \quad \therefore \quad \nabla f(2,1) = (1,\,-1)$$

$$\text{so} \quad f'_{\mathbf{u}} = (1,\,-1) \bullet (\cos\theta,\, \sin\theta) = \cos\theta - \sin\theta.$$

ii) f is increasing most rapidly at $(2,1)$ in the direction of $\nabla f(2,1) = (1,\,-1)$, using a result about gradient. Alternatively, one could maximise, as a function of θ,

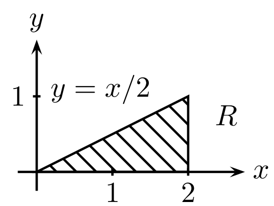
$$f'_{\mathbf{u}} = \cos\theta - \sin\theta = \sqrt{2}\,\sin\left(\tfrac{\pi}{4} - \theta\right)$$

$$\therefore \quad \text{max. at} \quad \frac{\pi}{4} - \theta = \frac{\pi}{2} + 2k\pi, \qquad k \in \mathbb{Z}$$

$$\therefore \quad \text{when} \quad \theta = -\frac{\pi}{4} + 2j\pi, \qquad j \in \mathbb{Z}$$

i.e. in the direction of $(1,\,-1)$.

d. Let $I = \int_0^1 \int_{2y}^2 \cosh(x^2)\, dx\, dy = \iint_R \cosh(x^2)\, dx\, dy$, where R is the region of integration:



$$\therefore \quad I = \int_0^2 \int_0^{x/2} \cosh(x^2)\, dy\, dx = \int_0^2 \frac{x}{2} \cosh(x^2)\, dx$$

$$= \left.\frac{\sinh(x^2)}{4}\right|_{x=0}^{x=2} = \frac{\sinh(4)}{4}.$$

2 **a.** For $f(x,y) = x^2 + y^2 + 2xy - x - y + 1$, critical points are where $\nabla f(x,y) = \underset{\sim}{\mathbf{0}}$, i.e. where

$$\frac{\partial f}{\partial x} = 2x + 2y - 1 = 0 \tag{1}$$

$$\frac{\partial f}{\partial y} = 3y^2 + 2x - 1 = 0 \tag{2}$$

Subtracting (1) from (2) gives

$$3y^2 - 2y = y(3y-2) = 0$$

and so $y = 0$ or $y = 2/3$. From (1), $x = 1/2 - y$ and so the critical points are

$$\left(\tfrac{1}{2},\, 0\right) \qquad \text{and} \qquad \left(-\tfrac{1}{6},\, \tfrac{2}{3}\right).$$

	$\left(\tfrac{1}{2},\, 0\right)$	$\left(-\tfrac{1}{6},\, \tfrac{2}{3}\right)$
$A = f_{xx} = 2$	2	2
$B = f_{xy} = 2$	2	2
$C = f_{yy} = 6y$	0	4
$D = B^2 - AC$	4	-4

Since $D > 0$, the point $\left(\tfrac{1}{2},\, 0\right)$ is a saddle point.
Since $D < 0$, and $A > 0$ the point $\left(-\tfrac{1}{6},\, \tfrac{2}{3}\right)$ is a local minimum point.

b. i)

$$Jf = \begin{pmatrix} 2x & -2y \end{pmatrix}$$

$$Jg = \begin{pmatrix} v\cos w & u\cos w & -uv\sin w \\ v\sin w & u\sin w & uv\cos w \end{pmatrix}$$

$$g(\mathbf{u}_0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad \text{and so} \qquad J_{g(\mathbf{u}_0)}f = \begin{pmatrix} -2 & 0 \end{pmatrix}$$

$$J_{\mathbf{u}_0}g = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

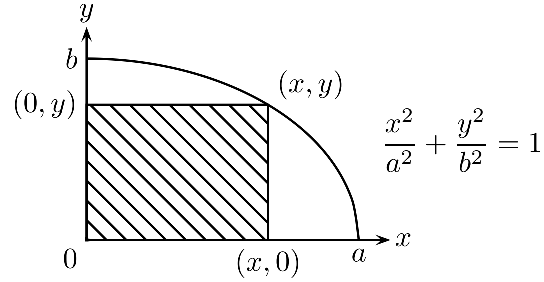
$$\text{Thus} \qquad J_{g(\mathbf{u}_0)}f\, J_{\mathbf{u}_0}g = \begin{pmatrix} -2 & 0 \end{pmatrix}\begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \end{pmatrix}$$

ii) Now $J_{\mathbf{u}_0}h = J_{g(\mathbf{u}_0)}f\, J_{\mathbf{u}_0}g = \begin{pmatrix} -2 & 0 \end{pmatrix}$ but $h = f \circ g$ is a mapping from $\mathbb{R}^3 \to \mathbb{R}$ and so

$$J_{\mathbf{u}_0}h = \nabla h(\mathbf{u}_0)$$

(identifying a 1×3 matrix to a row vector), hence $\nabla h(\mathbf{u}_0) = (2, 2, 0)$.

c.



The area of the rectangle is $A = xy$, thus we must maximise A subject to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad x > 0,\ y > 0.$$

$$\text{Let} \qquad L(x, y; \lambda) = xy - \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

$$\frac{\partial L}{\partial x} = y - \lambda\left(\frac{2x}{a^2}\right) = 0 \tag{1}$$

$$\frac{\partial L}{\partial y} = x - \lambda\left(\frac{2y}{b^2}\right) = 0 \tag{2}$$

$$\frac{\partial L}{\partial \lambda} = -\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = 0$$

$$\text{By (1) and (2),} \qquad 2\lambda = \frac{a^2 y}{x} = \frac{b^2 x}{y} \quad \therefore \quad a^2y^2 = b^2x^2$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{2x^2}{a^2} = 1$$

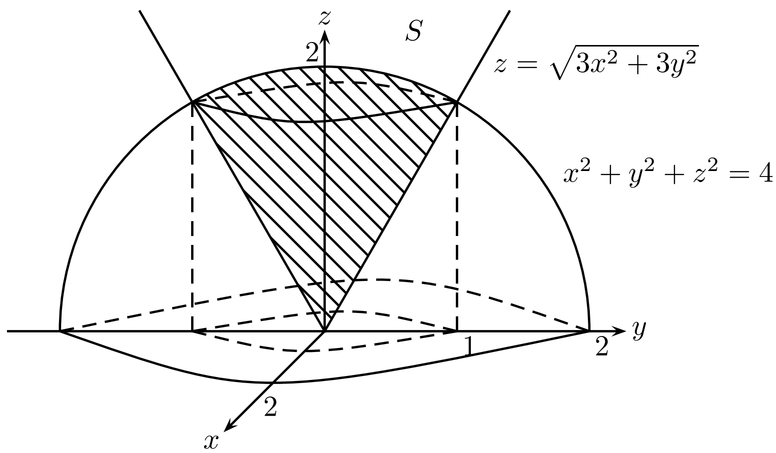
$$\therefore \quad x = \frac{a}{\sqrt{2}} \qquad (\text{since } x,\ a > 0)$$

$$\text{Also} \qquad y^2 = \frac{b^2x^2}{a^2} = \frac{b^2}{2}$$

$$\therefore \quad y = \frac{b}{\sqrt{2}} \qquad \text{(since } y,\, b > 0\text{)}$$

$$\text{So max. area} \quad = \frac{ab}{2} \qquad \text{when } x = \frac{a}{\sqrt{2}},\ y = \frac{b}{\sqrt{2}}.$$

3 **a.** i)



Note that the cone and sphere intersect in the circle $x^2+y^2=1,\ z=\sqrt{3}$ and the projection of S onto the xy–plane is the disk $x^2+y^2 \le 1$.

$$\alpha) \qquad V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{3x^2+3y^2}}^{\sqrt{4-x^2-y^2}} 1\ dz\,dy\,dx$$

$$\beta) \qquad V = \int_{0}^{2\pi}\int_{0}^{1}\int_{\sqrt{3}\,r}^{\sqrt{4-r^2}} r\ dz\,dr\,d\theta$$

$$\gamma) \qquad V = \int_{0}^{2\pi}\int_{0}^{\pi/6}\int_{0}^{2} \rho^2\sin\phi\ d\rho\,d\phi\,d\theta$$

ii) Using cylindrical co–ordinates,

$$\begin{aligned} V &= \int_{0}^{2\pi}\int_{0}^{1}\int_{\sqrt{3}\,r}^{\sqrt{4-r^2}} r\ dz\,dr\,d\theta = \int_{0}^{2\pi}\int_{0}^{1}\left(r\sqrt{4-r^2}-\sqrt{3}\,r^2\right)\,dr\,d\theta \\ &= \int_{0}^{2\pi}\left[-\frac{1}{3}(4-r^2)^{3/2}-\frac{\sqrt{3}\,r^3}{3}\right]\Bigg|_0^1 d\theta = 2\pi\left(-\frac{1}{3}\cdot 3^{3/2}-\frac{\sqrt{3}}{3}+\frac{8}{3}\right) \\ &= \frac{8\pi}{3}\,(2-\sqrt{3}) \end{aligned}$$

or, using spherical co–ordinates,

$$\begin{aligned} V &= \int_{0}^{2\pi}\int_{0}^{\pi/6}\int_{0}^{2} \rho^2\sin\phi\ d\rho\,d\phi\,d\theta = \frac{8}{3}\int_{0}^{2\pi}\int_{0}^{\pi/6}\sin\phi\ d\phi\,d\theta \\ &= \frac{8}{3}\cdot 2\pi\,[-\cos\phi]\big|_0^{\pi/6} = \frac{8}{3}\cdot 2\pi\left(1-\frac{\sqrt{3}}{2}\right) \\ &= \frac{8\pi}{3}\,(2-\sqrt{3}) \end{aligned}$$

b. Given $u = x + y,\ v = x - 2y$,

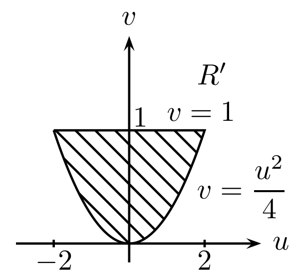
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -3$$

So if R is the region between the curves $(x+y)^2 = 4(x-2y)$ and $x - 2y = 1$ in the xy–plane, then

$$\text{Area}(R) = \iint_R 1\ dx\,dy = \iint_{R'} 1 \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\,dv$$

where R' is the region between the curves $u^2 = 4v$ and $v = 1$ in the uv–plane.

$$\begin{aligned} \therefore \quad \text{Area}(R) &= \iint_{R'} 1 \cdot \left|-\frac{1}{3}\right|\ du\,dv \\ &= \frac{1}{3}\int_{-2}^{2}\int_{u^2/4}^{1} dv\,du \\ &= \frac{1}{3}\int_{-2}^{2}\left(1 - \frac{u^2}{4}\right)\ du = \frac{1}{3}\left[u - \frac{u^3}{12}\right]\Bigg|_{-2}^{2} \\ &= \frac{1}{3}\cdot 2\left(2 - \frac{8}{12}\right) = \frac{2}{3}\cdot\frac{4}{3} = \frac{8}{9} \end{aligned}$$



4 a.
$\underset{\sim}{\mathbf{F}} = (2xy^3 + 1,\ 3x^2y^2 - 2y)$. Let $\underset{\sim}{\mathbf{F}} = \text{grad}\,\phi$ so

$$\frac{\partial\phi}{\partial x} = 2xy^3 + 1 \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = 3x^2y^2 - 2y$$

$$\begin{aligned} \frac{\partial\phi}{\partial x} &= 2xy^3 + 1 \Rightarrow \phi(x,y) = x^2y^3 + x + f(y) \\ \text{so} \quad \frac{\partial\phi}{\partial y} &= 3x^2y^2 + f'(y) = 3x^2y^2 - 2y \\ \therefore \quad f'(y) &= -2y \Rightarrow f(y) = -y^2 + \text{constant} \end{aligned}$$

Thus a scalar potential function ϕ for $\underset{\sim}{\mathbf{F}}$ is

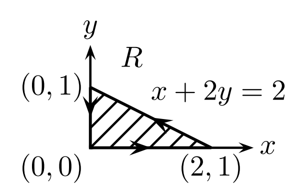
$$\phi(x,y) = x^2y^3 + x - y^2.$$

Since $\underset{\sim}{\mathbf{F}} = \text{grad}\,\phi$, the line integral is independent of path and the work done by $\underset{\sim}{\mathbf{F}}$ (as a force on an object) in moving that object along any path P from $(1,-2)$ to $(3,1)$ is

$$\int_P \underset{\sim}{\mathbf{F}} \bullet d\underset{\sim}{\mathbf{r}} = \phi(3,1) - \phi(1,-2) = (9 + 3 - 1) - (-8 + 1 - 4) = 22.$$

b.

$$\begin{aligned} &\oint_C (x+y)\ dx + x^2\ dy \\ &= \iint_R \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(x+y)\right)\ dx\,dy \\ &= \iint_R (2x - 1)\ dx\,dy = \int_0^1\int_0^{2-2y}(2x-1)\ dx\,dy \\ &= \int_0^1 \left[x^2 - x\right]\Big|_0^{2-2y}\ dy = \int_0^1 \left((2-2y)^2 - (2-2y)\right)\ dy \\ &= \left[-\frac{4}{3}(1-y)^3 - 2y + y^2\right]\Bigg|_0^1 = -2 + 1 + \frac{4}{3} = \frac{1}{3}. \end{aligned}$$



c. $\mathbf{r}(t) = (3t^2,\, t^3 - 9t)$ hence

$$\text{velocity} \qquad \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = (6t,\, 3t^2 - 9)$$

$$\text{acceleration} \qquad \mathbf{a}(t) = \frac{d^2\mathbf{r}}{dt^2} = (6,\, 6t)$$

The velocity is parallel to the acceleration (at some time t) if there is a scaler λ and some t such that $\mathbf{v}(t) = \lambda \mathbf{a}(t)$.

$$\therefore \qquad 3t^2 - 9 = 6\lambda t = 6t^2 \qquad \text{hence} \qquad 3t^2 = -9$$

which has no real solutions in t. Hence the velocity and the acceleration are never parallel.

d. The three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ form a parallelopiped if $\mathbf{u} \bullet \mathbf{v} \times \mathbf{w} \neq 0$ (i.e. if $\mathbf{u}$ is not perpendicular to the normal to the plane generated by $\mathbf{v}$ and $\mathbf{w}$, i.e. $\mathbf{u}$ is not parallel to this plane – nor are $\mathbf{v}$ and $\mathbf{w}$ parallel – so they do generate a plane). If this is, so the volume of the parallelopiped is given by $|\mathbf{u} \bullet \mathbf{v} \times \mathbf{w}|$.

$$\mathbf{u} \bullet \mathbf{v} \times \mathbf{w} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ -1 & 5 & -1 \end{vmatrix} = 1 \cdot (-3 - 5) - 1 \cdot (-2 + 1) + 1 \cdot (10 + 3)$$

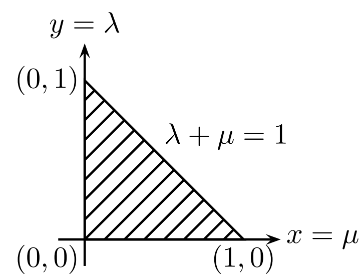
$$= -8 - (-1) + 13 = -8 + 1 + 13 = 6$$

$$\therefore \qquad \text{volume } = 6.$$

5 **a.** A parameterisation of the surface is

$$\mathbf{r}(\lambda,\, \mu) = (\lambda,\, \mu,\, 1 - \frac{2}{3}\mu^{3/2})$$

$$\text{where} \qquad 0 \le \lambda \le 1,\ 0 \le \mu \le 1 - \lambda.$$



$$d\mathbf{S} = \pm \left(\frac{\partial \mathbf{r}}{\partial \lambda} \times \frac{\partial \mathbf{r}}{\partial \mu} \right) d\lambda\, d\mu = \pm \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -\mu^{1/2} \end{vmatrix} d\lambda\, d\mu$$

$$= \pm (0,\, \mu^{1/2},\, 1)\, d\lambda\, d\mu.$$

$$\text{So} \qquad dS = ||d\mathbf{S}|| = \sqrt{\mu + 1}\, d\lambda\, d\mu.$$

$$\therefore \qquad \text{Area } = \int_0^1 \int_0^{1-\lambda} \sqrt{\mu + 1}\; d\mu\, d\lambda = \int_0^1 \left[\frac{2}{3}(\mu + 1)^{3/2} \right]\Bigg|_0^{1-\lambda}$$

$$= \frac{2}{3} \int_0^1 \left((2 - \lambda)^{3/2} - 1 \right)\, d\lambda = \frac{2}{3} \left[-\frac{2}{5}(2 - \lambda)^{5/2} - \lambda \right]\Bigg|_0^1$$

$$= \frac{2}{3} \left[-\frac{2}{5} - 1 + \frac{2}{5} \cdot 2^{5/2} \right] = \frac{2}{15}(8\sqrt{2} - 7).$$

b. i) $\underset{\sim}{\mathbf{F}} = (x^2 + yz,\ -2y(x+z),\ xy + z^2)$

$$\begin{aligned}\operatorname{div}\underset{\sim}{\mathbf{F}} &= \frac{\partial}{\partial x}(x^2+yz) + \frac{\partial}{\partial y}\big(-2y(x+z)\big) + \frac{\partial}{\partial z}(xy+z^2)\\ &= 2x - 2(x+z) + 2z = 0.\end{aligned}$$

Hence $\underset{\sim}{\mathbf{F}}$ is solenoidal.

ii) $\underset{\sim}{\mathbf{G}} = (-yz^2 - \frac{1}{2}xy^2,\ 0,\ x^2 + \frac{1}{2}y^2 z)$

$$\begin{aligned}\operatorname{curl}\underset{\sim}{\mathbf{G}} &= \begin{vmatrix} \underset{\sim}{\mathbf{i}} & \underset{\sim}{\mathbf{j}} & \underset{\sim}{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \\ -yz^2 - \dfrac{1}{2}xy^2 & 0 & x^2 y + \dfrac{1}{2}y^2 z \end{vmatrix}\\ &= \underset{\sim}{\mathbf{i}}\left[\frac{\partial}{\partial y}(x^2y + \frac{1}{2}y^2z) - 0\right] - \underset{\sim}{\mathbf{j}}\left[\frac{\partial}{\partial x}(x^2y + \frac{1}{2}y^2z)\right.\\ &\qquad \left. - \frac{\partial}{\partial z}(-yz^2 - \frac{1}{2}xy^2)\right] + \underset{\sim}{\mathbf{k}}\left[0 - \frac{\partial}{\partial y}(-yz^2 - \frac{1}{2}xy^2)\right]\\ &= (x^2 + yz)\underset{\sim}{\mathbf{i}} - (2xy + 2yz)\underset{\sim}{\mathbf{j}} + (z^2 + xy)\underset{\sim}{\mathbf{k}} = \underset{\sim}{\mathbf{F}}.\end{aligned}$$

c. By the divergence theorem,

$$\iint_S \underset{\sim}{\mathbf{F}} \bullet \underset{\sim}{\mathbf{n}}\, ds = \iiint_B \operatorname{div}\underset{\sim}{\mathbf{F}}\, dV$$

where we have renamed $B = [0,1] \times [0,1 \times [0,1]$ as the (solid) unit box and S is now the closed boundary surface for the cube.
Now $\underset{\sim}{\mathbf{F}} = (y - x,\ x - xz\, xy - z)$ so

$$\begin{aligned}\operatorname{div}\underset{\sim}{\mathbf{F}} &= \frac{\partial}{\partial x}(y-x) + \frac{\partial}{\partial y}(x - xz) + \frac{\partial}{\partial z}(xy - z)\\ &= -1 + 0 - 1 = -2.\end{aligned}$$

$$\therefore \qquad \iint_S \underset{\sim}{\mathbf{F}} \bullet \underset{\sim}{\mathbf{n}}\, ds = (-2)\iint_B dV = (-2) \times \text{Volume of } B = -2$$

6 **a.** i)

$$\begin{aligned}\nabla \times \underset{\sim}{\mathbf{F}} = \operatorname{curl}\underset{\sim}{\mathbf{F}} &= \left(\tfrac{\partial}{\partial x},\ \tfrac{\partial}{\partial y},\ \tfrac{\partial}{\partial z}\right) \times \underset{\sim}{\mathbf{F}}\\ &= \begin{vmatrix} \underset{\sim}{\mathbf{i}} & \underset{\sim}{\mathbf{j}} & \underset{\sim}{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \underset{\sim}{\mathbf{i}} & \underset{\sim}{\mathbf{j}} & \underset{\sim}{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz & xy \end{vmatrix}\\ &= (x - (-1),\ -(y - y),\ -z - z) = (2x,\ 0,\ -2z).\end{aligned}$$

ii) S should denote the (bounded flat) triangular surface with vertices $P(1,0,0)$, $Q(0,2,0)$ and $R(0,0,3)$ and not the (unbounded) plane containing the triangle PQR.

A normal vector to this plane is

$$\begin{aligned}\underset{\sim}{\mathbf{n}} &= \overrightarrow{PQ} \times \overrightarrow{PR} = \left(\overrightarrow{OQ} - \overrightarrow{OP}\right) \times \left(\overrightarrow{OR} - \overrightarrow{OP}\right) \\ &= (-1,\, 2,\, 0) \times (-1,\, 0,\, 3) \\ &= \begin{vmatrix} \underset{\sim}{\mathbf{e}}_1 & \underset{\sim}{\mathbf{e}}_2 & \underset{\sim}{\mathbf{e}}_3 \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = (6-0,\, -(-3-0),\, 0+2) = (6,\, 3,\, 2).\end{aligned}$$

Hence the plane containing triangle PQR has Point–normal form

$(\underset{\sim}{\mathbf{x}} - (1,0,0)) \bullet \underset{\sim}{\mathbf{n}} = 0, \quad \therefore \quad \underset{\sim}{\mathbf{x}} \bullet \underset{\sim}{\mathbf{n}} = (1,0,0) \bullet \underset{\sim}{\mathbf{n}}$

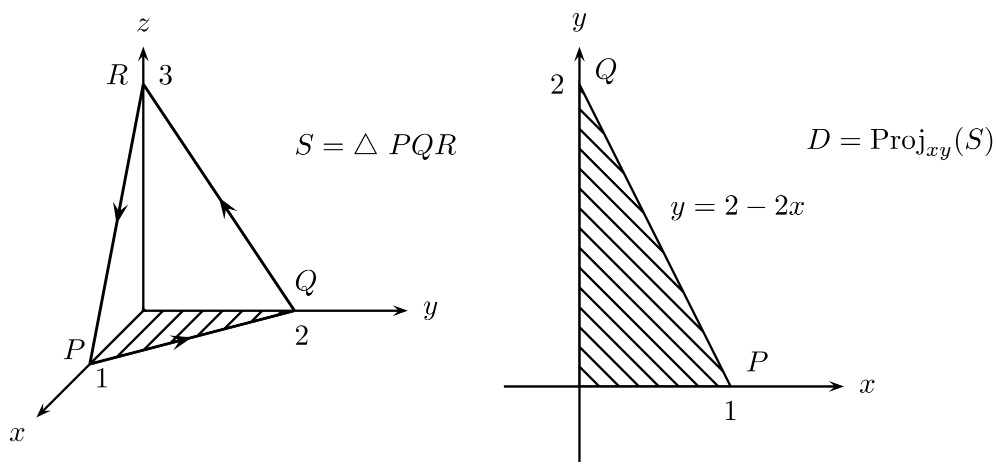
i.e. $6x + 3y + 2z = 6$ or $z = 3 - 3x - \frac{3}{2}y$.

Alternatively, a plane in $\mathbb{R}^3$ has cartesian form $\alpha x + \beta y + \gamma z = \delta$ with not all $\alpha,\ \beta,\ \gamma$ equal to 0. If the plane does not go through the origin, we must have $\delta \neq 0$ hence it can be expressed in the form

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1.$$

Substituting in the corners of the triangle P, Q, R which are the axes intercepts, we see $p = 1$, $q = 2$, $r = 3$ which gives the required equation.

iii) We want the parameterisation for the bounded flat triangle PQR which is what S should be. The intended parameterisation after ii) is to use x and y as parameters:



Hence the intended parameterisation with s for x and t for y and parameter domain $D = \mathrm{Proj}_{xy}(S)$ is:

$$S: \quad \underset{\sim}{\mathbf{r}}(s,t) = \left(s,\, t,\, 3 - 3s - \tfrac{3}{2}t\right), \qquad 0 \le s \le 1,\ 0 \le t \le 2 - 2s.$$

Alternatively, (and this was **not** the intended parameterisation!), one can use a parametric form for the plane, using say $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ as vectors parallel to the plane. Note these vectors "take in the vertices P, Q, R in cyclic order". This leads to the alternative parameterisation:

$$\begin{aligned} S: \quad \underset{\sim}{\mathbf{r}}(\lambda,\mu) &= \overrightarrow{OP} + \lambda\overrightarrow{PQ} + \mu\overrightarrow{QR} = (1,\, 0,\, 0) + \lambda(-1,\, 2,\, 0) + \mu(0,\, -2,\, 3) \\ &= (1-\lambda,\, 2\lambda - 2\mu,\, 3\mu), \qquad 0 \le \lambda \le 1,\ 0 \le \mu \le \lambda. \end{aligned}$$

iv) The closed boundary curve $C = PQR$ is made up of 3 smooth straight line segments, so

$$\begin{aligned} & C = C_{PQ} + C_{QR} + C_{RP} \quad \text{with} \\ C_{PQ}: \quad & \underset{\sim}{\mathbf{r}}(x) = (x,\, 2-2x,\, 0) \qquad x = 1 \text{ to } x = 0 \\ C_{QR}: \quad & \underset{\sim}{\mathbf{r}}(y) = \left(0,\, y,\, 3 - \tfrac{3}{2}y\right) \qquad y = 2 \text{ to } y = 0 \\ C_{RP}: \quad & \underset{\sim}{\mathbf{r}}(x) = (x,\, 0,\, 3-3x) \qquad x = 0 \text{ to } x = 1. \end{aligned}$$

These curve parameterisations use the relevant edges in the intended parameterisation of S, choosing either the 1st or 2nd co–ordinate as the curve parameter. Alternatively, one could use standard straight line parameterisations from scratch:

$$\begin{aligned} C_{PQ}: \quad \underset{\sim}{\mathbf{r}}(t) &= \overrightarrow{OP} + t\,\overrightarrow{PQ} = (1,\ 0,\ 0) + t(-1,\, 2,\, 0) \\ &= (1-t,\, 2t, 0) \qquad t = 0 \text{ to } t = 1 \\ C_{QR}: \quad \underset{\sim}{\mathbf{r}}(t) &= \overrightarrow{OQ} + t\,\overrightarrow{QR} = (0,\ 2,\ 0) + t(0,\, -2,\, 3) \\ &= (0,\, 2-2t, 3t) \qquad t = 0 \text{ to } t = 1 \\ C_{RP}: \quad \underset{\sim}{\mathbf{r}}(t) &= \overrightarrow{OR} + t\,\overrightarrow{RP} = (0,\ 0,\ 3) + t(1,\, 0,\, -3) \\ &= (t,\, 0, 3-3t) \qquad t = 0 \text{ to } t = 1. \end{aligned}$$

(The possible advantage of the alternative standard method is they all have $t = 0$ to $t = 1$.)

v) Let $d\underset{\sim}{\mathbf{s}} = \underset{\sim}{\hat{\mathbf{n}}}\, ds$ be the vector surface element, with ds the scalar surface element and $\underset{\sim}{\hat{\mathbf{n}}}$ a unit normal at the surface element. Stoke's theorem says here

$$\iint_S \nabla \times \underset{\sim}{\mathbf{F}} \bullet d\underset{\sim}{\mathbf{s}} = \int_C \underset{\sim}{\mathbf{F}} \bullet d\underset{\sim}{\mathbf{r}}.$$

Here for the open orientable surface $S = \triangle PQR$, with boundary the simple closed curve C, C must be positively oriented. This means if we choose the unit normal vector $\underset{\sim}{\hat{\mathbf{n}}}$ to be pointing from the side of S away from the origin (i.e. with positive z–component) then we must traverse C in the order P to Q, Q to R then R to P.

Calculating the LHS:

1. Using the intended parameterisation, for a correct choice of $\pm$,

$$\begin{aligned} d\underset{\sim}{\mathbf{s}} &= \pm\left(\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}\right) ds\,dt = \pm(1,\, 0,\, -3) \times \left(0,\, 1,\, -\tfrac{3}{2}\right) ds\,dt \\ &= \pm \begin{vmatrix} \underset{\sim}{\mathbf{e}}_1 & \underset{\sim}{\mathbf{e}}_2 & \underset{\sim}{\mathbf{e}}_3 \\ 1 & 0 & -3 \\ 0 & 1 & -\frac{3}{2} \end{vmatrix} ds\,dt = \pm\left(0+3,\ -\left(-\tfrac{3}{2} - 0\right),\, 1 - 0\right) ds\,dt \\ &= \pm\left(3,\, \tfrac{3}{2},\, 1\right) ds\,dt \end{aligned}$$

for the normal vector away from the origin, we choose $+$,

$$\therefore \quad \text{LHS} = \int_0^1 \int_0^{2-2s} (2x,\, 0,\, -2z) \bullet \left(3,\, \tfrac{3}{2},\, 1\right) dt\,ds$$

$$\text{with} \quad x = s,\ z = 3 - 3s - \tfrac{3}{2}t$$

$$= \int_0^1 \int_0^{2-2s} 6x - 2z \; dt \, ds = \int_0^1 \int_0^{2-2s} 6s - 2\left(3 - 3s - \tfrac{3}{2}t\right) \; dt \, ds$$

$$= \int_0^1 \int_0^{2-2s} 12s + 3t - 6 \; dt \, ds = \int_0^1 12st + \tfrac{3}{2}t^2 - 6t\Big|_{t=0}^{t=2(1-s)} \; ds$$

$$= \int_0^1 24s(1-s) + 6(1-s)^2 - 12(1-s) \; ds$$

$$= \int_0^1 24[1-(1-s)](1-s) + 6(1-s)^2 - 12(1-s) \; ds$$

$$= \int_0^1 12(1-s) - 18(1-s)^2 \; ds$$

$$= \int_0^1 12v - 18v^2 \; dv \qquad \text{after} \quad v = 1 - s$$

$$= (6v^2 - 6v^3)\Big|_0^1 = 0.$$

2. Using the alternative parameterisation, for a correct choice of $\pm$,

$$d\underset{\sim}{s} = \pm\left(\frac{\partial r}{\partial \lambda} \times \frac{\partial r}{\partial \mu}\right) d\lambda \, d\mu = \pm(-1,\, 2,\, 0) \times (0,\, -2,\, 3) \, d\lambda \, d\mu$$

$$= \pm \begin{vmatrix} \underset{\sim}{\mathbf{e}}_1 & \underset{\sim}{\mathbf{e}}_2 & \underset{\sim}{\mathbf{e}}_3 \\ -1 & 2 & 0 \\ 0 & -2 & 3 \end{vmatrix} d\lambda \, d\mu = \pm\,(6-0,\; -(-3-0),\; 2-0) \; d\lambda \, d\mu$$

$$= \pm(6,\, 3,\, 2) \, d\lambda \, d\mu$$

again, for the normal vector away from the origin, we choose $+$,

$$\therefore \quad \text{LHS} = \int_0^1 \int_0^{\lambda} (2x,\, 0,\, -2z) \bullet (6,\, 3,\, 2) \; d\mu \, d\lambda$$

$$\text{with} \quad x = 1 - \lambda, \;\; z = 3\mu$$

$$= \int_0^1 \int_0^{\lambda} 12x - 4z \; d\mu \, d\lambda = \int_0^1 \int_0^{\lambda} 12(1-\lambda) - 4(3\mu) \; d\mu \, d\lambda$$

$$= \int_0^1 \int_0^{\lambda} 12 - 12\lambda - 12\mu \; d\mu \, d\lambda = \int_0^1 12\mu - 12\lambda\mu - 6\mu^2\Big|_{\mu=0}^{\mu=\lambda} \; d\lambda$$

$$= \int_0^1 12\lambda - 12\lambda^2 - 6\lambda^2 \; d\lambda = \int_0^1 12\lambda - 18\lambda^2 \; d\lambda$$

$$= 6\lambda^2 - 6\lambda^3\Big|_{\lambda=0}^{\lambda=1} = 0.$$

Calculating the RHS:

$$\text{RHS} = \int_C \underset{\sim}{\mathbf{F}} \bullet d\underset{\sim}{\mathbf{r}} = \int_{C_{PQ}} \underset{\sim}{\mathbf{F}} \bullet d\underset{\sim}{\mathbf{r}} + \int_{C_{QR}} \underset{\sim}{\mathbf{F}} \bullet d\underset{\sim}{\mathbf{r}} + \int_{C_{RP}} \underset{\sim}{\mathbf{F}} \bullet d\underset{\sim}{\mathbf{r}}$$

1. Using the first parameterisation for C:

$$\begin{aligned}
\text{RHS} &= \int_1^0 (yz, -xz, xy) \bullet (1, -2, 0)\, dt \qquad \text{with} \quad x = t,\ y = 2 - 2t,\ z = 0 \\
&\quad + \int_2^0 (yz, -xz, xy) \bullet (0, 1, -\tfrac{3}{2})\, dt \qquad \text{with} \quad x = 0,\ y = t,\ z = 3 - \tfrac{3}{2}t \\
&\quad + \int_0^1 (yz, -xz, xy) \bullet (1, 0, -3)\, dt \qquad \text{with} \quad x = t,\ y = 0,\ z = 3 - 3t \\
&= \underset{\substack{\downarrow \\ z=0}}{\int_0^1 yz + 2xz\, dt} + \underset{\substack{\downarrow \\ x=0}}{\int_2^0 -xz - \tfrac{3}{2}xy\, dt} + \underset{\substack{\downarrow \\ y=0}}{\int_0^1 yz - 3xy\, dt} \\
&= \int_1^0 0\ dt + \int_2^0 0\ dt + \int_0^1 0\ dt = 0 + 0 + 0 = 0.
\end{aligned}$$

2. Using the alternative "standard" parameterisation for C:

$$\begin{aligned}
\text{RHS} &= \int_0^1 (yz, -xz, xy) \bullet (-1, 2, 0)\, dt \qquad \text{with} \quad x = 1 - t,\ y = 2t,\ z = 0 \\
&\quad + \int_0^1 (yz, -xz, xy) \bullet (0, -2, 3)\, dt \qquad \text{with} \quad x = 0,\ y = 2 - 2t,\ z = 3t \\
&\quad + \int_0^1 (yz, -xz, xy) \bullet (1, 0, -3)\, dt \qquad \text{with} \quad x = t,\ y = 0,\ z = 3 - 3t \\
&= \underset{\substack{\downarrow \\ z=0}}{\int_0^1 -yz - 2xz\, dt} + \underset{\substack{\downarrow \\ x=0}}{\int_0^1 2xz + 3xy\, dt} + \underset{\substack{\downarrow \\ y=0}}{\int_0^1 yz - 3xy\, dt} \\
&= \int_0^1 0\ dt + \int_0^1 0\ dt + \int_0^1 0\ dt = 0 + 0 + 0 = 0.
\end{aligned}$$

Note: Since S lies in a plane with a unit normal vector $\hat{\underset{\sim}{\mathbf{n}}} = \frac{1}{7}(6,\ 3,\ 2)$, it follows

$$d\underset{\sim}{\mathbf{s}} = \hat{\underset{\sim}{\mathbf{n}}}\, \phi(\alpha, \beta)\, d\alpha\, d\beta$$

for some scalar valued function $\phi(\alpha, \beta)$ of the parameters α, β. However it is not immediately obvious what this scalar function ϕ is without performing the cross product calculation for $d\underset{\sim}{\mathbf{s}}$. For the two parameterisations for S, this scalar function ϕ is a constant function of the parameters:

$$d\underset{\sim}{\mathbf{s}} = \pm\, \hat{\underset{\sim}{\mathbf{n}}} \cdot \tfrac{7}{2}\, ds\, dt = \pm\, \hat{\underset{\sim}{\mathbf{n}}} \cdot 7\, d\lambda\, d\mu.$$

b. The Divergence Theorem (or Gauss's Theorem) states

Divergence Theorem. *If R is a closed bounded region in $\mathbb{R}^3$ whose boundary is a closed piecewise smooth orientable surface $\mathcal{S}$ and $\underset{\sim}{\mathbf{F}}$ is a smooth vector field (i.e. has continuous first order partial derivatives) on an open set containing R, then*

$$\iiint_R \operatorname{div} \underset{\sim}{\mathbf{F}}\ dV = \iint_{\mathcal{S}} \underset{\sim}{\mathbf{F}} \bullet d\underset{\sim}{\mathbf{s}}$$

where $dV = dx\,dy\,dz$ is the volume element in $\mathbb{R}^3$, $d\underset{\sim}{\mathbf{s}} = \underset{\sim}{\hat{\mathbf{n}}}\,ds$ is the vector surface area element on $\mathcal{S}$, with outward unit normal $\underset{\sim}{\hat{\mathbf{n}}}$ on $\mathcal{S}$ at the surface element and $\operatorname{div} \underset{\sim}{\mathbf{F}}$ is the divergence of $\underset{\sim}{\mathbf{F}}$, defined by

$$\operatorname{div} \underset{\sim}{\mathbf{F}} = \nabla \bullet \underset{\sim}{\mathbf{F}} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

where $\underset{\sim}{\mathbf{F}} = (F_1, F_2, F_3)$, i.e. F_i is the i-th component function of $\underset{\sim}{\mathbf{F}}$.

Math2011 Several Variable Calculus — June 2002.

1 **a.** Let $f(x,y)$ be the function
$$f(x,y) = (x^2-1)^2 + y^2.$$

i) Find ∇f at $(0,0)$, and hence find the equation of the tangent plane to $z = f(x,y)$ at $(x,y,z) = (0,0,1)$.

ii) Find the Taylor series of $f(x,y)$ about the point $(1,0)$ up to and including the second order terms.

iii) Find the critical points of $z = f(x,y)$, and determine whether they are local maxima, local minima or saddle points.

b. Given that
$$\int_0^\infty \frac{1}{x^2+a^2}\,dx = \frac{\pi}{2a}\ ,$$
show, by differentiating both sides of the result with respect to a, that
$$\int_0^\infty \frac{1}{(x^2+a^2)^2}\,dx = \frac{\pi}{4a^3}.$$

c. Prove by induction that for $n \geq 1$
$$\int_0^\infty \frac{1}{(x^2+a^2)^{n+1}}\,dx = \binom{2n}{n}\frac{\pi}{(2a)^{2n+1}},$$
where $\binom{n}{r}$ is the binomial coefficient,
$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

2 **a.** The volume V of a cylindrical can of height h and radius r is given by
$$V = \pi r^2 h.$$

i) Find $\dfrac{\partial V}{\partial r}$ and $\dfrac{\partial V}{\partial h}$.

ii) Write down an expression for the differential dV in terms of dr and dh. Hence find an expression for $\dfrac{dV}{V}$ in terms of $\dfrac{dr}{r}$ and $\dfrac{dh}{h}$.

iii) If r is increased by 5% and h by 10%, what is the resulting approximate percentage increase in V?

b. The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by
$$f(x,y) = (u,v) = (x^2-y^2, 2xy)$$
and P is the point (a,b).

i) Calculate $\left(\dfrac{\partial u}{\partial x}\right)_P$, $\left(\dfrac{\partial u}{\partial y}\right)_P$, $\left(\dfrac{\partial v}{\partial x}\right)_P$, $\left(\dfrac{\partial v}{\partial y}\right)_P$, and hence write down
$$J_P f = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)_P.$$

ii) By using the chain rule, or otherwise, calculate $J_P(f \circ f)$.

3 **a.** Consider a triangular lamina R in the plane, bounded by $x = 0, y = 0$, $x + y = 1$, with density $\delta(x, y) = xy$.

i) Find the mass M of the lamina,

$$M = \iint\limits_R \delta(x, y)\, dA.$$

ii) Find the centre of mass $(\overline{x}, \overline{y})$ of the lamina, given by

$$\overline{x} = \iint\limits_R x\delta(x, y)\, dA/M,$$

$$\overline{y} = \iint\limits_R y\delta(x, y)\, dA/M.$$

b. A solid S is bounded below by the plane $z = 0$ and above by the sphere $z = \sqrt{4 - x^2 - y^2}$. It has density $\delta(x, y, z) = z$.

Using either spherical or cylindrical polar coordinates, calculate the centre of mass, $(0, 0, \overline{z})$ where

$$\overline{z} = \iiint\limits_S z\delta(x, y, z)\, dV \Big/ \iiint\limits_S \delta(x, y, z)\, dV.$$

4 **a.** Evaluate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ if the vector field $\boldsymbol{F}$ is defined by

$$\boldsymbol{F} = 2yz\, \boldsymbol{i} + (x + z)\, \boldsymbol{j} - \frac{y^2 e^x}{z}\, \boldsymbol{k}$$

and if the curve C has the parametric representation

$$\boldsymbol{r}(t) = t^2\, \boldsymbol{i} + t\, \boldsymbol{j} + \frac{2}{t}\, \boldsymbol{k} \quad \text{for } 1 \le t \le 2\ .$$

b. Find a scalar field ϕ such that the gradient of ϕ is

$$\nabla\phi = \left(y\cos(xy) - 2xe^z\right)\boldsymbol{i} + x\cos(xy)\, \boldsymbol{j} - x^2 e^z\, \boldsymbol{k}.$$

c. For the vector field

$$\boldsymbol{G} = xz^2\, \boldsymbol{i} - 2y^2 z\, \boldsymbol{j} + x^2 yz\, \boldsymbol{k}$$

find

i) $\operatorname{div} \boldsymbol{G}$;

ii) $\operatorname{curl} \boldsymbol{G}$;

iii) $\operatorname{div}(\operatorname{curl} \boldsymbol{G})$.

d. i) Show that if $\boldsymbol{A}$ is a *constant* vector field then

$$\nabla \cdot (\phi \boldsymbol{A}) = (\nabla\phi) \cdot \boldsymbol{A} \qquad (*)$$

for any scalar field ϕ.

ii) Verify that the identity $(*)$ in i) above holds for $\boldsymbol{A} = \boldsymbol{i} + \boldsymbol{j}$ and $\phi(x, y, z) = xyz$.

5 In this question two different methods are to be used to evaluate

$$I = \iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS$$

where S is the open surface defined by $z = x^2 + y^2$ and $0 \le z \le 4$, $\boldsymbol{n}$ is the unit normal to S directed upwards, and $\boldsymbol{F}$ is the vector field

$$\boldsymbol{F}(x, y, z) = (z - 4)y\,\boldsymbol{i} + (z - 4)x\,\boldsymbol{j}.$$

a. Parametrise the surface S, and hence evaluate the surface integral I directly.
b. State Stokes' theorem.
c. Evaluate the integral I using Stokes' theorem.

6 **a.** Find the Fourier series $S(x)$ of the function defined by

$$f(x) = x \quad \text{if } -\pi < x < \pi.$$

b. To what value does the Fourier series $S(x)$ converge at

$$\alpha) \quad x = \frac{\pi}{2}, \qquad\qquad \beta) \quad x = \pi?$$

c. Sketch the graph of the function to which the Fourier series $S(x)$ converges for the interval $-3\pi \le x \le 3\pi$, carefully indicating the value of the limit function at any point of discontinuity.
d. Use the Fourier series at $x = \pi/2$ to obtain an infinite series which converges to $\pi/4$.

Solutions to Math2011 June 2002 exam

1 **a.** i)

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\,\boldsymbol{i} + \frac{\partial f}{\partial y}\,\boldsymbol{j} = 2(x^2 - 1) \cdot 2x\,\boldsymbol{i} + 2y\,\boldsymbol{j}$$

$$\therefore \quad \boldsymbol{\nabla} f(0, 0) = 0\,\boldsymbol{i} + 0\,\boldsymbol{j} = \boldsymbol{0}$$

The tangent plane is $z - z_0 = \boldsymbol{\nabla} f(x_0, y_0) \bullet (\boldsymbol{x} - \boldsymbol{x}_0)$

i.e. $z - 1 = \boldsymbol{0} \bullet (\boldsymbol{x} - \boldsymbol{x}_0) = 0$

or $z = 1$.

ii)

$$\begin{aligned} f(x, y) = f(x_0, y_0) &+ \frac{1}{1!}\left[\frac{\partial f}{\partial x}(x_0, y_0)\,(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)\,(y - y_0)\right] \\ &+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(x_0, y_0)\,(x - x_0)^2 + 2\frac{\partial^2 f}{\partial x\,\partial y}(x_0, y_0)\,(x - x_0)(y - y_0)\right. \\ &\qquad \left. + \frac{\partial^2 f}{\partial y^2}(x_0, y_0)(y - y_0)^2\right] + \cdots \\ = 0 &+ \frac{1}{1!}\left[0(x - 0) + 0(y - 0)\right] \\ &+ \frac{1}{2!}\left[8(x - 0)^2 + 2 \cdot 0(x - 0)(y - 0) + 2(y - 0)^2\right] + \cdots \end{aligned}$$

$$\therefore \quad f(x, y) = 4(x - 1)^2 + y^2 + \cdots$$

Since

$$f(1,0) = (1-1)^2 + 0^2 = 0$$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 4x^3 - 4x & \therefore\quad \frac{\partial f}{\partial x}(1,0) &= 0 \\
\frac{\partial f}{\partial y} &= 2y & \therefore\quad \frac{\partial f}{\partial y}(1,0) &= 0 \\
\frac{\partial^2 f}{\partial x^2} &= 12x^2 - 4 & \therefore\quad \frac{\partial^2 f}{\partial x^2}(1,0) &= 12 - 4 = 8 \\
\frac{\partial^2 f}{\partial x\,\partial y} &= 0 & \therefore\quad \frac{\partial^2 f}{\partial x\,\partial y}(1,0) &= 0 \\
\frac{\partial^2 f}{\partial y^2} &= 2 & \therefore\quad \frac{\partial^2 f}{\partial y^2}(1,0) &= 2
\end{aligned}$$

OR let $u = x - 1,\ v = y$ so $x = u + 1,\ y = v$, hence

$$\begin{aligned}
f(x,y) &= (x^2-1)^2 + y^2 = \left[(u+1)^2 - 1\right]^2 + v^2 \\
&= (u^2+2u)^2 + v^2 = u^2(u+2)^2 + v^2 \\
&= u^2(u^2+4u+4) + v^2 \\
&= 4u^2 + v^2 + 4u^3 + u^4 \\
&= 4(x-1)^2 + y^2 + 4(x-1)^3 + (x-1)^4\ .
\end{aligned}$$

iii) Critical points are the points (x,y) satisfying $\boldsymbol{\nabla} f(x,y) = \mathbf{0}$. So solving

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 4x(x^2-1) = 0 \\
\frac{\partial f}{\partial y} &= 2y = 0
\end{aligned}$$

we see the critical points are $(0,0),\ (-1,0),\ (1,0)$. Now at any point (x,y), let

$$\begin{aligned}
A &= \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4 \\
B &= \frac{\partial^2 f}{\partial x\,\partial y} = 0 \\
C &= \frac{\partial^2 f}{\partial y^2} = 2
\end{aligned}$$

So applying the 2nd order test:

Critical Point	$A = \frac{\partial^2 f}{\partial x^2}$	$B = \frac{\partial^2 f}{\partial x\,\partial y}$	$C = \frac{\partial^2 f}{\partial y^2}$	$B^2 - AC$	Type
$(-1,0)$	8	0	2	-16	Local Min. Pt.
$(0,0)$	-4	0	2	8	Saddle Pt.
$(1,0)$	8	0	2	-16	Local Min. Pt.

b. Differentiating both sides w.r.t. a (and assuming a version of Leibniz's Rule for Differentiation under the Integral Sign for Improper Integrals*)

$$\int_0^\infty -\frac{2a}{(x^2+a^2)^2}\,dx = -\frac{\pi}{2a^2}$$

$$\therefore \quad \int_0^\infty \frac{1}{(x^2+a^2)^2}\,dx = \frac{\pi}{4a^3}\ .$$

c. For $n \in \mathbb{Z}^+$ (i.e. an integer and $n \geq 1$), let P_n be the proposition that

$$\int_0^\infty \frac{1}{(x^2+a^2)^{n+1}}\,dx = \binom{2n}{n}\frac{\pi}{(2a)^{2n+1}}$$

then P_1 is true by part b) above.
Suppose for some $n \in \mathbb{Z}^+$ that P_n is true. Again differentiate both sides w.r.t. a, (again assuming the appropriate version of Leibnz's Rule *),

$$\int_0^\infty -\frac{(n+1)\cdot 2a}{(x^2+a^2)^{n+2}}\,dx = \binom{2n}{n}\cdot\left[-(2n+1)\cdot\frac{\pi}{2a^{2n+2}}\cdot 2\right]$$

Dividing both sides by $-2a(n+1)$,

$$\therefore \quad \int_0^\infty \frac{1}{(x^2+a^2)^{n+2}}\,dx = \frac{2n+1}{n+1}\cdot\binom{2n}{n}\cdot 2\cdot\frac{\pi}{(2a)^{2n+3}}$$

Now $$\frac{2n+1}{n+1}\cdot\binom{2n}{n}\cdot 2 = \frac{2(2n+1)}{n+1}\cdot\frac{(2n)!}{n!\,n!} = \frac{2}{n+1}\cdot\frac{(2n+2)!}{n!\,n!}\cdot\frac{1}{2n+2}$$

$$= \frac{1}{(n+1)^2}\cdot\frac{(2n+2)!}{n!\,n!} = \frac{(2n+2)!}{(n+1)!\,(n+1)!} = \binom{2(n+1)}{n+1}$$

$$\therefore \quad \int_0^\infty \frac{1}{(x^2+a^2)^{(n+1)+1}}\,dx = \binom{2(n+1)}{n+1}\frac{\pi}{(2a)^{2(n+1)+1}}$$

So P_{n+1} is true, so as P_1 is true and for any $n \in \mathbb{Z}^+$, if P_n is true then P_{n+1} is true, hence by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.

[* **Technical Note** (Not expected in solution) The version of Leibniz's rule required here:

$$\frac{\partial}{\partial a}\int_0^\infty f(x,a)\,dx = \int_0^\infty \frac{\partial f}{\partial a}(x,a)\,dx$$

requires the LHS improper integral to converge for at least one value of a in some interval, and the RHS improper integral to be **uniformly** convergent for that interval for a, i.e. given $\epsilon > 0$, there is an X such that for all $x_0 > X$ and all a in the interval,

$$\left|\int_{x_0}^\infty \frac{\partial f}{\partial a}(x,a)\,dx\right| < \epsilon$$

– in other words, this X is independent of a and the same X works for each value of a in the a–interval.
This is usually checked by showing $\left|\frac{\partial f}{\partial a}(x,a)\right| \leq g(x)$ for all values of a in the a–interval, for some function $g(x)$ which is independent of a and for which $\int_0^\infty g(x)\,dx$ is convergent.
These conditions can be verified in this case with $f(x,a) = 1/(x^2+a^2)^n$ when we take the interval for a to be any finite closed interval not including 0.]

2 **a.** $V = \pi r^2 h$.

i)

$$\frac{\partial V}{\partial r} = 2\pi r h, \qquad \frac{\partial V}{\partial h} = \pi r^2 \,.$$

ii)

$$dV = \frac{\partial V}{\partial r}\,dr + \frac{\partial V}{\partial h}\,dh = 2\pi r h\,dr + \pi r^2\,dh$$

$$\therefore \qquad \frac{dV}{V} = \frac{2\pi r h\,dr}{\pi r^2 h} + \frac{\pi r^2\,dh}{\pi r^2 h} = 2\frac{dr}{r} + \frac{dh}{h}\,.$$

iii) With $\dfrac{dr}{r} = 5\%$, $\dfrac{h}{h} = 10\%$ we have $\dfrac{dV}{V} = 2 \times 5\% + 10\% = 20\%$, so the approximate percentage increase in V is 20%.

b. $f(x,y) = (u,v) = (x^2 - y^2, 2xy)$. P is the point (a,b).

i)

$$\frac{\partial u}{\partial x} = 2x, \qquad \left(\frac{\partial u}{\partial x}\right)_P = 2a$$

$$\frac{\partial u}{\partial y} = -2y, \qquad \left(\frac{\partial u}{\partial y}\right)_P = -2b$$

$$\frac{\partial v}{\partial x} = 2y, \qquad \left(\frac{\partial v}{\partial x}\right)_P = 2b$$

$$\frac{\partial v}{\partial y} = 2x, \qquad \left(\frac{\partial v}{\partial y}\right)_P = 2a$$

$$\therefore \qquad J_P(f) = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)_P = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}_P = \begin{pmatrix} 2a & -2b \\ 2b & 2a \end{pmatrix}$$

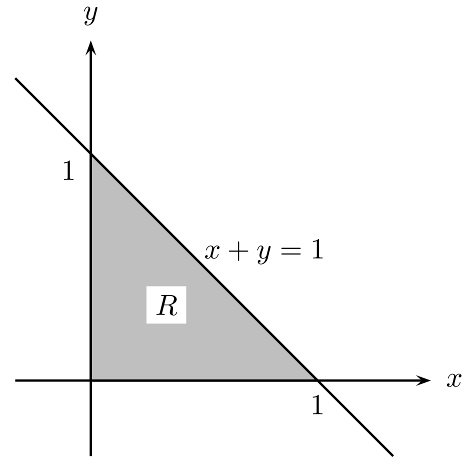
ii)

$$\begin{aligned} J_P(f\circ f) &= J_{f(P)}(f)\,J_P(f) \\ &= \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}_{(x,y)=(a^2-b^2,2ab)} \begin{pmatrix} 2a & -2b \\ 2b & 2a \end{pmatrix} \\ &= \begin{pmatrix} 2(a^2-b^2) & -4ab \\ 4ab & 2(a^2-b^2) \end{pmatrix} \begin{pmatrix} 2a & -2b \\ 2b & 2a \end{pmatrix} \\ &= \begin{pmatrix} 4a(a^2-b^2) - 8ab^2 & -4b(a^2-b^2) - 8a^2b \\ 8a^2b + 4b(a^2-b^2) & -8ab^2 + 4a(a^2-b^2) \end{pmatrix} \\ &= \begin{pmatrix} 4a^3 - 12ab^2 & -12a^2b + 4b^3 \\ 12a^2b - 4b^3 & 4a^3 - 12ab^2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{Alternatively} \quad f\circ f(x,y) &= f(f(x,y)) = f(x^2-y^2,\ 2xy) \\ &= ((x^2-y^2)^2 - (2xy)^2,\ 2(x^2-y^2)(2xy)) \\ &= (x^4 - 6x^2y^2 + y^4,\ 4x^3y - 4xy^3) \end{aligned}$$

$$\therefore \qquad J_P(f\circ f) = \begin{pmatrix} 4a^3 - 12ab^2 & -12a^2b + 4b^2 \\ 12a^2b - 4b^3 & 4a^3 - 12ab^2 \end{pmatrix}\,.$$

3 a.



i)

$$\begin{aligned} M &= \iint_R \delta \; dA = \int_0^1 \int_0^{1-x} xy \; dy\, dx \\ &= \int_0^1 x \left[\frac{y^2}{2}\right]_{y=0}^{y=1-x} dx = \frac{1}{2}\int_0^1 x(1-x)^2 \; dx \\ &= \frac{1}{2}\int_0^1 u^2(1-u)\; du \qquad \text{(after } u = 1-x\text{)} \\ &= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{1}{24}\;. \end{aligned}$$

ii)

$$\begin{aligned} \overline{x} &= \frac{1}{M}\iint_R x\delta(x,y)\; dA = 24\int_0^1 \int_0^{1-x} x^2 y \; dy\, dx \\ &= 24\int_0^1 x^2 \left[\frac{y^2}{2}\right]_{y=0}^{y=1-x} dx = 12\int_0^1 x^2(1-x)^2 \; dx \\ &= 12\left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1 = 12\left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right] \\ &= 12\left[\frac{10-15+6}{30}\right] = 12\cdot\frac{1}{30} = \frac{2}{5}\;. \end{aligned}$$

By symmetry, (i.e. R and δ are both symmetric under relection $(x,y) \to (y,x)$ about the line $y = x$), $\overline{y} = \frac{2}{5}$ also, so centre of mass is $(\overline{x}, \overline{y}) = \left(\frac{2}{5}, \frac{2}{5}\right)$.

b. Using cylindricals,

$$\begin{aligned} M &= \iiint_S \delta(x,y,z)\; dV = \int_0^{2\pi}\int_0^2\int_0^{\sqrt{4-r^2}} z\cdot r \; dz\, dr\, d\theta \\ &= 2\pi\int_0^2 r\left[\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{4-r^2}} dr = \pi\int_0^2 r(4-r^2)\; dr \\ &= \pi\left[2r^2 - \frac{r^4}{4}\right]_0^2 = 4\pi \end{aligned}$$

$$\therefore \quad \overline{z} = \frac{1}{M}\iiint_S z\delta(x,y,z)\, dV = \frac{1}{4\pi}\int_0^{2\pi}\int_0^2\int_0^{\sqrt{4-r^2}} z^2\cdot r\; dz\, dr\, d\theta$$
$$= \frac{1}{4\pi}\cdot 2\pi\cdot\int_0^2 r\left[\frac{z^3}{3}\right]_{z=0}^{z=\sqrt{4-r^2}} dr = \frac{1}{6}\int_0^2 r(4-r^2)^{3/2}\; dr$$
$$= \frac{1}{6}\left[-\frac{1}{5}(4-r^2)^{5/2}\right]_0^2 = \frac{1}{30}\cdot 4^{5/2} = \frac{32}{30} = \frac{16}{15}$$
$$\therefore \quad \text{C. of M.} = (0,0,\overline{z}) = (0,0,16/15)\ .$$

4 a.

$$\int_C \boldsymbol{F}\bullet d\boldsymbol{r} = \int_1^2\left[\left(2\cdot t\cdot\frac{2}{t}\right)\boldsymbol{i} + \left(t^2+\frac{2}{t}\right)\boldsymbol{j} - \frac{t^3e^{t^2}}{2}\boldsymbol{k}\right]\bullet\left(2t\,\boldsymbol{i}+\boldsymbol{j}-\frac{2}{t^2}\boldsymbol{k}\right)\; dt$$
$$= \int_1^2\left[8t + \left(t^2+\frac{2}{t}\right) + te^{t^2}\right]\; dt = \left[4t^2 + \frac{t^3}{3} - 2\ln t + \frac{1}{2}e^{t^2}\right]_1^2$$
$$= 4(2^2-1^1) + \frac{1}{3}(2^3-1^3) - 2(\ln 2 - \ln 1) + \frac{1}{2}(e^4-e^1)$$
$$= 12 + \frac{7}{3} - 2\ln 2 + \frac{1}{2}(e^4-e) = \frac{43}{3} - 2\ln 2 + \frac{1}{2}(e^4-e)\ .$$

b.

$$\text{Solving}\quad \frac{\partial\phi}{\partial x} = y\cos(xy) - 2xe^z \qquad (1)$$
$$\frac{\partial\phi}{\partial y} = x\cos(xy) \qquad (2)$$
$$\frac{\partial\phi}{\partial z} = -x^2e^z \qquad (3)$$

Integrating (1) w.r.t. x, gives $\phi = \sin(xy) - x^2e^z + h(y,z)$. Subst. this into (2) now gives

$$x\sin(xy) - 0 + \frac{\partial h}{\partial y}(y,z) = x\cos(xy)$$
$$\therefore \quad \frac{\partial h}{\partial y}(y,z) = 0$$
$$\therefore \quad h(y,z) = g(z)$$

Finally subst. $\phi = \sin(xy) - x^2e^z + g(z)$ into (3), then

$$0 - x^2e^z + g'(z) = -x^2e^z$$
$$\therefore \quad g'(z) = 0$$
$$\therefore \quad g(z) = c \quad \text{a constant}$$
$$\text{hence}\quad \phi(x,y,z) = \sin(xy) - x^2e^z + \text{const.}$$

(All equations hold for all x, y, z).

c. $\boldsymbol{G} = xz^2\,\boldsymbol{i} - 2y^2z\,\boldsymbol{j} + x^2yz\,\boldsymbol{k}$.

i)

$$\operatorname{div}\boldsymbol{G} = \boldsymbol{\nabla}\bullet\boldsymbol{G} = \frac{\partial}{\partial x}(xz^2) + \frac{\partial}{\partial y}(2y^2z) + \frac{\partial}{\partial z}(x^2yz)$$
$$= z^2 - 4yz + x^2z$$

ii)

$$\operatorname{curl}\boldsymbol{G} = \boldsymbol{\nabla}\times\boldsymbol{G} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz^2 & -2y^2z & x^2yz \end{vmatrix}$$

$$= (x^2z - (-2y^2))\,\boldsymbol{i} - (2xyz - 2xz)\,\boldsymbol{j} + (0-0)\,\boldsymbol{k}$$

$$= (x^2z + 2y^2))\,\boldsymbol{i} - (2xyz - 2xz)\,\boldsymbol{j}$$

iii)

$$\operatorname{div}(\operatorname{curl}\boldsymbol{G}) = \frac{\partial}{\partial x}(x^2z+2y^2)\,\boldsymbol{i} + \frac{\partial}{\partial y}(-2xyz+2xz)\,\boldsymbol{j} + \frac{\partial}{\partial z}(0)\,\boldsymbol{k}$$
$$= (2xz)\,\boldsymbol{i} - (2xz)\,\boldsymbol{j}$$

d. i)

$$\text{Let}\quad \boldsymbol{A} = A_1\,\boldsymbol{i} + A_2\,\boldsymbol{j} + A_3\,\boldsymbol{k}$$

$$\therefore\quad \boldsymbol{\nabla}\bullet(\phi\boldsymbol{A}) = \frac{\partial}{\partial x}(\phi A_1) + \frac{\partial}{\partial y}(\phi A_2) + \frac{\partial}{\partial z}(\phi A_3)$$
$$= \phi\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + A_1\frac{\partial\phi}{\partial x} + A_2\frac{\partial\phi}{\partial y} + A_3\frac{\partial\phi}{\partial z}$$

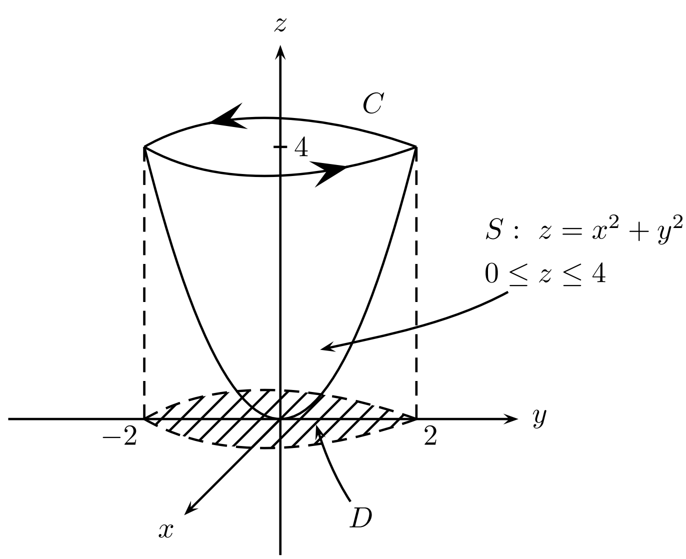
(by the product rule for differentiation)

$$= \phi\cdot 0 + \boldsymbol{\nabla}\phi\bullet\boldsymbol{A} \qquad (\text{as } \boldsymbol{A} \text{ is constant})$$
$$= (\boldsymbol{\nabla}\phi)\bullet\boldsymbol{A}$$

ii)

$$\text{With}\quad \boldsymbol{A} = \boldsymbol{i} + \boldsymbol{j} \quad\text{and}\quad \phi(x,y,z) = xyz$$
$$\text{then}\quad \phi\boldsymbol{A} = xyz\,\boldsymbol{i} + xyz\,\boldsymbol{j} + 0\,\boldsymbol{k}$$
$$\therefore\quad \boldsymbol{\nabla}\bullet\phi\boldsymbol{A} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(0) = yz + xz$$
$$\text{and}\quad (\boldsymbol{\nabla}\phi)\bullet\boldsymbol{A} = \left(\frac{\partial}{\partial x}(xyz)\,\boldsymbol{i} + \frac{\partial}{\partial y}(xyz)\,\boldsymbol{j} + \frac{\partial}{\partial z}(0)\,\boldsymbol{k}\right)\bullet(\boldsymbol{i}+\boldsymbol{j})$$
$$= (yz\,\boldsymbol{i} + xz\,\boldsymbol{j})\bullet(\boldsymbol{i}+\boldsymbol{j}) = yz + xz\ .$$

5 **a.** [Sketch not required]



i) Two possible parameterisations for S are

1. Cartesian : $\boldsymbol{r} = (x, y, x^2 + y^2)$, $\quad (x, y) \in D = \{(x, y) \mid 0 \le x^2 + y^2 \le 4\}$
2. Polar : $\boldsymbol{r} = (r\cos\theta, r\sin\theta, r^2)$, $\quad 0 \le \theta \le 2\pi,\ 0 \le r \le 2$

Using 1. Cartesian, (where the parameter domain has been given the name D),

$$d\boldsymbol{S} = \boldsymbol{n}\,dS = \pm\left(\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y}\right) dx\,dy = \pm\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} dx\,dy$$

$$= \pm(-2x, -2y, 1)\,dx\,dy$$

and $\pm$ must be taken to be $+$ for upward normal

$$\therefore \quad d\boldsymbol{S} = \boldsymbol{n}\,dS = (-2x, -2y, 1)\,dx\,dy$$

$$\text{and} \quad \boldsymbol{\nabla} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz - 4y & xz - 4x & 0 \end{vmatrix}$$

$$= (0 - x, -(0 - y), ((z - 4) - (z - 4)))$$

$$= (-x, y, 0)$$

$$\therefore \quad \iint_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \bullet \boldsymbol{n}\,dS = \iint_D (-x, y, 0) \bullet (-2x, -2y, 1)\,dx\,dy$$

$$= \iint_D 2x^2 - 2y^2\,dx\,dy$$

$$= 0$$

by symmetry of the parameter domain D about the line $y = x$ under reflection $T : (x, y) \mapsto (y, x)$ and antisymmetry of the integrand under this reflection, i.e. if $f(x, y) = 2x^2 - 2y^2$ then

$$f(T(x, y)) = f(y, x) = 2y^2 - 2x^2 = -f(x, y)\ .$$

OR by evaluating this double integral by polars,

$$\iint_D 2x^2 - 2y^2\,dx\,dy = \int_0^{2\pi}\int_0^2 (2r^2\cos^2\theta - 2r^2\sin^2\theta)\cdot r\,dr\,d\theta$$

$$= \left(\int_0^2 2r^3\,dr\right)\cdot\left(\int_0^{2\pi} \cos 2\theta\,d\theta\right)$$

$$= 0, \qquad \text{as} \quad \int_0^{2\pi} \cos 2\theta\,d\theta = \left[\frac{\sin 2\theta}{2}\right]_0^{2\pi} = (0 - 0) = 0.$$

Alternatively using 2. Polar,

$$d\boldsymbol{S} = \boldsymbol{n}\,dS = \pm\left(\frac{\partial \boldsymbol{r}}{\partial r} \times \frac{\partial \boldsymbol{r}}{\partial \theta}\right) dr\,d\theta$$

$$= \pm\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} dr\,d\theta$$

$$= \pm(-2r^2\cos\theta, -2r^2\sin\theta, r)\,dr\,d\theta$$

and $\pm$ must be taken to be $+$ for upward normal

$$\therefore \quad d\boldsymbol{S} = \boldsymbol{n}\,dS = (-2r^2\cos\theta, -2r^2\sin\theta, r)\,dr\,d\theta$$

$$\begin{aligned} \text{and} \quad \boldsymbol{\nabla}\times\boldsymbol{F} &= (-x, y, 0) \qquad \text{(as found above)} \\ &= (-r\cos\theta, r\sin\theta, 0) \end{aligned}$$

$$\begin{aligned} \therefore \quad \iint_S (\boldsymbol{\nabla}\times\boldsymbol{F}) \bullet \boldsymbol{n}\;dS &= \int_0^{2\pi}\int_0^2 (-r\cos\theta, r\sin\theta, 0) \bullet (-2r^2\cos\theta, -2r^2\sin\theta, r)\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^2 2r^3\cos^2\theta - 2r^3\sin^2\theta\;dr\,d\theta \\ &= \left(\int_0^2 2r^3\;dr\right)\cdot\left(\int_0^{2\pi}\cos 2\theta\;d\theta\right) \\ &= 0 \quad \text{as above.} \end{aligned}$$

b.

Stokes' Theorem. *If $\boldsymbol{F}$ is a continuously differentiable vector field on an open subset of $\mathbb{R}^3$ containing an open smooth (orientable) connected surface $\mathcal{S}$ with a positively oriented boundary curve C made up of finitely many smooth curves, then*

$$\int_C \boldsymbol{F} \bullet d\boldsymbol{r} = \int_{\mathcal{S}} \operatorname{curl}\boldsymbol{F} \bullet d\boldsymbol{S} = \int_{\mathcal{S}} (\boldsymbol{\nabla}\times\boldsymbol{F}) \bullet d\boldsymbol{S}$$

c. Let C be the boundary curve $x^2+y^2=4,\ z=4$ to S with the orientation shown as in the above diagram, i.e. anticlockwise as seen looking down the z-axis towards the origin. With the choice of upward normals on S, then C is positively oriented w.r.t. this orientation of S. A parameterisation for C with this orientation is

$$C\ :\ \boldsymbol{r} = (2\cos\theta, 2\sin\theta, 4), \qquad \theta = 0 \quad \text{to} \quad \theta = 2\pi\ .$$

On C we observe, however, that $\boldsymbol{F} = \big((z-4)y, (z-4)x, 0\big) = (0,0,0)$ (as C lies in the plane $z=4$), so we find that

$$\begin{aligned} \therefore \quad \int_C \boldsymbol{F} \bullet d\boldsymbol{r} &= \int_0^{2\pi} \big((z-4)y, (z-4)x, 0\big) \bullet \boldsymbol{r}'(\theta)\;d\theta \\ &= \int_0^{2\pi} (0,0,0) \bullet \boldsymbol{r}'(\theta)\;d\theta = \int_0^{2\pi} 0\;d\theta = 0\ . \end{aligned}$$

6 **a.** Since f is odd on $(-\pi,\pi)$, its Fourier series is

$$S(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned} \text{with} \quad b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\;dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\;dx \quad \text{for } n \ge 1 \\ &= \frac{2}{\pi}\int_0^{\pi} x\frac{d}{dx}\left(\frac{-\cos nx}{n}\right)\;dx = \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \int \frac{\cos nx}{n}\cdot\frac{d}{dx}(x)\;dx\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] \\ &= \frac{2(-1)^{n+1}}{n} \end{aligned}$$

$$\text{so} \quad S(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx\ .$$

b. The 2π–periodic extension $\tilde{f}$ of f is piecewise continuous (its discontinuities are at odd multiples of π), it is differentiable everywhere except at points of discontinuity $a = (2p+1)\pi$ for $p \in \mathbb{Z}$, (where $\tilde{f}(a)$ is undefined), but there $D^+\tilde{f}(a) = 1$ and $D^-\tilde{f}(a) = 1$, so exist and are finite.
Hence by the Pointwise Convergence Theorem, for all $x \in \mathbb{R}$, the Fourier series $S(x)$ converges and

$$S(x) = \frac{1}{2}\left[\tilde{f}(x^-) + \tilde{f}(x^-)\right] .$$

where $\tilde{f}(x^-) = \lim_{y \to x^-} \tilde{f}(y)$ and $\tilde{f}(x^+) = \lim_{y \to x^+} \tilde{f}(y)$ are respectively the left and right hand limits of $\tilde{f}$ at x. Hence

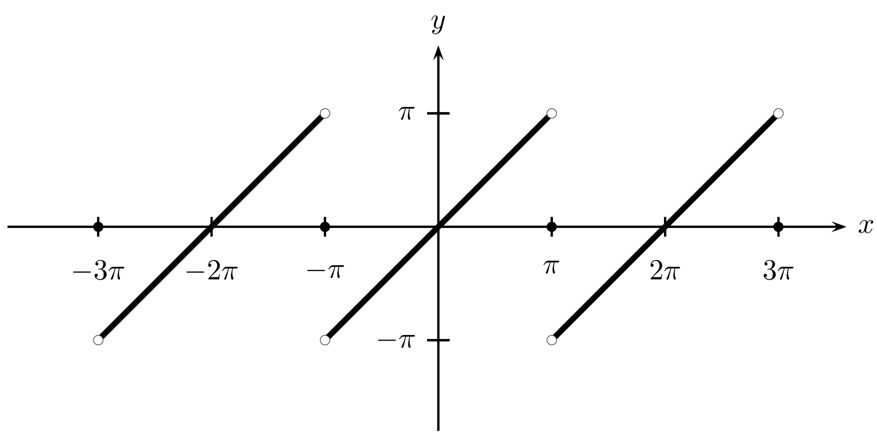
i)

$$S(\pi/2) = \frac{1}{2}\left[\tilde{f}((\pi/2)^-) + \tilde{f}((\pi/2)^+)\right] = f(\pi/2) = \frac{\pi}{2} .$$

ii)

$$S(\pi) = \frac{1}{2}\left[\tilde{f}(\pi^-) + \tilde{f}(\pi^+)\right] = \frac{1}{2}\left[\pi + (-\pi)\right] = 0 .$$

c.



d.

$$\begin{aligned}
\frac{\pi}{2} = f(\pi/2) = S(\pi/2) &= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{2}\right) \\
&= 2\sum_{k=0}^{\infty} \frac{(-1)^{2k+1+1}}{2k+1} \sin\left(\frac{(2k+1)\pi}{2}\right) \\
&\quad \text{(as the sine terms for } n \text{ even are all zero)} \\
&= 2\sum_{k=0}^{\infty} \frac{1}{2k+1} \cdot (-1)^k
\end{aligned}$$

$$\therefore \quad \frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Math2011 Several Variable Calculus — November 2002.

1 **a.** Let $\phi(x, y, z)$ be the function

$$\phi(x, y, z) = -2xy + x\ln(y + z).$$

i) Find $\nabla\phi$ at $(1, 3, -2)$, and hence find the parametric equation of the line normal to the surface $\phi(x, y, z) = -6$ at $(x, y, z) = (1, 3, -2)$.

ii) Find all points (x, y, z) such that $\nabla\phi = \mathbf{o}$ at (x, y, z).

iii) Find the directional derivative of ϕ at the point P with coordinates $(1, 3, -2)$ in the direction of the vector $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$

b. If $z = f(t)$ and $t = \dfrac{(x+y)}{xy}$, show that

$$x^2\frac{\partial z}{\partial x} = y^2\frac{\partial z}{\partial y}.$$

c. Use the method of Lagrange Multipliers to find the minimum value of $x + 2y$ subject to the conditions $xy = 2$, $x > 0$, $y > 0$.

2 **a.** For a lens with focal length f, the object distance u and image distance v are related by

$$\frac{1}{v} = \frac{1}{f} - \frac{1}{u}$$

If f is measured as 2 meters and u as 3 meters, each with an error of at most $\pm 0.5\%$, estimate the maximum possible percentage error in the calculated value of v.

b. The function $g : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ x^2 + y^2 \end{pmatrix} .$$

i) Find the Jacobian matrix $J\,g$.

ii) By using the chain rule, or otherwise, calculate $J(g \circ g)$.

c. Find the Taylor series expansion of $f(x, y) = \ln(1 + x + xy + y^2)$ about the point $(0, 0)$ including all terms up to second order.

3 **a.** i) Sketch the region of integration of

$$I = \int_0^1 \int_x^1 ye^{-y^3}\,dy\,dx$$

ii) Evaluate the integral in i) by changing the order of integration.

b. i) Write down iterated integrals in both Cartesian and spherical coordinates for $\int_\Omega f dV$ where

$$f(x, y, z) = z/\sqrt{x^2 + y^2 + z^2} \quad \text{and}$$

$$\Omega = \{(x, y, z) : 0 \le z \le \sqrt{4 - x^2 - y^2},\ 0 \le y \le x\}$$

ii) Evaluate one of the integrals in b) i) above.

c. Use the substitution $u = x^2 + y^2/4$, $v = y/x$ to evaluate $\int_\Omega f dA$ where

$$f(x, y) = x^2 + y^2/4 + y/x$$

and Ω is the region in the first quadrant bounded by $x^2+y^2/4 = 1$, $x^2+y^2/4 = 3$, $y = 2x$, $y = 2\sqrt{3}\,x$.

4 **a.** Let $\boldsymbol{F}$ be the vector field

$$\boldsymbol{F}(x,y,z) = (e^y,\ xe^y + \ln z,\ y/z)$$

i) Find a potential for $\boldsymbol{F}$.

ii) Calculate $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}$ where C is the curve $\boldsymbol{r}(t) = (\cos t, \sin t, 1+t), 0 \le t \le 2\pi$.

b. i) Let $\boldsymbol{G}$ be the vector field

$$\boldsymbol{G}(x,y,z) = (-yz,\ xz,\ x^2)$$

Calculate $\operatorname{curl} \boldsymbol{G}$.

ii) For a vector field $\boldsymbol{F}(\boldsymbol{x}) = (P(\boldsymbol{x}),\ Q(\boldsymbol{x}),\ R(\boldsymbol{x}))$ and a scalar field ϕ on $\mathbb{R}^3$, express $\operatorname{div}(\phi\boldsymbol{F})$ in terms of derivatives of ϕ and $\boldsymbol{F}$, where $\phi\boldsymbol{F}$ is defined as

$$(\phi\boldsymbol{F})(\boldsymbol{x}) = (\phi(\boldsymbol{x})P(\boldsymbol{x}),\ \phi(\boldsymbol{x})Q(\boldsymbol{x}),\ \phi(\boldsymbol{x})R(\boldsymbol{x}))$$

c. Find the flux $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$ out of the unit cube

$$\{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$$

where $\boldsymbol{F} = (xy,\ xz,\ z^2)$.

5 **a.** Let $\boldsymbol{F}(x,y) = (1,x)$ and let C be the curve $\{(x,y) : x^2/4 + y^2 = 1\}$ oriented counter clockwise.

i) Evaluate $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}$

ii) Is $\boldsymbol{F}$ conservative? Explain your answer.

b. State Stokes' theorem.

c. Let S be the portion of the surface

$$z = 1 + e^{(x+y)}$$

lying above $\{(x,y) : x \ge 0,\ y \ge 0,\ x+y \le 1\}$ and let $\boldsymbol{F}$ be the vector field $\boldsymbol{F}(x,y,z) = (y,\ z,\ x)$. Calculate $\int_S \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{n}\, dS$ where $\boldsymbol{n}$ is the upward pointing unit normal to S

i) directly, by parametrising S, and

ii) by evaluating an appropriate line integral.

6 Consider the function defined by

$$f(x) = \pi - x \quad \text{for} \quad 0 \le x \le \pi.$$

a. Suppose that f were to be extended to an even function $f_e(x)$ with period 2π. Thus $f_e(x) = f(x)$ for $0 \le x \le \pi$ and

$$f_e(-x) = f_e(x) \quad \text{and} \quad f_e(x+2\pi) = f_e(x) \quad \text{for all} \quad x$$

i) Sketch the graph of the function $f_e(x)$ for $-4\pi \le x \le 4\pi$.

ii) Find the coefficients in the cosine half-range expansion

$$f_e(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)\ .$$

iii) Deduce the value of

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$

b. Suppose now that we were to find the Fourier series of period 2π of the **odd** periodic extension $f_o(x)$ of f. (Note : You are **not** asked to actually determine this series). Sketch for $-4\pi \le x \le 4\pi$ the function to which this series would converge, indicating clearly the behaviour at any points of discontinuity.

Solutions to Math2011 November 2002 exam

1 **a.** $\phi(x, y, z) = -2xy + x\ln(y + z)$

i) $\nabla\phi = (-2y + \ln(y + z))\mathbf{i} + (-2x + \frac{x}{y+z})\mathbf{j} + \frac{x}{y+z}\mathbf{k}$

At $(1, 3, -2)$, $\nabla\phi = -6\mathbf{i} - \mathbf{j} + \mathbf{k}$.

Thus, if (x, y, z) are the coordinates of a typical point on the normal line:

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} - (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = \lambda(-6\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$\text{i.e.,} \quad x = 1 - 6\lambda,\ y = 3 - \lambda,\ z = -2 + \lambda.$$

ii) If $\nabla\phi = \mathbf{0}$, then $-2y + \ln(y + z) = 0$, $-2x + \dfrac{x}{y+z} = 0$ and $\dfrac{x}{y+z} = 0$.

The second two imply $x = 0$ and the first implies $2y = \ln(y + z)$, i.e. $z = -y + e^{2y}$.

Thus, the required points lie on the **curve** $z = -y + e^{2y}$ in the plane $x = 0$.

iii) The unit vector in the direction of $\mathbf{a}$ is

$$\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}| = (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})/\sqrt{14}.$$

The required directional derivative is then

$$(\nabla\phi)_p \cdot \hat{\mathbf{a}} = (-6\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot \hat{\mathbf{a}} = (-6 - 3 - 2)/\sqrt{14} = -11/\sqrt{14}$$

b.

$$\frac{\partial z}{\partial x} = \frac{df}{dt} \cdot \frac{\partial t}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{df}{dt} \cdot \frac{\partial t}{\partial y}.$$

$$\text{Since} \quad t = \frac{1}{x} + \frac{1}{y},\ \frac{\partial t}{\partial x} = -\frac{1}{x^2} \quad \text{and} \quad \frac{\partial t}{\partial y} = -\frac{1}{y^2}.$$

$$\therefore\ x^2\frac{\partial z}{\partial x} = y^2\frac{\partial z}{\partial y} \quad (= -f'(t)) \quad \text{as required.}$$

c. The Lagrangian for this situation is

$$\mathcal{L}(x, y, \lambda) = x + 2y - \lambda(xy - 2).$$

Hence setting to zero the derivatives of $\mathcal{L}$ with respect to x, y and λ gives

$$1 - \lambda y = 0,\ 2 - \lambda x = 0 \quad \text{and} \quad xy = 2.$$

Hence $y = \lambda^{-1}$ and $x = 2\lambda^{-1} = 2y$. Substituting this into the constraint gives $2y^2 = 2$, so $y = +1$ (as $y > 0$) and $x = 2$. Thus the minimum occurs at $(2, 1)$ and the minimum value is $2 + 2 \times 1 = 4$.

2 **a.**

$$\frac{1}{v} = \frac{1}{f} - \frac{1}{u}.$$

$$\therefore\ -\frac{1}{v^2}\frac{\partial v}{\partial f} = -\frac{1}{f^2} \quad \text{and} \quad -\frac{1}{v^2}\frac{\partial v}{\partial u} = \frac{1}{u^2}$$

$$\therefore\ \frac{\partial v}{\partial f} = \frac{v^2}{f^2} \quad \text{and} \quad \frac{\partial v}{\partial u} = -v^2/u^2.$$

Now,

$$\begin{aligned}\Delta v &\simeq \frac{\partial v}{\partial f}\Delta f + \frac{\partial v}{\partial u}\Delta u \\ &= \frac{v^2}{f^2}\Delta f - \frac{v^2}{u^2}\Delta u\end{aligned}$$

$\therefore$ We have, approximately,

$$|\Delta v| \le \frac{v^2}{f^2}|\Delta f| + \frac{v^2}{u^2}|\Delta u|.$$

The calculated value of v would come from

$$\frac{1}{v} = \frac{1}{f} - \frac{1}{u} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}, \text{ so } v = 6.$$

Further, $|\Delta f|/f \le \dfrac{1}{200}$ and $|\Delta u|/u \le \dfrac{1}{200}$. Thus $100|\Delta v|/v \le \dfrac{1}{2} \times 6 \times (\dfrac{1}{2} + \dfrac{1}{3}) = 5/2$.
Thus, the calculated value ($v = 6$) could be in error by as much as $\pm 2\frac{1}{2}\%$ (approximately)

b. i) $Jg = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} 2x & -2y \\ 2x & 2y \end{bmatrix}$.

ii) Using Chain rule:

$$\begin{aligned} J(g \circ g) &= J_g g \ J_g \\ &= \begin{bmatrix} 2(x^2 - y^2) & -2(x^2 + y^2) \\ 2(x^2 - y^2) & 2(x^2 + y^2) \end{bmatrix} \begin{bmatrix} 2x & -2y \\ 2x & 2y \end{bmatrix} \\ &= 4 \begin{bmatrix} x(x^2 - y^2) - x(x^2 + y^2) & -y(x^2 - y^2) - y(x^2 + y^2) \\ x(x^2 - y^2) + x(x^2 + y^2) & -y(x^2 - y^2) + y(x^2 + y^2) \end{bmatrix} \\ &= 8 \begin{bmatrix} -xy^2 & -yx^2 \\ x^3 & y^3 \end{bmatrix} \end{aligned}$$

Alternatively,

$$\begin{aligned} (g \circ g)\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{bmatrix} (x^2 - y^2)^2 - (x^2 + y^2)^2 \\ (x^2 - y^2)^2 + (x^2 + y^2)^2 \end{bmatrix} \\ &= \begin{bmatrix} -4x^2y^2 \\ 2x^4 + 2y^4 \end{bmatrix} \\ \therefore \ J(g \circ g) &= \begin{bmatrix} -8xy^2 & -8yx^2 \\ 8x^3 & 8y^3 \end{bmatrix} \end{aligned}$$

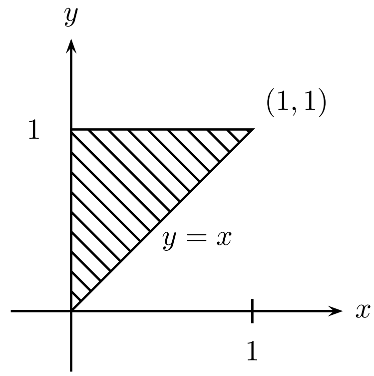
c.

$$\begin{aligned} f(x, y) &= \ln(1 + x + xy + y^2); & f(0,0) &= 0 \\ f_x &= \frac{1 + y}{1 + x + xy + y^2}; & f_x(0,0) &= 1 \\ f_y &= \frac{x + 2y}{1 + x + xy + y^2}; & f_y(0,0) &= 0 \\ f_{xx} &= \frac{-(1 + y)^2}{(1 + x + xy + y^2)^2}; & f_{xx}(0,0) &= -1 \\ f_{xy} &= \frac{(1 + x + xy + y^2)1 - (1 + y)(x + 2y)}{(1 + x + xy + y^2)^2}; & f_{xy}(0,0) &= 1 \\ f_{yy} &= \frac{2(1 + x + x + y^2) - (x + 2y)^2}{(1 + x + xy + y^2)^2}; & f_{yy}(0,0) &= 2. \end{aligned}$$

$$\therefore \quad f(x, y) = 0 + 1(x) + 0(y) + \frac{1}{2!}\{(-1)x^2 + (2 \times 1)xy + 2y^2\} + .$$

i.e. $\ln(1 + x + xy + y^2) = x - \frac{1}{2}x^2 + xy + y^2 + \cdots$

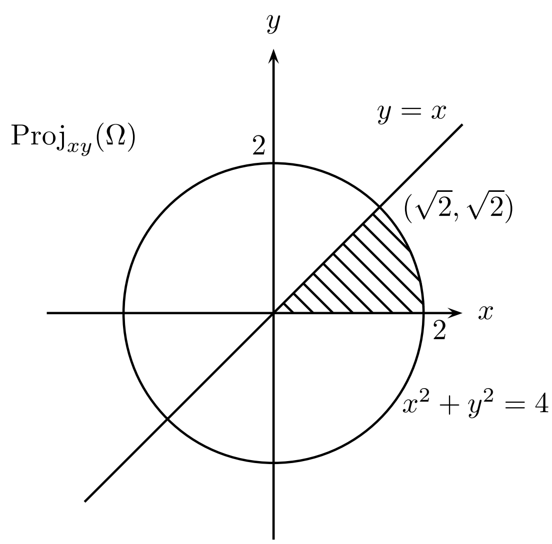
3 a. i)

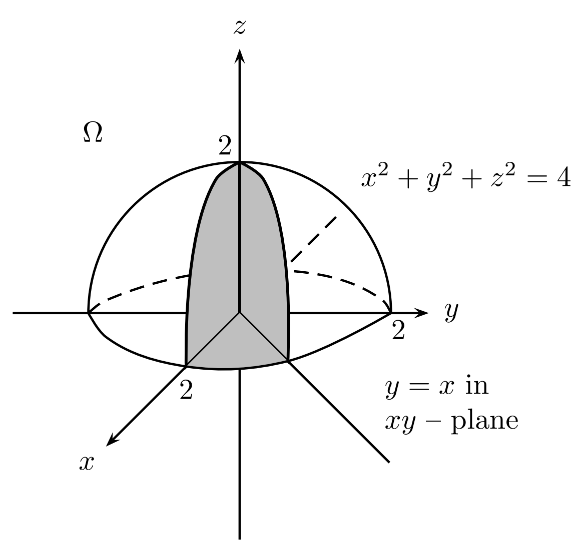


ii)

$$
\begin{aligned}
I &= \int_0^1 \int_x^1 y e^{-y^3}\, dy\, dx \\
&= \int_0^1 \int_0^y y e^{-y^3}\, dx\, dy = \int_0^1 y^2 e^{-y^3}\, dy \\
&= \left[-\frac{1}{3} e^{-y^3} \right]_0^1 = \frac{1}{3}(1 - e^{-1}) \ .
\end{aligned}
$$

b. i) & ii)
[The following sketches were not required in the solution]





$$
\begin{aligned}
I = \int_\Omega f\, dV &= \int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{z}{(x^2 + y^2 + z^2)^{1/2}}\, dz\, dx\, dy \\
&= \int_0^{\pi/4} \int_0^{\pi/2} \int_0^2 \frac{\rho \cos\phi}{\rho} \cdot \rho^2 \sin\phi \ d\rho d\phi d\theta \\
&= \frac{\pi}{4} \left[\frac{\rho^3}{3} \right]_0^2 \int_0^{\pi/2} \sin\phi \cos\phi \ d\phi = \frac{\pi}{4} \cdot \frac{8}{3} \left[\frac{1}{2} \sin^2\phi \right]_0^{\pi/2} \\
&= \frac{\pi}{4} \cdot \frac{8}{3} \cdot \frac{1}{2}(1 - 0) = \pi/3 \ .
\end{aligned}
$$

c. Need Jacobian determinant

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & y/2 \\ -y/x^2 & 1/x \end{vmatrix}$$

$$= 2 + y^2/2x^2 = 2 + v^2/2 = \frac{1}{2}(v^2+4).$$

Hence $\displaystyle \frac{\partial(x,y)}{\partial(u,v)} = \frac{2}{4+v^2}$

Now $$\int_\Omega f\,dx\,dy = \int_{\Omega^*} f \left|\frac{\partial(x,y)}{\partial(u,u)}\right| dv\,du$$

$$= \int_1^3 \int_2^{2\sqrt{3}} 2\frac{(u+v)}{4+v^2}\,dv\,du$$

$$= 2\int_1^3 \left\{ \left[\frac{u}{2}\tan^{-1}\frac{v}{2} + \frac{1}{2}\ln(4+v^2) \right]_2^{2\sqrt{3}} \right\} du$$

$$= \int_1^3 \left\{ u[\tan^{-1}\sqrt{3} - \tan^{-1}(1)] + \ln\left(\frac{4+12}{4+4}\right) \right\} du$$

$$= \int_1^3 \left\{ u\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + \ln 2 \right\} du$$

$$= \left[\frac{1}{2}u^2\frac{\pi}{12} - u\ln 2 \right]_1^3 = \frac{\pi}{24}(9-1) + 2\ln 2$$

$$= \frac{\pi}{3} + 2\ln 2\,.$$

4 a. i) $\mathbf{F} = \boldsymbol{\nabla}\phi \quad\Rightarrow\quad \phi_x = e^y;\ \phi_y = xe^y + \ln z;\ \phi_z = y/z.$
Integrating the first $\Rightarrow \quad \phi = xe^y + H(y,z) \quad\Rightarrow\quad \phi_y = xe^y + H_y.$
Comparing with the second gives $H_y = \ln z$, so $H = y\ln z + G(z)$.
So $\phi = xe^y + y\ln z + G(z)$ which $\Rightarrow \quad \phi_z = y/z + G'(z)$.
Thus $G'(z) = 0$, so $G = C = 0$ w.l.o.g.
Hence, $\phi = xe^y + y\ln z$ will suffice.

ii)

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int \boldsymbol{\nabla}\phi\cdot d\mathbf{r} = \phi_{\text{end}} - \phi_{\text{start}}$$

$$= \phi(1,0,1+2\pi) - \phi(1,0,1) = 1+0-1-0 = 0\,.$$

b. i)

$$\boldsymbol{\nabla}\times\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -yz & xz & x^2 \end{vmatrix}$$

$$= \mathbf{i}(-x) - \mathbf{j}(2x+y) + \mathbf{k}(z+z) = -x\mathbf{i} - (2x+y)\mathbf{j} + 2z\mathbf{k}\,.$$

ii)

$$\boldsymbol{\nabla}\cdot(\phi\mathbf{F}) = \frac{\partial}{\partial x}(\phi P) + \frac{\partial}{\partial y}(\phi Q) + \frac{\partial}{\partial z}(\phi R)$$

$$= \phi_x P + P\phi_x + \phi_y Q + \phi Q_y + \phi_z R + R_z\phi$$

$$= (\nabla\phi)\cdot\mathbf{F} + \phi(\boldsymbol{\nabla}\cdot\mathbf{F})\,.$$

c. Use Gauss' Divergence Theorem

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,dV = \int_0^1\int_0^1\int_0^1 (y+2z)dx\,dy\,dz$$
$$= \int_0^1\int_0^1 (y+2z)dy\,dz = \int_0^1 \left\{\left[\frac{1}{2}y^2+2yz\right]_0^1\right\} dz$$
$$= \int_0^1 \left(\frac{1}{2}+2z\right) dz = \left[z^2+\frac{1}{2}z\right]_0^1 = 3/2.$$

5 **a.** i) Represent curve as $x = 2\cos\theta,\ y=\sin\theta;\ 0\le\theta<2\pi$.
So, on $\mathcal{C}$, $\mathbf{r} = 2\cos\theta\mathbf{i}+\sin\theta\mathbf{j}$,

$$\text{i.e.}\quad d\mathbf{r} = (-2\sin\theta\mathbf{i}+\cos\theta\mathbf{j})\ d\theta \quad\text{and}\quad \mathbf{F}\cdot d\mathbf{r} = (-2\sin\theta+2\cos^2\theta)\ d\theta$$

$$\therefore\qquad \oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}(-2\sin\theta+2\cos^2\theta)\ d\theta$$
$$= \int_0^{2\pi}(-2\sin\theta+1+\cos 2\theta)\ d\theta$$
$$= \left[2\cos\theta+\theta+\frac{1}{2}\sin 2\theta\right]_0^{2\pi} = 2\pi\ .$$

ii) If $\mathbf{F}$ were conservative, the integral would be zero since we are integrating around a closed curve. Hence $\mathbf{F}$ cannot be conservative.

b. Let S be a piecewise smooth orientable surface with boundary $\mathcal{C}$ (which is a piecewise smooth simple closed curve). Then if $\mathbf{F}$ is differentiable on a domain containing S,

$$\int_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\mathbf{n}\,dS = \oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$$

where $\mathbf{n}$ is a continuous non-vanishing normal to S and $\mathcal{C}$ is oriented positively with respect to $\mathbf{n}$.

c. i) Parametrise S as the set of all points $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+(1+e^{x+y})\mathbf{k}$ with

$$x\ge 0,\ y\ge 0,\ x+y\le 1.$$

Then a normal is

$$\mathbf{N} = \mathbf{r}_x\times\mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & e^{x+y}\\ 0 & 1 & e^{x+y}\end{vmatrix}$$
$$= -e^{x+y}\mathbf{i}-e^{x+y}\mathbf{j}+\mathbf{k}\ .$$

(**Note:** This is upward, as required)

$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ y & z & x\end{vmatrix} = -(\mathbf{i}+\mathbf{j}+\mathbf{k})$$

So, on S, $\boldsymbol{\nabla}\times\mathbf{F}\cdot\mathbf{N} = 2e^{x+y}-1$ and

$$\int_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^{1-x}(2e^{x+y}-1)\ dy\,dx$$
$$= \int_0^1 \left([2e^{x+y}-y]_0^{1-x}\right) dx = \int_0^1 (2e-(1-x)-2e^x)\,dx$$
$$= [2ex+\frac{1}{2}x^2-x-2e^x]_0^1 = 2e+\frac{1}{2}-1-2e+2 = 3/2\ .$$

ii) The boundary curve consists of 3 segments

C_1: From $(0,0,1)$ to $(1,0,1+e)$ along the curve $z = 1+e^x$ in the plane $y=0$.
C_2: From $(1,0,1+e)$ to $(0,1,1+e)$ along the curve $z = 1+e$ in the plane $x+y=1$.
C_3: From $(0,1,1+e)$ back to $(0,0,1)$ along the curve $z=1+e^y$ in the plane $x=0$.

Now $\mathbf{F}\cdot d\mathbf{r} = y\,dx + z\,dy + x\,dz$.

On $\mathbf{C_1}$: $\mathbf{r} = x\mathbf{i} + (1+x)\mathbf{k}$, $(y=0)$, $\quad 0 \le x \le 1$.

$$\therefore \quad \int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} x\,dz = \int_0^1 xe^x\,dx = [(x-1)e^x]_0^1 = 1\ .$$

On C_2 : $\mathbf{r} = (1-y)\mathbf{i} + y\mathbf{j} + (1+e)\mathbf{k}$. $\quad 0 \le y \le 1,\ \ x+y=1$.

$$\therefore \quad d\mathbf{r} = (-\mathbf{i}+\mathbf{j})dy$$

$$\therefore \quad \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (-y + (1+e))dy = \left[-\frac{1}{2}y^2 + (1+e)y\right]_0^1$$

$$= 1 + e - \frac{1}{2} = e + \frac{1}{2}\ .$$

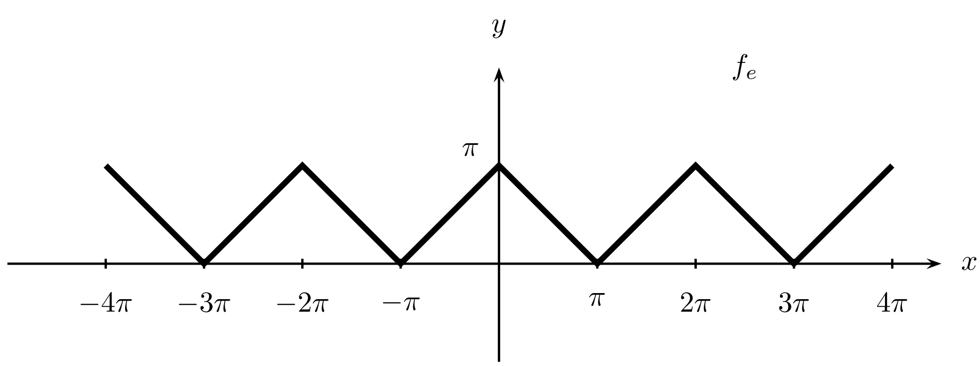
On C_3 : $\mathbf{r} = y\mathbf{j} + (1+e^y)\mathbf{k}$, $\quad x=0,\ 1 \ge y \ge 0$.

$$\therefore \quad \int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \int_1^0 z\,dy = \int_1^0 (1+e^y)dy$$

$$= [y+e^y]_1^0 = 0 + 1 - 1 - e = -e\ .$$

$$\therefore \quad \oint_C \mathbf{F}\cdot \mathbf{dr} = 1 + e + \frac{1}{2} - e = 3/2 \text{ (which agrees!)}$$

6 a. i)



ii)

$$f_e(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where $\quad a_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi f_e(x)\cos nx\ dx = \dfrac{2}{\pi}\int_0^\pi (\pi - x)\cos nx\,dx$

So $\quad a_0 = \dfrac{2}{\pi}\displaystyle\int_0^\pi (\pi - x)\,dx = \dfrac{2}{\pi}\left[\pi x - \dfrac{1}{2}x^2\right]_0^\pi = \dfrac{2}{\pi}[\pi^2 - \dfrac{1}{2}\pi^2] = \pi$

and for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx \\ &= \frac{2}{\pi}\left[\frac{(\pi - x)\sin\,nx}{n}\right]_0^{\pi} + \frac{2}{\pi}\int_0^{\pi}\frac{\sin nx}{n}dx \\ &= 0 - 0 + \frac{2}{\pi}\left[\frac{-\cos nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi n^2}[1 - \cos n\pi] \\ &= \frac{2}{\pi n^2}(1 - (-1)^n) = \frac{1}{\pi n^2} \times \begin{cases} 0: & \text{if } n \text{ is even} \\ 4: & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

$$\therefore \qquad f_e(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{\substack{n=0 \\ n \text{ odd}}}^{\infty}\frac{\cos nx}{n^2} = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}\ .$$

iii) Put $x = 0$. Since $f_e(0) = \pi$ and f_e continuous everywhere and $D^+f_e(0)$ and $D^-f_e(0)$ exist,

$$\pi = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \quad \Rightarrow \quad \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \pi^2/8\ .$$

c.