CONTROL 3

1.-

a) $$\frac{\partial^2 f}{\partial x \partial y} - 3y\frac{\partial f}{\partial x} = xy \cdot f \qquad \text{con} \quad \begin{array}{l}\phi(0)=0\\ \phi'(0)=1\end{array}$$

$$f(x,y) = \phi(x^2+y^2)$$

$$\frac{\partial f}{\partial x} = \phi' \cdot 2x \qquad \frac{\partial f}{\partial y} = \phi' \cdot 2y \Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \phi'' \cdot 2y \cdot 2x$$

$$\Rightarrow \quad 4xy \cdot \phi'' - 3 \cdot 2xy\,\phi' = xy \cdot \phi$$

$$4\phi'' - 6\phi' - \phi = 0 \Rightarrow 4\lambda^2 - 6\lambda - 1 = 0 \Rightarrow \begin{array}{l}\lambda_1 = \frac{3+\sqrt{13}}{4}\\ \lambda_2 = \frac{3-\sqrt{13}}{4}\end{array}$$

$$\Rightarrow \phi(t) = A \cdot e^{\frac{1}{4}(3+\sqrt{13})t} + B\,e^{\frac{1}{4}(3-\sqrt{13})t}$$

Pero

$$\phi(0) = 0 \Rightarrow A + B = 0 \Rightarrow \boxed{A = -B}$$

$$\phi'(0) = 1 \Rightarrow \frac{1}{4}(3+\sqrt{13})A + \frac{1}{4}(3-\sqrt{13})B = 1$$

$$(\cancel{3}+\sqrt{13})A - (\cancel{3}-\sqrt{13})A = 4$$

$$2\sqrt{13}A = 4$$

$$\boxed{A = \frac{2}{\sqrt{13}} = -B}$$

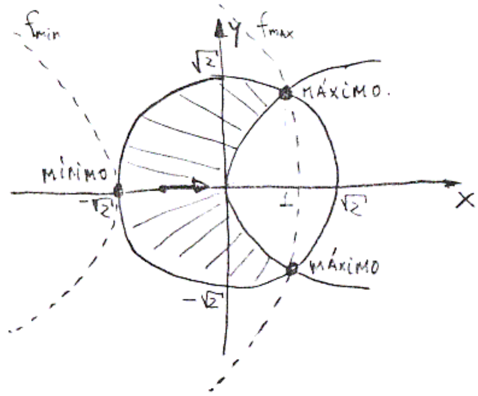
$$\phi(t) = \frac{2}{\sqrt{13}}\left[e^{\frac{1}{4}(3+\sqrt{13})t} - e^{\frac{1}{4}(3-\sqrt{13})t}\right]$$

$$\Rightarrow f(x,y) = \frac{2}{\sqrt{13}}\left[e^{\frac{1}{4}(3+\sqrt{13})(x^2+y^2)} - e^{\frac{1}{4}(3-\sqrt{13})(x^2+y^2)}\right]$$

1- b) $f(x,y) = 2x + y^2$ $\qquad K = \{(x,y) \;/\; x^2+y^2 \leq 2 \;;\; y^2 - x \geq 0\}$

$$\Downarrow$$

$$y^2 \geq x$$



$\nabla f = (2, 2y)$

$y^2 = -2x + f$

$y^2 = -2\left(x - \frac{f}{2}\right)$ Parábola vértice $\left(\frac{f}{2}, 0\right)$

$f_{mín}(-\sqrt{2}, 0) = 2 \cdot -\sqrt{2} + 0^2 = -2\sqrt{2}$

$f_{máx}(1,1) = f_{máx}(1,-1) = 2 \cdot 1 + (-1)^2 = 3$

Mínimo $= (-\sqrt{2}, 0)$ $\qquad$ Máximo $= \{(1,1) \;;\; (1,-1)\}$

$$\left.\begin{array}{c} x^2 + y^2 = 2 \\ y^2 = x \end{array}\right| \Rightarrow x^2 + x - 2 = 0$$

$$x = \frac{-1 \pm \sqrt{1+4\cdot 2}}{2} \begin{cases} -2 \Rightarrow y^2 = -2 \text{ No sirve!!} \\ \boxed{1} \Rightarrow y^2 = 1 \Rightarrow \boxed{\begin{array}{c} y = 1 \\ y = -1 \end{array}} \end{cases}$$

2.-

a) i) $f(x,y) = (x+y)^p \Rightarrow \nabla f = \left(p(x+y)^{p-1} \, ; \, p(x+1)^{p-1} \right)$

$$H_f = \begin{bmatrix} p(p-1)(x+y)^{p-2} & p(p-1)(x+y)^{p-2} \\ p(p-1)(x+y)^{p-2} & p(p-1)(x+y)^{p-2} \end{bmatrix} = p(p-1)(x+y)^{p-2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

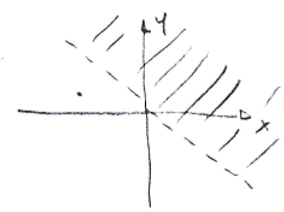
Valores propios $\lambda = 0, \lambda = 2$

Para que sea definida positiva

$$p(p-1)(x+y)^{p-2} > 0$$

Si p es par $p-2$ es par $\Rightarrow \boxed{\mathbb{R}^2}$.

si p es impar $\Rightarrow x+y>0 \Rightarrow \boxed{y>-x}$.



ii) $g(x,y) = e^{xy} \Rightarrow \nabla g = (y, x)\, e^{xy}$.

$$H_g = e^{xy} \begin{bmatrix} y^2 & 1+xy \\ 1+xy & x^2 \end{bmatrix}$$

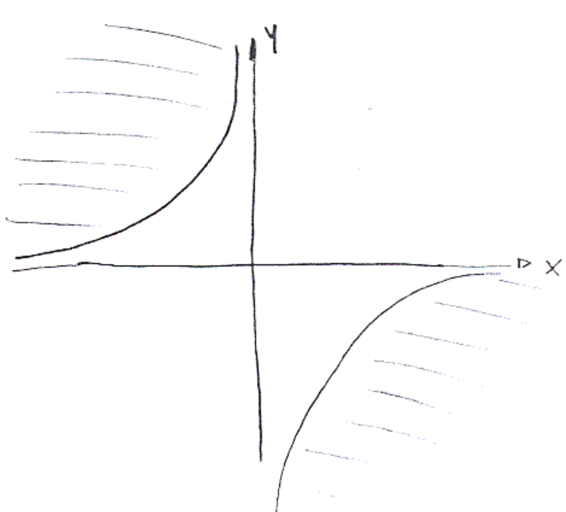
Def positiva $\Rightarrow y^2 > 0 \quad x^2y^2 - (1+xy)^2 > 0$

$$x^2y^2 - (1 + 2xy + x^2y^2) > 0$$

$$-1 - 2xy > 0$$

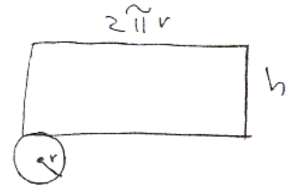
$$-2xy > 1$$

$$\boxed{xy < -\tfrac{1}{2}}$$



Control # 3

2- b) Se desea



$$\min \quad A(r,h) = 2\pi r h + \pi r^2$$

$$\text{sa} \qquad \pi r^2 \cdot h = V_0$$

$$L(r,h,\lambda) = 2\pi r h + \pi r^2 - \lambda(\pi r^2 h - V_0)$$

$$\frac{\partial L}{\partial r} = \cancel{2}\pi h + \cancel{2}\pi r - \cancel{2}\pi \lambda r h = 0 \Rightarrow \boxed{h + r = \lambda r h} \quad ①$$

$$\frac{\partial L}{\partial h} = 2\pi r - \lambda \pi r^2 = 0 \Rightarrow \boxed{2 = \lambda r} \quad ②$$

$$\frac{\partial L}{\partial \lambda} = -(\pi r^2 h - V_0) = 0 \Rightarrow \boxed{\pi r^2 h = V_0} \quad ③$$

$$\Rightarrow r = h \Rightarrow r^3 = \frac{V_0}{\pi} \quad \therefore \boxed{r = h = \sqrt[3]{\frac{V_0}{\pi}}}$$

c) $$\min \sum_{i=1}^{2} \sum_{j=1}^{3} c_{ij} x_{ij}$$

Sea x_{ij}: Cant de product que se fabrica en i y se envia a la ciuda j.

Restricciones.

Oferta $\sum_{j=1}^{3} x_{ij} \leq O_i \quad \forall i$

Demanda $\sum_{i=1}^{2} x_{ij} \leq D_j \quad \forall j$

No negatividad $x_{ij} \geq 0$

3-

a) $F:\mathbb{R}^n \to \mathbb{R}^n \qquad f:\mathbb{R}^n \to \mathbb{R}$

PDQ: $\quad \mathrm{div}(f\cdot F) = f\cdot \mathrm{div}(F) + \langle F, \nabla f\rangle$

Sea $F = (F_1, F_2, \dots, F_n)$

$$\Rightarrow \mathrm{div}(f\cdot F) = \sum_{i=1}^{n} \frac{\partial (f F_i)}{\partial x_i} = \sum_{i=1}^{n} f\cdot\frac{\partial F_i}{\partial x_i} + F_i\cdot\frac{\partial f}{\partial x_i}$$

$$= f\sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} + \sum_{i=1}^{n} F_i\cdot\frac{\partial f}{\partial x_i} = f\cdot\mathrm{div}(F) + \langle F, \nabla f\rangle$$

b) $F:\mathbb{R}^3\to\mathbb{R}^3 \qquad G:\mathbb{R}^3\to\mathbb{R}^3$

PDQ. $\mathrm{div}(F\times G) = \langle G, \mathrm{rot}(F)\rangle - \langle F, \mathrm{rot}(G)\rangle$

Sean $F = (F_1, F_2, F_3) \qquad G = (G_1, G_2, G_3)$

$$F\times G = (F_2G_3 - F_3G_2,\ F_3G_1 - F_1G_3,\ F_1G_2 - F_2G_1)$$

$$\Rightarrow \mathrm{div}(F\times G) = \frac{\partial(F_2G_3 - F_3G_2)}{\partial x} + \frac{\partial(F_3G_1 - F_1G_3)}{\partial y} + \frac{\partial(F_1G_2 - F_2G_1)}{\partial z}$$

$$= \left(\frac{\partial F_2}{\partial x}G_3 + F_2\frac{\partial G_3}{\partial x} - \frac{\partial F_3}{\partial x}G_2 - F_3\frac{\partial G_2}{\partial x}\right) + \left(\frac{\partial F_3}{\partial y}G_1 + F_3\frac{\partial G_1}{\partial y} - \frac{\partial F_1}{\partial y}G_3 - F_1\frac{\partial G_3}{\partial y}\right)$$

$$+ \left(\frac{\partial F_1}{\partial z}G_2 + F_1\frac{\partial G_2}{\partial z} - \frac{\partial F_2}{\partial z}G_1 - F_2\frac{\partial G_1}{\partial z}\right)$$

$$= G_1\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] + G_2\left[\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right] + G_3\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

$$- F_1\left[\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right] - F_2\left[\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right] - F_3\left[\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right]$$

$$= \langle G, \mathrm{rot}\,F\rangle - \langle F, \mathrm{rot}\,G\rangle$$

c) $F:\mathbb{R}^3\to\mathbb{R}^3$ de $C^2(\mathbb{R}^3)$ $\qquad$ PDQ $\quad \mathrm{div}(\mathrm{rot}\,F) = 0$

$$\mathrm{div}(\mathrm{rot}\,F) = \mathrm{div}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} + \frac{\partial^2 F_1}{\partial y\partial z} - \frac{\partial^2 F_3}{\partial y\partial x} + \frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_1}{\partial z\partial y}$$

$$= 0$$

Pero $\dfrac{\partial^2 F_i}{\partial x_k\partial x_j} = \dfrac{\partial^2 F_i}{\partial x_j\partial x_k}$

3- e)

i) $\phi_2' = \phi_3 \Rightarrow \phi_2' = \dfrac{1}{t+C_2} \Rightarrow \dfrac{d\phi_2}{dt} = \dfrac{1}{t+C_2}$

$$\phi_2 = \int \frac{dt}{t+C_2} + C_3 = \operatorname{Ln}(t+C_2) + C_3$$

$\therefore$ $\phi_1(t) = C_1 e^{2t}$

$\phi_2(t) = \dfrac{1}{t+C_2}$

$\phi_3(t) = \operatorname{Ln}(t+C_2) + C_3$

ii) $\phi(t) = (e^{2t}, \operatorname{Ln} t, 1/t)$

Es solución basta tomar $C_1 = 1$ $C_2 = 0$ $C_3 = 0$

iii) $F(x,y,z) = (2x, z, -z^2)$

$$\operatorname{div}(F) = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(-z^2) = 2 + 0 - 2z$$

$$\boxed{\operatorname{div}(F) = 2 - 2z}$$

$$\operatorname{rot}(F) = (0-1, 0-0, 0-0) = (-1, 0, 0)$$

$$\boxed{\operatorname{rot}(F) = (-1, 0, 0)}$$

3-

d) $f: \mathbb{R}^n \to \mathbb{R}$ $\quad g: \mathbb{R}^n \to \mathbb{R}$

- $\operatorname{div}(f \cdot \nabla g - g \nabla f) = f \Delta g - g \Delta f$

$$\operatorname{div}(f \nabla g - g \nabla f) = \operatorname{div}\left(f\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) - g\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\right) = \operatorname{div}\left(f \frac{\partial g}{\partial x} - g \frac{\partial f}{\partial x} , f \frac{\partial g}{\partial y} - g\right.$$

$$\Rightarrow = \frac{\partial}{\partial x}\left(f \cdot \frac{\partial g}{\partial x} - g \frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(f \frac{\partial g}{\partial y} - g \frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial x} + f \frac{\partial^2 g}{\partial x^2} - \cancel{\frac{\partial g}{\partial x} \frac{\partial f}{\partial x}} - g \frac{\partial^2 f}{\partial x^2} + \cancel{\frac{\partial f}{\partial y} \frac{\partial g}{\partial y}} + f \frac{\partial^2 g}{\partial y} - \cancel{\frac{\partial g}{\partial y} \frac{\partial f}{\partial y}} - g \frac{\partial^2 f}{\partial y^2}$$

$$= f\left[\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right] - g\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right] = f \cdot \Delta g - g \Delta f$$

$\blacksquare$

- $\Delta(f \cdot g) = f \Delta g + g \Delta f + 2 \langle \nabla f, \nabla g \rangle$

$$\Delta(f \cdot g) = \langle \nabla, \nabla(f \cdot g)\rangle = \langle \nabla, f \nabla g + \nabla f \cdot g \rangle$$

$$= \nabla f \cdot \nabla g + f \Delta g + \Delta f \, g + \nabla f \, \nabla g$$

$$= f \cdot \Delta g + g \Delta f + 2 \langle \nabla f, \nabla g \rangle$$

$\blacksquare$

e) $F(x,y,z) = (2x, z, -z^2)$

i) $\phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t))$ Si ϕ es línea de flujo

$$\phi'(t) = F(\phi(t)) \Rightarrow \begin{pmatrix} \phi_1' \\ \phi_2' \\ \phi_3' \end{pmatrix} = \begin{pmatrix} 2\phi_1 \\ \phi_3 \\ -\phi_3^2 \end{pmatrix}$$

$$\Rightarrow \quad \phi_1' = 2\phi_1 \Rightarrow \boxed{\phi_1(t) = c_1 e^{2t}}$$

$$\Rightarrow \quad \phi_3' = -\phi_3^2 \Rightarrow \frac{d\phi_3}{dt} = -\phi_3^2 \Rightarrow -\frac{d\phi_3}{\phi_3^2} = dt \Rightarrow \frac{1}{\phi_3} = t + c_2$$

$$\Rightarrow \quad \boxed{\phi_3(t) = \frac{1}{t + c_2}}$$