LINEAR FUNCTIONAL ANALYSIS

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Chapter 1

INTRODUCTION TO METRIC SPACES

1.1. Introduction

We begin by looking at two very familiar examples.

EXAMPLE 1.1.1. Consider a function $f : \mathbb{R} \to \mathbb{R}$. Suppose that f is continuous at a point $a \in \mathbb{R}$. Then given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

EXAMPLE 1.1.2. Consider a function $g : \mathbb{R}^2 \to \mathbb{R}$. Suppose that g converges to a limit L as (x_1, x_2) approaches a point $(a_1, a_2) \in \mathbb{R}^2$. Then given any $\epsilon > 0$, there exists $\delta > 0$ such that $|g(x_1, x_2) - L| < \epsilon$ whenever $0 < \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} < \delta$.

Let us now replace all the Greek from these two examples by plain English. In Example 1.1.1, we basically say that $f(x)$ can be made arbitrarily close to $f(a)$ provided that x is close enough to a. In Example 1.1.2, we basically say that $g(x_1, x_2)$ can be made arbitrarily close to L provided that (x_1, x_2) is close enough to (a_1, a_2) while remaining different from (a_1, a_2). Here what is important is not the algebraic nature of numbers in $\mathbb{R}$ or number pairs in $\mathbb{R}^2$, but the fact that distance from one point to another is well defined and has certain properties. We now generalize this concept of distance to an arbitrary set.

DEFINITION. Suppose that X is a non-empty set. By a metric or distance function on X, we mean a real valued function $\rho : X \times X \to \mathbb{R}$ which satisfies the following conditions:

(MS1) For every $\mathbf{x}, \mathbf{y} \in X$, we have $\rho(\mathbf{x}, \mathbf{y}) \geq 0$.
(MS2) For every $\mathbf{x}, \mathbf{y} \in X$, we have $\rho(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$.
(MS3) For every $\mathbf{x}, \mathbf{y} \in X$, we have $\rho(\mathbf{x}, \mathbf{y}) = \rho(\mathbf{y}, \mathbf{x})$.
(MS4) (TRIANGLE INEQUALITY) For every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$, we have $\rho(\mathbf{x}, \mathbf{z}) \leq \rho(\mathbf{x}, \mathbf{y}) + \rho(\mathbf{y}, \mathbf{z})$.

DEFINITION. By a metric space (X, ρ), we mean a non-empty set X together with a metric ρ.

EXAMPLE 1.1.3. In $\mathbb{R}$, we can define a metric $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by writing $\rho(x, y) = |x - y|$ for every $x, y \in \mathbb{R}$. It is easy to check that (MS1)–(MS4) are all satisfied. This is called the euclidean metric or usual metric on $\mathbb{R}$.

EXAMPLE 1.1.4. In $\mathbb{C}$, we can define a metric $\rho : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ by writing $\rho(z, w) = |z - w|$ for every $z, w \in \mathbb{C}$. It is easy to check that (MS1)–(MS4) are all satisfied. This is called the euclidean metric or usual metric on $\mathbb{C}$.

EXAMPLE 1.1.5. In $\mathbb{R}^2$, we can define a metric $\rho : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by writing

$$\rho(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

for every $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. It is easy to check that (MS1)–(MS3) are satisfied. To check (MS4), simply note that if we interpret the points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ as points on the complex plane $\mathbb{C}$, then we have $\rho(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$, the modulus of the complex number $\mathbf{x} - \mathbf{y}$. In this way, (MS4) is simply the Triangle inequality in $\mathbb{C}$.

EXAMPLE 1.1.6. Suppose that $r \in \mathbb{N}$. Then in $\mathbb{R}^r$, we can define a metric $\rho : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}$ by writing

$$\rho(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_r - y_r)^2}$$

for every $\mathbf{x} = (x_1, \ldots, x_r) \in \mathbb{R}^r$ and $\mathbf{y} = (y_1, \ldots, y_r) \in \mathbb{R}^r$. This metric is called the euclidean metric or usual metric on $\mathbb{R}^r$, and the metric space $(\mathbb{R}^r, \rho)$ is called the r-dimensional euclidean space.

Let us now look beyond euclidean spaces for a few other simple examples of metric spaces.

EXAMPLE 1.1.7. Suppose that X is a non-empty set. For every $\mathbf{x}, \mathbf{y} \in X$, write

$$\rho(\mathbf{x}, \mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{y}, \\ 1 & \text{if } \mathbf{x} \neq \mathbf{y}. \end{cases}$$

It is easy to check that (MS1)–(MS3) are satisfied. To check (MS4), note that the result is trivial if $\mathbf{x} = \mathbf{z}$. On the other hand, if $\mathbf{x} \neq \mathbf{z}$, then $\rho(\mathbf{x}, \mathbf{z}) = 1$. But then either $\mathbf{y} \neq \mathbf{x}$ or $\mathbf{y} \neq \mathbf{z}$, so that $\rho(\mathbf{x}, \mathbf{y}) + \rho(\mathbf{y}, \mathbf{z}) \geq 1$. This metric is known as the discrete metric.

EXAMPLE 1.1.8. Suppose that $r \in \mathbb{N}$. Consider the set $\mathbb{Z}_2^r$ of all ordered r-tuples with entries 0 or 1. For every $\mathbf{x} = (x_1, \ldots, x_r) \in \mathbb{Z}_2^r$ and $\mathbf{y} = (y_1, \ldots, y_r) \in \mathbb{Z}_2^r$, write

$$\rho(\mathbf{x}, \mathbf{y}) = \#\{i = 1, \ldots, r : x_i \neq y_i\}.$$

It is easy to check that (MS1)–(MS3) are satisfied. To check (MS4), note that $x_i \neq z_i$ implies that $x_i \neq y_i$ or $y_i \neq z_i$. This metric is known as the Hamming metric, and has applications in coding theory.

EXAMPLE 1.1.9. Consider the set $\mathbb{R}^2$. For every $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$, write

$$\rho_1(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2| \qquad \text{and} \qquad \rho_2(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

For ρ_1, it is easy to check that (MS1)–(MS4) are all satisfied. For ρ_2, it is easy to check that (MS1)–(MS3) are satisfied. To check (MS4), note that

$$\begin{aligned} |x_1 - z_1| &\leq |x_1 - y_1| + |y_1 - z_1| \leq \max\{|x_1 - y_1|, |x_2 - y_2|\} + \max\{|y_1 - z_1|, |y_2 - z_2|\} \\ &= \rho_2(\mathbf{x}, \mathbf{y}) + \rho_2(\mathbf{y}, \mathbf{z}). \end{aligned}$$

A similar argument gives $|x_2 - z_2| \leq \rho_2(\mathbf{x}, \mathbf{y}) + \rho_2(\mathbf{y}, \mathbf{z})$. Hence

$$\rho_2(\mathbf{x}, \mathbf{z}) = \max\{|x_1 - z_1|, |x_2 - z_2|\} \leq \rho_2(\mathbf{x}, \mathbf{y}) + \rho_2(\mathbf{y}, \mathbf{z})$$

as required. Observe also that

$$\rho_2(\mathbf{x},\mathbf{y}) \le \rho_1(\mathbf{x},\mathbf{y}) \le 2\rho_2(\mathbf{x},\mathbf{y}) \qquad \text{and} \qquad \rho(\mathbf{x},\mathbf{y}) \le \rho_1(\mathbf{x},\mathbf{y}) \le 2\rho(\mathbf{x},\mathbf{y}),$$

where ρ is the euclidean metric on $\mathbb{R}^2$.

EXAMPLE 1.1.10. Suppose that $r \in \mathbb{N}$. Then in $\mathbb{R}^r$, we can define metrics ρ_1 and ρ_2 in $\mathbb{R}^r$ by writing

$$\rho_1(\mathbf{x},\mathbf{y}) = |x_1 - y_1| + \ldots + |x_r - y_r| \qquad \text{and} \qquad \rho_2(\mathbf{x},\mathbf{y}) = \max\{|x_1 - y_1|, \ldots, |x_r - y_r|\}$$

for every $\mathbf{x} = (x_1, \ldots, x_r) \in \mathbb{R}^r$ and $\mathbf{y} = (y_1, \ldots, y_r) \in \mathbb{R}^r$. Observe also that

$$\rho_2(\mathbf{x},\mathbf{y}) \le \rho_1(\mathbf{x},\mathbf{y}) \le r\rho_2(\mathbf{x},\mathbf{y}) \qquad \text{and} \qquad \rho(\mathbf{x},\mathbf{y}) \le \rho_1(\mathbf{x},\mathbf{y}) \le r\rho(\mathbf{x},\mathbf{y}),$$

where ρ is the euclidean metric on $\mathbb{R}^r$.

EXAMPLE 1.1.11. Suppose that (X_1, d_1) and (X_2, d_2) are metric spaces. We can define metrics ρ, ρ_1 and ρ_2 on the set $X_1 \times X_2$ by writing, for every $\mathbf{x} = (x_1, x_2) \in X_1 \times X_2$ and $\mathbf{y} = (y_1, y_2) \in X_1 \times X_2$,

$$\rho(\mathbf{x},\mathbf{y}) = \sqrt{d_1^2(x_1, y_1) + d_2^2(x_2, y_2)},$$

and

$$\rho_1(\mathbf{x},\mathbf{y}) = d_1(x_1, y_1) + d_2(x_2, y_2) \qquad \text{and} \qquad \rho_2(\mathbf{x},\mathbf{y}) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

It can also be checked that

$$\rho_2(\mathbf{x},\mathbf{y}) \le \rho_1(\mathbf{x},\mathbf{y}) \le 2\rho_2(\mathbf{x},\mathbf{y}) \qquad \text{and} \qquad \rho(\mathbf{x},\mathbf{y}) \le \rho_1(\mathbf{x},\mathbf{y}) \le 2\rho(\mathbf{x},\mathbf{y}).$$

EXAMPLE 1.1.12. Suppose that $r \in \mathbb{N}$, and that $(X_1, d_1), \ldots, (X_r, d_r)$ are metric spaces. We can define metrics ρ, ρ_1 and ρ_2 on the set $X_1 \times \ldots \times X_r$ by writing, for every $\mathbf{x} = (x_1, \ldots, x_r) \in X_1 \times \ldots \times X_r$ and $\mathbf{y} = (y_1, \ldots, y_r) \in X_1 \times \ldots \times X_r$,

$$\rho(\mathbf{x},\mathbf{y}) = \sqrt{d_1^2(x_1, y_1) + \ldots + d_r^2(x_r, y_r)},$$

and

$$\rho_1(\mathbf{x},\mathbf{y}) = d_1(x_1, y_1) + \ldots + d_r(x_r, y_r) \qquad \text{and} \qquad \rho_2(\mathbf{x},\mathbf{y}) = \max\{d_1(x_1, y_1), \ldots, d_r(x_r, y_r)\}.$$

It can also be checked that

$$\rho_2(\mathbf{x},\mathbf{y}) \le \rho_1(\mathbf{x},\mathbf{y}) \le r\rho_2(\mathbf{x},\mathbf{y}) \qquad \text{and} \qquad \rho(\mathbf{x},\mathbf{y}) \le \rho_1(\mathbf{x},\mathbf{y}) \le r\rho(\mathbf{x},\mathbf{y}).$$

We next turn our attention to metric spaces arising from certain types of sequences.

EXAMPLE 1.1.13. Consider the set ℓ^∞ of all bounded infinite sequences of complex numbers. For any two such sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ and $\mathbf{y} = (y_1, y_2, y_3, \ldots)$, write

$$\rho(\mathbf{x},\mathbf{y}) = \sup_{i \in \mathbb{N}} |x_i - y_i|.$$

It is easy to check that (MS1)–(MS3) are satisfied. To check (MS4), note that for every $i \in \mathbb{N}$, we have $|x_i - z_i| \le |x_i - y_i| + |y_i - z_i| \le \rho(\mathbf{x},\mathbf{y}) + \rho(\mathbf{y},\mathbf{z})$. The result follows on taking the supremum of the left hand side over $\mathbb{N}$.

EXAMPLE 1.1.14. Consider the set ℓ^2 of all infinite sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ of complex numbers such that the series

$$\sum_{i=1}^{\infty} |x_i|^2$$

is convergent. For any two such sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ and $\mathbf{y} = (y_1, y_2, y_3, \ldots)$, write

$$\rho_2(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^2 \right)^{1/2}.$$

To see that the series on the right hand side is convergent, note that $|x_i - y_i|^2 \le 2(|x_i|^2 + |y_i|^2)$ for every $i \in \mathbb{N}$, and apply the Comparison test. It is easy to check that (MS1)–(MS3) are satisfied. The proof of (MS4) is omitted here.

EXAMPLE 1.1.15. Suppose that $p \in \mathbb{N}$. Consider the set ℓ^p of all infinite sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ of complex numbers such that the series

$$\sum_{i=1}^{\infty} |x_i|^p$$

is convergent. We can define a metric ρ_p on ℓ^p by writing

$$\rho_p(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p}$$

for any two sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ and $\mathbf{y} = (y_1, y_2, y_3, \ldots)$ in ℓ^p. We omit the details here.

Our last set of examples arise from certain types of functions.

EXAMPLE 1.1.16. Consider the set $B[a, b]$ of all bounded complex valued functions defined on an interval $[a, b]$. For every $f, g \in B[a, b]$, write

$$\rho(f, g) = \sup_{t \in [a,b]} |f(t) - g(t)|.$$

It is easy to check that (MS1)–(MS3) are satisfied. To check (MS4), note that for every $t \in [a, b]$, we have $|f(t) - h(t)| \le |f(t) - g(t)| + |g(t) - h(t)| \le \rho(f, g) + \rho(g, h)$. The result follows on taking the supremum of the left hand side over $[a, b]$.

EXAMPLE 1.1.17. Consider the set $C[a, b]$ of all continuous complex valued functions defined on an interval $[a, b]$. This is a subset of $B[a, b]$ discussed in Example 1.1.16, and clearly forms a metric space with the metric ρ discussed previously. On the other hand, for every $f, g \in C[a, b]$, write

$$\rho_1(f, g) = \int_a^b |f(t) - g(t)| \, \mathrm{d}t,$$

noting that any continuous complex valued function on a closed interval is Riemann integrable over the interval. It is easy to check that (MS1) and (MS3) are satisfied. For (MS2), see Problem 8. To check (MS4), note that for every $t \in [a, b]$, we have

$$|f(t) - h(t)| \le |f(t) - g(t)| + |g(t) - h(t)|.$$

The result follows on integrating over the interval $[a, b]$. We can also define a metric ρ_2 on $C[a, b]$ by writing

$$\rho_2(f, g) = \left(\int_a^b |f(t) - g(t)|^2 \, \mathrm{d}t \right)^{1/2}$$

for every $f, g \in C[a, b]$. Indeed, for every $p \in \mathbb{N}$, we can define a metric ρ_p on $C[a, b]$ by writing

$$\rho_p(f, g) = \left(\int_a^b |f(t) - g(t)|^p \, \mathrm{d}t \right)^{1/p}$$

for every $f, g \in C[a, b]$. We omit the details here.

EXAMPLE 1.1.18. We can generalize Example 1.1.16 by replacing the real interval by an arbitrary set. More precisely, suppose that X is a given non-empty set. Consider the set $B(X)$ of all bounded complex valued functions defined on the set X. We can define a metric ρ on $B(X)$ by writing

$$\rho(f, g) = \sup_{\mathbf{x} \in X} |f(\mathbf{x}) - g(\mathbf{x})|,$$

for every $f, g \in B(X)$. An interesting special case is when $X = \mathbb{N}$; note that $B(\mathbb{N})$ is essentially ℓ^∞.

EXAMPLE 1.1.19. In fact, we can further generalize Example 1.1.18 by permitting the functions to take values in a metric space. More precisely, suppose that X is a given non-empty set, and that (Y, d) is a metric space. Consider the set $B(X, Y)$ of all bounded functions with domain X and codomain Y. We can define a metric ρ on $B(X, Y)$ by writing

$$\rho(f, g) = \sup_{\mathbf{x} \in X} d(f(\mathbf{x}), g(\mathbf{x})),$$

for every $f, g \in B(X, Y)$. However, to check the details for this last generalization, we need the notion of boundedness in a metric space which we have not yet defined. The subset $C(X, Y)$ of all bounded continuous functions with domain X and codomain Y also forms a metric space under the same metric. Again, we have not yet defined the notion of continuity in a metric space.

EXAMPLE 1.1.20. Suppose that $k \in \mathbb{N}$. Consider the set $C^k[a, b]$ of all k-times continuously differentiable complex valued functions defined on an interval $[a, b]$. We can define a metric ρ on $C^k[a, b]$ by writing

$$\rho(f, g) = \sup_{t \in [a,b]} \max\{|f(t) - g(t)|, |f'(t) - g'(t)|, \ldots, |f^{(k)}(t) - g^{(k)}(t)|\},$$

for every $f, g \in C^k[a, b]$. We omit the details here.

1.2. Convergence in a Metric Space

DEFINITION. A sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ of points in a metric space (X, ρ) is said to converge to $\mathbf{x} \in X$, denoted by $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$ or

$$\lim_{n \to \infty} \mathbf{x}_n = \mathbf{x},$$

if the sequence of non-negative real numbers $\rho(\mathbf{x}_n, \mathbf{x}) \to 0$ as $n \to \infty$; in other words, given any $\epsilon > 0$, there exists N such that $\rho(\mathbf{x}_n, \mathbf{x}) < \epsilon$ whenever $n > N$.

It is not difficult to show that the limit of a sequence, if it exists, is unique. Note also that the definition reduces to our familiar definition of convergence of a real sequence when (X, ρ) is $\mathbb{R}$ with the euclidean metric.

EXAMPLE 1.2.1. Consider a non-empty set X together with the discrete metric. Then the only convergent sequences are those which take the same value for all large n.

EXAMPLE 1.2.2. Consider the set $C[1,2]$ of all continuous complex valued functions defined on the interval $[1,2]$. For every $n \in \mathbb{N}$ and $t \in [1,2]$, write $f_n(t) = (1+t^n)^{1/n}$ and $f(t) = t$. Consider first of all the metric ρ given by

$$\rho(f,g) = \sup_{t\in[1,2]} |f(t) - g(t)|.$$

Note that

$$0 \le f_n(t) - f(t) = (1+t^n)^{1/n} - t \le (t^n + t^n)^{1/n} - t = t(2^{1/n} - 1) \le 2(2^{1/n} - 1)$$

for every $n \in \mathbb{N}$ and $t \in [1,2]$, so that $\rho(f_n, f) \le 2(2^{1/n} - 1) \to 0$ as $n \to \infty$. Consider next the metric ρ_p given by

$$\rho_p(f,g) = \left(\int_1^2 |f(t) - g(t)|^p \,\mathrm{d}t \right)^{1/p},$$

where $p \in \mathbb{N}$ is fixed. Since

$$\left(\int_1^2 |f(t) - g(t)|^p \,\mathrm{d}t \right)^{1/p} \le \left(\int_1^2 (\rho(f,g))^p \,\mathrm{d}t \right)^{1/p} = \rho(f,g) \left(\int_1^2 \mathrm{d}t \right)^{1/p} = \rho(f,g),$$

it follows that $\rho_p(f_n, f) \le \rho(f_n, f) \to 0$ as $n \to \infty$. Hence $f_n \to f$ as $n \to \infty$ under the metric ρ as well as under the metric ρ_p for any $p \in \mathbb{N}$. Finally, consider the discrete metric. Since the sequence $(f_n)_{n\in\mathbb{N}}$ is non-constant, it follows from Example 1.2.1 that $(f_n)_{n\in\mathbb{N}}$ does not converge under the discrete metric. Note that this example illustrates the important point that a convergent sequence under one metric may be divergent under a different metric.

1.3. Open Sets and Closed Sets

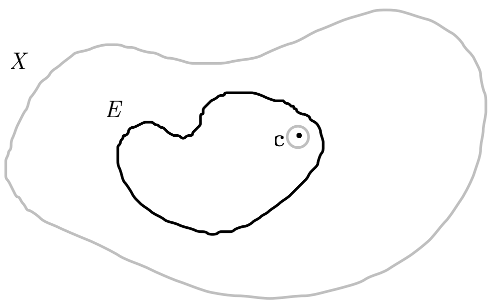
In this section, we fix the metric space (X, ρ) and may sometimes omit reference to the metric ρ.

DEFINITION. By an open ball $B(\mathbf{a}, r)$ with centre $\mathbf{a}$ and radius r in X, we mean a set of the form

$$B(\mathbf{a}, r) = \{\mathbf{x} \in X : \rho(\mathbf{a}, \mathbf{x}) < r\},$$

where $\mathbf{a} \in X$ and $r \in \mathbb{R}$ with $r > 0$.

DEFINITION. Suppose that $E \subseteq X$. A point $\mathbf{c} \in E$ is said to be an interior point of E if there exists $\epsilon > 0$ such that $B(\mathbf{c}, \epsilon) \subseteq E$.



DEFINITION. A set $G \subseteq X$ is said to be open (in X) if every point of G is an interior point of G.

DEFINITION. Suppose that $E \subseteq X$.
(1) A point $\mathbf{c} \in X$ is said to be an accumulation point of E if, for every $\epsilon > 0$, there exists $\mathbf{x} \in E$ such that $0 < \rho(\mathbf{c}, \mathbf{x}) < \epsilon$; in other words, every punctured open ball $B(\mathbf{c}, \epsilon) \setminus \{\mathbf{c}\}$ contains a point of E.
(2) A point $\mathbf{c} \in E$ is said to be an isolated point of E if it is not an accumulation point of E.

THEOREM 1A. *Suppose that (X, ρ) is a metric space. Suppose further that $E \subseteq X$. Then the following three statements are equivalent:*
(a) The point $\mathbf{c} \in X$ is an accumulation point of E.
(b) For every $\epsilon > 0$, the open ball $B(\mathbf{c}, \epsilon)$ contains infinitely many points of E.
(c) There exists a sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ of points in E such that $\mathbf{x}_n \neq \mathbf{c}$ and $\mathbf{x}_n \to \mathbf{c}$ as $n \to \infty$.

PROOF. It is clear that (c)$\Rightarrow$(b)$\Rightarrow$(a). To show that (a)$\Rightarrow$(c), note that since $\mathbf{c}$ is an accumulation point of E, it follows that for every $n \in \mathbb{N}$, there exists $\mathbf{x}_n \in E$ such that $0 < \rho(\mathbf{x}_n, \mathbf{c}) < 1/n$. (c) follows immediately. ♣

DEFINITION. A set $F \subseteq X$ is said to be closed (in X) if F contains all its accumulation points.

REMARK. We sometimes call a point $\mathbf{c} \in X$ a limit point of E if it is an accumulation point of E or an isolated point of E. Then $\mathbf{c} \in X$ is a limit point of E if and only if there exists a sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ of points in E such that $\mathbf{x}_n \to \mathbf{c}$ as $n \to \infty$. It is easily seen that a set $F \subseteq X$ is closed (in X) if and only if F contains all its limit points.

EXAMPLE 1.3.1. In $\mathbb{R}$ and $\mathbb{R}^2$ with the euclidean metric, an open ball is simply an open interval and an open disc respectively.

EXAMPLE 1.3.2. Every element of an open ball $B(\mathbf{a}, r)$ is an interior point of $B(\mathbf{a}, r)$, so that an open ball is open. To see this, suppose that $\mathbf{c} \in B(\mathbf{a}, r)$. We shall show that $B(\mathbf{c}, r - \rho(\mathbf{a}, \mathbf{c})) \subseteq B(\mathbf{a}, r)$. Suppose that $\mathbf{x} \in B(\mathbf{c}, r - \rho(\mathbf{a}, \mathbf{c}))$. Then $\rho(\mathbf{c}, \mathbf{x}) < r - \rho(\mathbf{a}, \mathbf{c})$. It follows from the Triangle inequality that $\rho(\mathbf{a}, \mathbf{x}) \leq \rho(\mathbf{a}, \mathbf{c}) + \rho(\mathbf{c}, \mathbf{x}) < r$, so that $\mathbf{x} \in B(\mathbf{a}, r)$.

EXAMPLE 1.3.3. Every point in a discrete metric space forms an open ball. Clearly $B(\mathbf{x}, 1) = \{\mathbf{x}\}$ for every $\mathbf{x} \in X$. Consequently, every set in a discrete metric space is open. On the other hand, no set in a discrete metric space has accumulation points, so that every set contains all its accumulation points (by default). Consequently, every set in a discrete metric space is closed.

EXAMPLE 1.3.4. In every metric space X, the sets $\emptyset$ and X are both open and closed. The set $\emptyset$ has no points, so that every point is an interior point (by default). On the other hand, it has no accumulation points, so that it contains all its accumulation points (by default). The set X contains everything, so contains all open balls, whence every point is an interior point. Clearly it also contains all its accumulation points.

EXAMPLE 1.3.5. A finite set in a metric space cannot have accumulation points. This follows immediately from Theorem 1A.

EXAMPLE 1.3.6. The set $\mathbb{N}$ has no accumulation points in $\mathbb{R}$ with the euclidean metric. This again follows immediately from Theorem 1A.

EXAMPLE 1.3.7. Consider the metric space $\mathbb{R}$ with the euclidean metric. Let $E = [0, 1)$. The set of all accumulation points of E is $[0, 1]$, so E is not closed. The set of all interior points of E is $(0, 1)$, so E is not open. On the other hand, recall that the sets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$. It follows that the set of all accumulation points of $\mathbb{Q}$ is $\mathbb{R}$ itself, while $\mathbb{Q}$ has no interior points.

REMARK. We say that a sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ in X is bounded if there exists an open ball $B(\mathbf{a}, r)$ such that $\mathbf{x}_n \in B(\mathbf{a}, r)$ for every $n \in \mathbb{N}$. We also say that a subset E of X is bounded if there exists an open ball $B(\mathbf{a}, r)$ such that $E \subseteq B(\mathbf{a}, r)$.

THEOREM 1B. *Suppose that (X,ρ) is a metric space.*
(a) The union of any collection of open sets in X is open (in X).
(b) The intersection of any finite collection of open sets in X is open (in X).

PROOF. (a) Suppose that $\mathcal{G}$ is a collection of open sets in X. Denote by U their union. Suppose that $\mathbf{x} \in U$. Then $\mathbf{x} \in G$ for some $G \in \mathcal{G}$. Since G is open, it follows that $\mathbf{x}$ is an interior point of G, and so there exists $\epsilon > 0$ such that $B(\mathbf{x},\epsilon) \subseteq G \subseteq U$. It follows that $\mathbf{x}$ is an interior point of U.

(b) Suppose that the open sets are $G_1,\ldots,G_n$. Denote by V their intersection. Suppose that $\mathbf{x} \in V$. Then $\mathbf{x} \in G_k$ for every $k = 1,\ldots,n$. Since G_k is open, it follows that $\mathbf{x}$ is an interior point of G_k, and so there exists $\epsilon_k > 0$ such that $B(\mathbf{x},\epsilon_k) \subseteq G_k$. Now let $\epsilon = \min\{\epsilon_1,\ldots,\epsilon_n\} > 0$. Then for every $k = 1,\ldots,n$, we have $B(\mathbf{x},\epsilon) \subseteq B(\mathbf{x},\epsilon_k) \subseteq G_k$, so that $B(\mathbf{x},\epsilon) \subseteq G_1 \cap \ldots \cap G_n = V$. It follows that $\mathbf{x}$ is an interior point of V. ♣

To obtain the analogous results for closed sets, we prove the following intermediate result.

THEOREM 1C. *Suppose that (X,ρ) is a metric space. A set $F \subseteq X$ is closed (in X) if and only if its complement $F' = X \setminus F$ is open (in X).*

PROOF. ($\Rightarrow$) Suppose that F is closed. Then F contains all its accumulation points, so that for every $\mathbf{x} \in F'$, $\mathbf{x}$ is not an accumulation point of F. Hence there exists $\epsilon > 0$ such that $B(\mathbf{x},\epsilon) \cap F = \emptyset$, so that $B(\mathbf{x},\epsilon) \subseteq F'$.

($\Leftarrow$) Suppose that $\mathbf{x} \notin F$. Then $\mathbf{x} \in F'$. Since F' is open, it follows that there exists $\epsilon > 0$ such that $B(\mathbf{x},\epsilon) \subseteq F'$, so that $B(\mathbf{x},\epsilon) \cap F = \emptyset$. Hence $\mathbf{x}$ is not an accumulation point of F. It now follows that F must contain all its accumulation points. ♣

The theorem below now follows immediately from Theorems 1B and 1C.

THEOREM 1D. *Suppose that (X,ρ) is a metric space.*
(a) The intersection of any collection of closed sets in X is closed (in X).
(b) The union of any finite collection of closed sets in X is closed (in X).

PROOF. Note simply De Morgan's law, that

$$\bigcap_{F\in\mathcal{F}} F = X \setminus \left(\bigcup_{F\in\mathcal{F}} (X \setminus F) \right)$$

for any collection $\mathcal{F}$ of sets in X. ♣

EXAMPLE 1.3.8. In $\mathbb{R}$ with the euclidean metric, the set $I_n = (-1/n, 1/n)$ is open for every $n \in \mathbb{N}$, but the intersection

$$\bigcap_{n=1}^{\infty} I_n = \{0\}$$

is not open.

EXAMPLE 1.3.9. In $\mathbb{R}$ with the euclidean metric, the set $J_n = [0,(n-1)/n]$ is closed for every $n \in \mathbb{N}$, but the union

$$\bigcup_{n=1}^{\infty} J_n = [0,1)$$

is not closed.

DEFINITION. Suppose that $E \subseteq X$.
(1) By the interior of E, we mean the set E° which contains precisely all the interior points of E.
(2) By the closure of E, we mean the set $\overline{E}$ which contains precisely all the points of E and all the accumulation points of E.

THEOREM 1E. *Suppose that (X, ρ) is a metric space, and that $E \subseteq X$. Then the interior E° is open (in X) and the closure $\overline{E}$ is closed (in X).*

PROOF. Suppose that $\mathbf{c} \in E^\circ$. Then there exists $\epsilon > 0$ such that $B(\mathbf{c}, \epsilon) \subseteq E$. Since $B(\mathbf{c}, \epsilon)$ is open, every point $\mathbf{x} \in B(\mathbf{c}, \epsilon)$ is an interior point of $B(\mathbf{c}, \epsilon)$ and hence of E. It follows that $B(\mathbf{c}, \epsilon) \subseteq E^\circ$, so that $\mathbf{c}$ is an interior point of E°. This proves the first assertion. The second assertion follows immediately from $(\overline{E})' = (E')^\circ$. The proof of this last identity is left as an exercise. ♣

DEFINITION. Suppose that $E \subseteq X$. We say that $\mathbf{c} \in X$ is a boundary point of E if, for every $\epsilon > 0$, the open ball $B(\mathbf{c}, \epsilon)$ contains at least one point of E and one point of E'; in other words, $B(\mathbf{c}, \epsilon) \cap E \neq \emptyset$ and $B(\mathbf{c}, \epsilon) \cap E' \neq \emptyset$. The set of all boundary points of E is denoted by ∂E.

REMARK. It can be shown that $\partial E = \overline{E} \setminus E^\circ = \overline{E} \cap \overline{E'}$.

EXAMPLE 1.3.10. In $\mathbb{R}$ with the euclidean metric, let $E = (0, 1]$. Then $E^\circ = (0, 1)$, $\overline{E} = [0, 1]$ and $\partial E = \{0, 1\}$.

EXAMPLE 1.3.11. In $\mathbb{Z}$ with the euclidean metric, let $E = B(c, 1)$. Then $E^\circ = \overline{E} = E$, and E has no boundary. Note that $E = \{c\}$.

1.4. Limits and Continuity

We now generalize our definition of limits of a function to functions whose domain and range lie in arbitrary metric spaces.

DEFINITION. Suppose that (X, ρ) and (Y, σ) are metric spaces, and that $E \subseteq X$. Suppose further that $\mathbf{c}$ is an accumulation point of E, and that $\mathbf{l} \in Y$. For any function $f : E \to Y$, we say that $f(\mathbf{x}) \to \mathbf{l}$ as $\mathbf{x} \to \mathbf{c}$, or

$$\lim_{\mathbf{x} \to \mathbf{c}} f(\mathbf{x}) = \mathbf{l},$$

if, given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sigma(f(\mathbf{x}), \mathbf{l}) < \epsilon \qquad \text{whenever } \mathbf{x} \in E \text{ and } 0 < \rho(\mathbf{x}, \mathbf{c}) < \delta.$$

REMARKS. (1) It does not matter whether the point $\mathbf{c}$ belongs to E or not.
(2) Note the special case when $X = Y = \mathbb{R}$ with ρ and σ the euclidean metric.

EXAMPLE 1.4.1. This example illustrates the importance of the subset E in the definition. Suppose that $X = Y = \mathbb{R}$ with the euclidean metric. Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

If $E = \mathbb{Q}$, then $f(x) \to 0$ as $x \to 0$. If $E = \mathbb{R} \setminus \mathbb{Q}$, then $f(x) \to 1$ as $x \to 0$. If $E = \mathbb{R}$, then $f(x)$ does not have a limit as $x \to 0$.

EXAMPLE 1.4.2. Let X be a set and ρ the discrete metric. Then for every $E \subseteq X$, the set E has no accumulation points. Hence there is no theory of limits of functions defined on E or on X.

THEOREM 1F. *Suppose that (X, ρ) and (Y, σ) are metric spaces, and that $E \subseteq X$. Suppose further that $\mathbf{c}$ is an accumulation point of E, and that $\mathbf{l} \in Y$. For any function $f : E \to Y$, we have $f(\mathbf{x}) \to \mathbf{l}$ as $\mathbf{x} \to \mathbf{c}$ if and only if $f(\mathbf{x}_n) \to \mathbf{l}$ as $n \to \infty$ for every sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ in $E \setminus \{\mathbf{c}\}$ such that $\mathbf{x}_n \to \mathbf{c}$ as $n \to \infty$.*

PROOF. The proof follows exactly the same lines as that for the corresponding result in real analysis. Suppose first of all that $f(\mathbf{x}) \to \mathbf{l}$ as $\mathbf{x} \to \mathbf{c}$. Then given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sigma(f(\mathbf{x}), \mathbf{l}) < \epsilon \qquad \text{whenever } \mathbf{x} \in E \text{ and } 0 < \rho(\mathbf{x}, \mathbf{c}) < \delta.$$

Let $(\mathbf{x}_n)_{n\in\mathbb{N}}$ be any sequence in $E \setminus \{\mathbf{c}\}$ such that $\mathbf{x}_n \to \mathbf{c}$ as $n \to \infty$. Then there exists $N \in \mathbb{R}$ such that

$$0 < \rho(\mathbf{x}_n, \mathbf{c}) < \delta \qquad \text{whenever } n > N.$$

Hence

$$\sigma(f(\mathbf{x}_n), \mathbf{l}) < \epsilon \qquad \text{whenever } n > N.$$

This shows that $f(\mathbf{x}_n) \to \mathbf{l}$ as $n \to \infty$.

Suppose next that $f(\mathbf{x}) \not\to \mathbf{l}$ as $\mathbf{x} \to \mathbf{c}$. Then there exists $\epsilon > 0$ such that for every $n \in \mathbb{N}$, there exists $\mathbf{x}_n \in E \setminus \{\mathbf{c}\}$ such that

$$0 < \rho(\mathbf{x}_n, \mathbf{c}) < \frac{1}{n} \qquad \text{and} \qquad \sigma(f(\mathbf{x}_n), \mathbf{l}) \ge \epsilon.$$

Clearly $\mathbf{x}_n \to \mathbf{c}$ as $n \to \infty$. However, it is not difficult to see that $f(\mathbf{x}_n) \not\to \mathbf{l}$ as $n \to \infty$. ♣

We next extend our definition of continuity to the same setting.

DEFINITION. Suppose that (X, ρ) and (Y, σ) are metric spaces, and that $\mathbf{c} \in X$. A function $f : X \to Y$ is said to be continuous at $\mathbf{c}$ if, given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sigma(f(\mathbf{x}), f(\mathbf{c})) < \epsilon \qquad \text{whenever } \rho(\mathbf{x}, \mathbf{c}) < \delta. \tag{1}$$

The function f is said to be continuous on X if f is continuous at every $\mathbf{c} \in X$.

REMARKS. (1) Note that (1) is equivalent to $f(B(\mathbf{c}, \delta)) \subseteq B(f(\mathbf{c}), \epsilon)$.
(2) If $\mathbf{c}$ is an accumulation point of X, then f is continuous at $\mathbf{c}$ if $f(\mathbf{x}) \to f(\mathbf{c})$ as $\mathbf{x} \to \mathbf{c}$.
(3) Note that f is continuous at every isolated point of X.

EXAMPLE 1.4.3. Suppose that (X, ρ) is a metric space, and that $\mathbf{a} \in X$. Consider the function $f : X \to \mathbb{R}$, where $\mathbb{R}$ has the euclidean metric, by letting $f(\mathbf{x}) = \rho(\mathbf{x}, \mathbf{a})$ for every $\mathbf{x} \in X$. Then f is continuous on X. To see this, note that $|f(\mathbf{x}) - f(\mathbf{c})| = |\rho(\mathbf{x}, \mathbf{a}) - \rho(\mathbf{c}, \mathbf{a})| \le \rho(\mathbf{x}, \mathbf{c})$ for every $\mathbf{x}, \mathbf{c} \in X$.

EXAMPLE 1.4.4. Any function defined on a metric space with the discrete metric is continuous on the metric space, since every point in the metric space is an isolated point.

THEOREM 1G. (CHAIN RULE) *Suppose that (X, ρ), (Y, σ) and (W, τ) are metric spaces. Suppose further that the function $f : X \to Y$ is continuous at the points $\mathbf{c} \in X$ and the function $g : Y \to W$ is continuous at the point $\mathbf{d} = f(\mathbf{c}) \in Y$. Then the function $g \circ f : X \to W$ is continuous at $\mathbf{c}$.*

PROOF. Since g is continuous at $\mathbf{d} = f(\mathbf{c})$, given any $\epsilon > 0$, there exists $\eta > 0$ such that

$$\tau(g(f(\mathbf{x})), g(f(\mathbf{c}))) < \epsilon \qquad \text{whenever } \sigma(f(\mathbf{x}), f(\mathbf{c})) < \eta.$$

Since f is continuous at $\mathbf{c}$, there exists $\delta > 0$ such that

$$\sigma(f(\mathbf{x}), f(\mathbf{c})) < \eta \qquad \text{whenever } \rho(\mathbf{x}, \mathbf{c}) < \delta.$$

It follows that

$$\tau((g \circ f)(\mathbf{x}), (g \circ f)(\mathbf{c})) < \epsilon \qquad \text{whenever } \rho(\mathbf{x}, \mathbf{c}) < \delta.$$

The result follows immediately. ♣

The next result is essentially a simple corollary of Theorem 1F.

THEOREM 1H. *Suppose that (X,ρ) and (Y,σ) are metric spaces, and that $\mathbf{c}\in X$. Then a function $f:X\to Y$ is continuous at $\mathbf{c}$ if and only if $f(\mathbf{x}_n)\to f(\mathbf{c})$ as $n\to\infty$ for every sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in X such that $\mathbf{x}_n\to\mathbf{c}$ as $n\to\infty$.*

We conclude this chapter by formulating continuity in terms of open sets and closed sets.

Definition. Suppose that (X,ρ) and (Y,σ) are metric spaces. For every set $E\subseteq Y$ and every function $f:X\to Y$, we denote by $f^{-1}(E)$ the pre-image of E, so that

$$f^{-1}(E)=\{\mathbf{x}\in X:f(\mathbf{x})\in E\}.$$

THEOREM 1J. *Suppose that (X,ρ) and (Y,σ) are metric spaces. For any function $f:X\to Y$, the following three statements are equivalent:*
(a) The function f is continuous on X.
(b) If G is open in Y, then $f^{-1}(G)$ is open in X.
(c) If F is closed in Y, then $f^{-1}(F)$ is closed in X.

Proof. ((a)⇒(b)) If $f^{-1}(G)$ is empty, then it is open. Otherwise, let $\mathbf{c}\in f^{-1}(G)$, so that $f(\mathbf{c})\in G$. Since G is open, there exists $\epsilon>0$ such that $B(f(\mathbf{c}),\epsilon)\subseteq G$, so that

$$\mathbf{y}\in G\qquad\text{whenever } \sigma(\mathbf{y},f(\mathbf{c}))<\epsilon.$$

Since f is continuous at $\mathbf{c}$, there exists $\delta>0$ such that

$$\sigma(f(\mathbf{x}),f(\mathbf{c}))<\epsilon\qquad\text{whenever } \rho(\mathbf{x},\mathbf{c})<\delta.$$

It follows that $f(\mathbf{x})\in G$ whenever $\rho(\mathbf{x},\mathbf{c})<\delta$. In other words, we have $\mathbf{x}\in f^{-1}(G)$ whenever $\mathbf{x}\in B(\mathbf{c},\delta)$. Hence $B(\mathbf{c},\delta)\subseteq f^{-1}(G)$, so that $f^{-1}(G)$ is open.

((b)⇒(a)) Let $\mathbf{c}\in X$. Given any $\epsilon>0$, the open ball $B(f(\mathbf{c}),\epsilon)$ is open in Y, so that its pre-image $f^{-1}(B(f(\mathbf{c}),\epsilon))$ is open in X. Since $\mathbf{c}\in f^{-1}(B(f(\mathbf{c}),\epsilon))$, it follows that there exists $\delta>0$ such that $B(\mathbf{c},\delta)\subseteq f^{-1}(B(f(\mathbf{c}),\epsilon))$. In other words, we have

$$f(\mathbf{x})\in B(f(\mathbf{c}),\epsilon)\qquad\text{whenever } \mathbf{x}\in B(\mathbf{c},\delta).$$

It follows that

$$\sigma(f(\mathbf{x}),f(\mathbf{c}))<\epsilon\qquad\text{whenever } \rho(\mathbf{x},\mathbf{c})<\delta,$$

and so f is continuous at $\mathbf{c}$. Note now that this argument is valid for every $\mathbf{c}\in X$. Hence f is continuous on X.

((b)⇔(c)) We simply take complements. Given any $E\subseteq Y$, we have

$$f^{-1}(E)\cap f^{-1}(E')=\emptyset\qquad\text{and}\qquad f^{-1}(E)\cup f^{-1}(E')=X.$$

We now appeal to Theorem 1C. ♣

Problems for Chapter 1

1. Suppose that X is a non-empty set, and that a function $\rho:X\times X\to\mathbb{R}$ satisfies the following two conditions:
   a) For every $\mathbf{x},\mathbf{y}\in X$, we have $\rho(\mathbf{x},\mathbf{y})=0$ if and only if $\mathbf{x}=\mathbf{y}$.
   b) For every $\mathbf{x},\mathbf{y},\mathbf{z}\in X$, we have $\rho(\mathbf{x},\mathbf{y})\le\rho(\mathbf{x},\mathbf{z})+\rho(\mathbf{y},\mathbf{z})$.

   Prove that ρ is a metric on X.

2. Suppose that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ and $(\mathbf{y}_n)_{n\in\mathbb{N}}$ are two sequences in a metric space (X,ρ) such that $\mathbf{x}_n \to \mathbf{x}$ and $\mathbf{y}_n \to \mathbf{y}$ as $n\to\infty$. Prove that the real sequence $(\rho(\mathbf{x}_n,\mathbf{y}_n))_{n\in\mathbb{N}}$ converges to $\rho(\mathbf{x},\mathbf{y})$ as $n\to\infty$.

3. a) For every $x,y\in\mathbb{R}$, let $\rho(x,y)=|x^2-y^2|$. Is ρ a metric on $\mathbb{R}$?
   b) For every $x,y\in[0,\infty)$, let $\rho(x,y)=|x^2-y^2|$. Is ρ a metric on $[0,\infty)$?

4. Suppose that (X,ρ) is a metric space.
   a) Prove that $|\rho(\mathbf{x},\mathbf{z})-\rho(\mathbf{y},\mathbf{z})|\le\rho(\mathbf{x},\mathbf{y})$ for every $\mathbf{x},\mathbf{y},\mathbf{z}\in X$.
   b) Prove that $|\rho(\mathbf{x},\mathbf{y})-\rho(\mathbf{z},\mathbf{w})|\le\rho(\mathbf{x},\mathbf{z})+\rho(\mathbf{y},\mathbf{w})$ for every $\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}\in X$.

5. Suppose that $X=\{z\in\mathbb{C}:|z|=1\}$ is the circle of radius 1 on the complex plane and centred at the origin. For any $z,w\in X$, let $\rho(z,w)=0$ if $z=w$, let $\rho(z,w)=\pi$ if $z=-w$, and let $\rho(z,w)$ denote the length of the shorter arc joining z and w if $z\ne\pm w$. Prove that ρ is a metric on X.

6. Let $C(\mathbb{R})$ denote the collection of all continuous and bounded complex valued functions defined on the set $\mathbb{R}$. For every $f,g\in C(\mathbb{R})$, let

$$\rho(f,g)=\sup_{\substack{x\in\mathbb{R}\\ h\in(0,1]}}\left|\int_x^{x+h}(f(t)-g(t))\,\mathrm{d}t\right|.$$

   Prove that ρ is a metric on C.

7. Suppose that (X,ρ) is a metric space. Prove that each of the following functions σ is a metric on the set X:
   a) For every $\mathbf{x},\mathbf{y}\in X$, $\sigma(\mathbf{x},\mathbf{y})=2\rho(\mathbf{x},\mathbf{y})$.
   b) For every $\mathbf{x},\mathbf{y}\in X$, $\sigma(\mathbf{x},\mathbf{y})=\min\{1,\rho(\mathbf{x},\mathbf{y})\}$.
   c) For every $\mathbf{x},\mathbf{y}\in X$, $\sigma(\mathbf{x},\mathbf{y})=\dfrac{\rho(\mathbf{x},\mathbf{y})}{1+\rho(\mathbf{x},\mathbf{y})}$.

8. Let C denote the set of all real valued functions continuous on $[0,1]$. For every $f,g\in C$, let

$$\rho(f,g)=\sup_{t\in[0,1]}|f(t)-g(t)|,$$

   and let

$$\sigma(f,g)=\left(\int_0^1|f(t)-g(t)|^2\,\mathrm{d}t\right)^{1/2}\qquad\text{and}\qquad\tau(f,g)=\int_0^1|f(t)-g(t)|\,\mathrm{d}t.$$

   a) Suppose that $f\ne g$. Then there exists $t_0\in[0,1]$ such that $f(t_0)\ne g(t_0)$. Show that there exists an interval $I\subseteq[0,1]$ such that $|f(t)-g(t)|\ge\frac{1}{2}|f(t_0)-g(t_0)|$ for every $t\in I$. Hence deduce that $\sigma(f,g)>0$ and $\tau(f,g)>0$.
   b) Prove that ρ, σ and τ are metrics on C.
   [HINT: In the case of σ, you will need Schwarz's inequality for integrals, that

$$\left(\int_0^1\phi(t)\psi(t)\,\mathrm{d}t\right)^2\le\left(\int_0^1\phi^2(t)\,\mathrm{d}t\right)\left(\int_0^1\psi^2(t)\,\mathrm{d}t\right).]$$

   c) Prove that for every $f,g\in C$, we have $\rho(f,g)\ge\sigma(f,g)\ge\tau(f,g)$.
   d) For every $n\in\mathbb{N}$, let

$$f_n(t)=\begin{cases}1-nt & \text{if } 0\le t\le n^{-1},\\ 0 & \text{if } n^{-1}<t\le 1,\end{cases}$$

   and let $f(t)=0$ for every $t\in[0,1]$.
   (i) Prove that $(f_n)_{n\in\mathbb{N}}$ converges to f as $n\to\infty$ in (C,σ).
   (ii) Does $(f_n)_{n\in\mathbb{N}}$ converge as $n\to\infty$ in (C,τ)? Justify your assertion.
   (iii) Prove that $(f_n)_{n\in\mathbb{N}}$ does not converge as $n\to\infty$ in (C,ρ).

e) For every $n \in \mathbb{N}$, let

$$g_n(t) = \begin{cases} n^{1/2}(1-nt) & \text{if } 0 \le t \le n^{-1}, \\ 0 & \text{if } n^{-1} < t \le 1. \end{cases}$$

(i) Prove that $(g_n)_{n\in\mathbb{N}}$ converges as $n \to \infty$ in (C,τ).
(ii) Prove that $(g_n)_{n\in\mathbb{N}}$ does not converge as $n \to \infty$ in (C,σ).
(iii) Does $(g_n)_{n\in\mathbb{N}}$ converge as $n \to \infty$ in (C,ρ)? Justify your assertion.

9. Give an example of a metric space (X,ρ) and two open balls $B(\mathbf{a}_1,r_1)$ and $B(\mathbf{a}_2,r_2)$ in X such that $B(\mathbf{a}_1,r_1) \subseteq B(\mathbf{a}_2,r_2)$ and $r_1 > r_2$.

10. Consider the metric space (X,ρ), where $X = [0,3)\cup[4,5]\cup(6,7)\cup\{8\}$ and ρ is the euclidean metric in $\mathbb{R}$ restricted to X. For each of the following subsets, indicate whether it is open and whether it is closed, and justify your assertions:
a) $[0,3)$ b) $[4,5)$ c) $(6,7)$ d) $\{8\}$
e) $[0,3)\cup[4,5)$ f) $[0,3)\cup(6,7)$ g) $(6,7)\cup\{8\}$ h) $[1,2)$
i) $(1,2)$ j) $[1,2]$

11. Prove that any rectangle of the form $(a,b)\times(c,d)$ is open in the set $\mathbb{R}^2$ with the euclidean metric.

12. Prove that in a metric space (X,ρ), a set $G \subseteq X$ is open if and only if it is a union of open balls in (X,ρ).

13. Is it possible for a metric space (X,ρ) to contain more than one point and be such that the only open sets are $\emptyset$ and X? Justify your assertion.

14. Prove that in any metric space (X,ρ), a closed ball $\{\mathbf{x} \in X : \rho(\mathbf{a},\mathbf{x}) \le r\}$ is closed.

15. Prove that any non-empty bounded and closed set of real numbers contains its supremum and infimum.

16. Suppose that X is an infinite set. Show that there exists a metric ρ on X such that there is an accumulation point in the metric space (X,ρ) by following the steps below.
a) It is obvious that X has a countable subset $Z = \{\mathbf{z}_0,\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3,\ldots\}$. Show that the function $\rho : X\times X \to \mathbb{R}$, defined by for every $\mathbf{x},\mathbf{y} \in X$ by

$$\rho(\mathbf{x},\mathbf{y}) = \rho(\mathbf{y},\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x}=\mathbf{y}, \\ |m^{-1}-n^{-1}| & \text{if } \mathbf{x}=\mathbf{z}_m \text{ and } \mathbf{y}=\mathbf{z}_n \text{ for some } m,n\in\mathbb{N}, \\ n^{-1} & \text{if } \mathbf{x}=\mathbf{z}_0 \text{ and } \mathbf{y}=\mathbf{z}_n \text{ for some } n\in\mathbb{N}, \\ 1 & \text{otherwise,} \end{cases}$$

forms a metric on X.
b) Show that $\mathbf{z}_0$ is an accumulation point in the metric space (X,ρ).

17. Consider the metric space $\mathbb{C}$ with the euclidean metric. For any subset $A \subseteq \mathbb{C}$ and every point $z \in \mathbb{C}$, let $\rho(z,A) = \inf\limits_{a\in A} |z-a|$.
a) Prove that if $z \in A$, then $\rho(z,A) = 0$.
b) Is the converse of part (a) true? Justify your assertion.
c) Prove that for fixed A, $\rho(z,A)$ is a continuous function of z.
d) Prove that $\rho(z,A) = 0$ if and only if $z \in \overline{A}$.

18. Let E be a set in a metric space (X,ρ).
    a) Prove that
$$E^\circ = \bigcup_{\substack{G\subseteq E\\ G \text{ open}}} G \qquad \text{and} \qquad \overline{E} = \bigcap_{\substack{F\supseteq E\\ F \text{ closed}}} F;$$
    in other words, E° is the union of all open sets which are contained in E, and $\overline{E}$ is the intersection of all closed sets containing E.
    b) Deduce that $(\overline{E})' = (E')^\circ$.

19. Let E be a set in a metric space (X,ρ). Show that the set E^* of all accumulation points of E is closed.
    [Hint: Let $\mathbf{c}$ be an accumulation point of E^*. Show that $\mathbf{c}$ is an accumulation point of E.]

20. a) Let E_1 and E_2 be sets in a metric space (X,ρ). Prove that $\overline{E_1 \cup E_2} = \overline{E_1} \cup \overline{E_2}$.
    b) Let $\mathcal{E}$ be any collection of sets in a metric space (X,ρ). Prove that
$$\overline{\bigcup_{E\in\mathcal{E}} E} \supseteq \bigcup_{E\in\mathcal{E}} \overline{E} \qquad \text{and} \qquad \overline{\bigcap_{E\in\mathcal{E}} E} \subseteq \bigcap_{E\in\mathcal{E}} \overline{E}.$$
    c) Give an example of an infinite collection $\mathcal{E}$ of sets in a metric space (X,ρ) such that
$$\overline{\bigcup_{E\in\mathcal{E}} E} \neq \bigcup_{E\in\mathcal{E}} \overline{E}.$$
    d) Give an example of two sets E_1 and E_2 in a metric space (X,ρ) such that $\overline{E_1 \cap E_2} \neq \overline{E_1} \cap \overline{E_2}$.
    e) Deduce the results for interiors corresponding to parts (a) and (b).
    [Hint: Use Problem 18(b) and the De Morgan laws.]

21. Suppose that (X,ρ) and (Y,σ) are metric spaces, and that $E \subseteq X$. Suppose further that $\mathbf{c}$ is an accumulation point of E, and that $\mathbf{d} \in Y$.
    a) Prove that if a function $f : E \to Y$ is such that $f(\mathbf{x}) \to \mathbf{d}$ as $\mathbf{x} \to \mathbf{c}$, then $\mathbf{d} \in \overline{f(E)}$.
    b) Suppose in addition that f is injective. Prove that $\mathbf{d}$ is an accumulation point of $f(E)$.

22. Suppose that (X,ρ) and (Y,σ) are metric spaces. Prove that a function $f : X \to Y$ is continuous at $\mathbf{c}$ if and only if, given any open set $N \subseteq Y$ such that $f(\mathbf{c}) \in N$, there exists an open set $M \subseteq X$ such that $\mathbf{c} \in M$ and $f(M) \subseteq N$.

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