Control #2

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PAUTA

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1-

a)

i) $L(\mu,\sigma) = \prod_{i=1}^{n} f(x_i) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \prod_{i=1}^{n} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2} = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$

ii) $F = \mathrm{Ln}(L(\mu,\sigma)) = -n\,\mathrm{Ln}(\sqrt{2\pi}\,\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 = -n\,\mathrm{Ln}(\sqrt{2\pi}) - n\,\mathrm{Ln}(\sigma) - \frac{1}{2\sigma^2}\sum(x_i-\mu$

$$\frac{\partial F}{\partial \mu} = -\frac{1}{2\sigma^2}\sum_{i=1}^{n} 2(x_i-\mu)\cdot -1 = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)$$

$$\frac{\partial F}{\partial \mu} = 0 \implies \boxed{\hat{\mu} = \frac{\sum_{i=1}^{n} x_i}{n}}$$

$$\frac{\partial F}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial F}{\partial \sigma} = 0 \implies -\frac{n}{\hat{\sigma}} + \frac{1}{\hat{\sigma}^3}\sum_{i=1}^{n}(x_i-\hat{\mu})^2 = 0 \implies \boxed{\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i-\hat{\mu})^2}{n}}$$

$$H_F = \begin{bmatrix} -\frac{n}{\sigma^2} & -\frac{2}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu) \\ -\frac{2}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu) & \frac{n}{\sigma^2} - \frac{3}{\sigma^4}\sum(x_i-\mu)^2 \end{bmatrix}$$

Pero $n\hat{\mu} = \sum x_i$

$\implies \sum(x_i-\hat{\mu}) = \sum x_i - n\hat{\mu} = 0$

$$H_f(\hat{\mu},\hat{\sigma}) = \begin{bmatrix} -\frac{n}{\hat{\sigma}^2} & 0 \\ 0 & -\frac{2n}{\hat{\sigma}^2} \end{bmatrix}$$ definida negativa $\implies (\hat{\mu},\hat{\sigma})$ es un máximo

Control #2

1.-

b) • $f(x,y) = \dfrac{\sqrt{1-x^2-y^2}}{x^2+y^2}$ $\quad D = \{(x,y)\in\mathbb{R}^2 / \ x^2+y^2 \le 1 \quad (x,y)\ne(0,0)\}$

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x} = \dfrac{x(x^2+y^2-2)}{(x^2+y^2)^2\sqrt{1-x^2-y^2}} = 0 \Rightarrow x=0 \ \vee \ \boxed{x^2+y^2=2 \notin D} \\ \dfrac{\partial f}{\partial y} = \dfrac{y(x^2+y^2-2)}{(x^2+y^2)^2\sqrt{1-x^2-y^2}} = 0 \Rightarrow y=0 \ \vee \ \boxed{x^2+y^2=2 \notin D} \end{array}\right\} (x,y)=(0,0)\notin D$$

f no tiene puntos críticos en el interior de D.

Veamos el borde del conjunto D.

Sea $x^2+y^2=r^2 \Rightarrow f(x,y) = f(r) = \dfrac{\sqrt{1-r^2}}{r^2}$

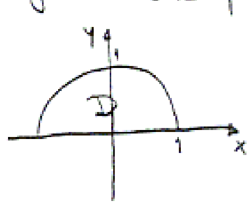
Cuando $(x,y)\to(0,0) \Rightarrow r^2\to 0 \Rightarrow f(r)\to+\infty \Rightarrow f$ <u>no</u> tiene máximo en su dominio.

Como $f(x,y)\ge 0$ y en $r=1$ $f(x,y)=0 \Rightarrow f$ tiene un mínimo en $x^2+y^2=1$ y vale 0.

• $g(x,y) = \dfrac{x\sqrt{1-x^2-y^2}}{y}$ $\quad D=\{(x,y)\in\mathbb{R}^2 / \ x^2+y^2\le 1 \ , y>0 \ , (x,y)\ne(0,0)\}$

$$\left.\dfrac{\partial g}{\partial x} = \dfrac{1-3x^2-y^2}{y\sqrt{1-x^2-y^2}} \qquad \dfrac{\partial g}{\partial y} = \dfrac{x(-1+x^2-y^2)}{y^2\sqrt{1-x^2-y^2}}\right\} \nabla g(x,y)=(0,0) \text{ No tiene solución}$$

g no tiene puntos críticos en D. Veamos el borde.



Para $x>0$ $\lim\limits_{y\to 0} g(x,y)\to+\infty \Rightarrow g$ <u>no</u> tiene máximo sobre D.

Para $x<0$ $\lim\limits_{y\to 0} g(x,y)\to-\infty \Rightarrow g$ <u>no</u> tiene mínimo sobre D.

Control#2

2.-

i) Como f, g son diferenciables, por regla de la cadena $\nabla(g\circ f)$ existe y vale.

$$\frac{\partial}{\partial x_i}(g\circ f)(x) = \sum_{j=1}^{m} \frac{\partial g}{\partial x_j}(f(x)) \cdot \frac{\partial f_j}{\partial x_i}(x)$$

ii) $g\circ f$ es de clase C^1

Tenemos que $\forall i$

$$\frac{\partial g\circ f}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial g}{\partial x_j}(f(x)) \frac{\partial f_j}{\partial x_i}(x) \quad \text{existe}$$

Como f y g son de clase C^1 $\Rightarrow$ $\frac{\partial f_i}{\partial x_i}, \frac{\partial g}{\partial x_i}$ son de clase C^0, es decir continuas.

Composición de continuas, suma y producto de continuas es continua.

$\Rightarrow$ $\frac{\partial}{\partial x_i}(g\circ f)$ es continua $\Rightarrow$ C^0 $\Rightarrow$ $(g\circ f)$ es de clase C^1

iii) Probemos que $(g\circ f)$ es de clase C^k.

$\Rightarrow$ a) $\frac{\partial g\circ f}{\partial x_i}$ existe

b) $\frac{\partial g\circ f}{\partial x_i}$ es de clase C^{k-1}

Como f y g son de clase C^k entonces existe.

$$\frac{\partial g\circ f}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial g}{\partial x_j}(f(x)) \cdot \frac{\partial f_j}{\partial x_i}(x)$$

Probemos que $\frac{\partial}{\partial x_i} g\circ f$ es de clase C^{k-1}

En efecto

Como $f \in C^k \Rightarrow \frac{\partial f_j}{\partial x_i} \in C^{k-1}$, $g \in C^k \Rightarrow \frac{\partial g}{\partial x_i} \in C^{k-1}$

$\Rightarrow \frac{\partial g}{\partial x_i}(f(x)) \in C^{k-1}$ por composición. Finalmente por suma y producto de C^{k-1}

$\Rightarrow (g\circ f) \in C^k$. Si $k=0$ es directo, ya que composición de funciones continuas es continua.

Control #2

3.-

i) $\displaystyle J_0(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\cos xt}{\sqrt{1-t^2}}\,dt$

PDQ $\quad J_0''(x) + \frac{1}{x} J_0'(x) + J_0(x) = 0 \quad \forall\, x>0$

Usando la regla de Leibniz se tiene

$$J_0'(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\partial}{\partial x}\left(\frac{\cos xt}{\sqrt{1-t^2}}\right) dt = \frac{1}{\pi}\int_{-1}^{1} \frac{-t\,\mathrm{sen}(xt)}{\sqrt{1-t^2}}\,dt$$

$$J_0''(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\partial}{\partial x}\left(\frac{-t\,\mathrm{sen}(xt)}{\sqrt{1-t^2}}\right) dt = \frac{1}{\pi}\int_{-1}^{1} \frac{-t^2\cos(xt)}{\sqrt{1-t^2}}\,dt$$

$$\Rightarrow J_0''(x) + \frac{1}{x}J_0'(x) + J_0(x) = \frac{1}{\pi}\int_{-1}^{1}\left[-t^2\cos xt - \frac{t}{x}\,\mathrm{sen}\,xt + \cos xt\right]\frac{1}{\sqrt{1-t^2}}\,dt$$

$$= \frac{1}{\pi}\int_{-1}^{1}\left(\sqrt{1-t^2}\,\cos(xt) - \frac{t\,\mathrm{sen}(xt)}{x\sqrt{1-t^2}}\right) dt$$

Calculemos

$$\int_{-1}^{1}\frac{-t\,\mathrm{sen}\,xt}{\sqrt{1-t^2}}\,dt = \overbrace{\mathrm{sen}(xt)\cdot\sqrt{1-t^2}\,\Big|_{-1}^{1}}^{0} - \int_{-1}^{1}\sqrt{1-t^2}\;x\cos xt\;dt$$

$u = \mathrm{sen}(xt) \longrightarrow du = x\cos xt\,dt$

$dv = \frac{-t}{\sqrt{1-t^2}}\,dt \longrightarrow v = \sqrt{1-t^2}$

Luego

$$J_0''(x) + \frac{1}{x}J_0'(x) + J_0(x) = \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}\cos(xt)\,dt + \frac{1}{\pi \cancel{x}}\cdot -\int \cancel{x}\sqrt{1-t^2}\cos(xt)\,dt = 0$$

Control #2

3-.

ii) $f(\vec{x}) = g\left(\sqrt{x_1^2+x_2^2+\dots+x_n^2}\right) = g(r) \qquad n \geq 3 \qquad g \in C^2(\mathbb{R})$

a) $$\frac{\partial f}{\partial x_k} = g'(r) \cdot \frac{\partial r}{\partial x_k}$$

$$\frac{\partial^2 f}{\partial x_k^2} = \frac{\partial r}{\partial x_k} \cdot g''(r) \cdot \frac{\partial r}{\partial x_k} + g'(r) \cdot \frac{\partial^2 r}{\partial x_k^2} = g''(r) \cdot \left(\frac{\partial r}{\partial x_k}\right)^2 + g'(r) \frac{\partial^2 r}{\partial x_k^2}$$

Pero

$$\frac{\partial r}{\partial x_k} = \frac{2x_k}{2\sqrt{x_1^2+\dots+x_n^2}} = \frac{x_k}{\sqrt{x_1^2+\dots+x_n^2}} \Rightarrow \left(\frac{\partial r}{\partial x_k}\right)^2 = \frac{x_k^2}{x_1^2+\dots+x_n^2}$$

$$\frac{\partial^2 r}{\partial x_k^2} = \frac{\sqrt{x_1^2+\dots+x_n^2} - \frac{x_k^2}{\sqrt{x_1^2+\dots+x_n^2}}}{x_1^2+\dots+x_n^2} = \frac{x_1^2+\dots+x_n^2 - x_k^2}{(x_1^2+\dots+x_n^2)^{3/2}} = \frac{r^2-x_k^2}{r^3}$$

Luego

$$\Delta f = \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2} = g''(r) \cdot \frac{\overbrace{\sum_{k=1}^{n} x_k^2}^{r^2}}{r^2} + g'(r) \sum_{k=1}^{n} \frac{r^2-x_k^2}{r^3}$$

$$= g''(r) + g'(r) \cdot \frac{nr^2 - \Sigma x_k^2}{r^3} = g''(r) + g'(r)\left(\frac{nr^2-r^2}{r^3}\right)$$

$$\boxed{\Delta f = g''(r) + g'(r)\left[\frac{n-1}{r}\right]}$$

b) $\Delta f = 0 \Rightarrow g''(r) + \frac{n-1}{r} g'(r) = 0$

$$\frac{d}{dr}\left(r^{n-1} \cdot g'(r)\right) = 0$$

$$r^{n-1} \cdot g'(r) = A$$

$$g'(r) = \frac{A}{r^{n-1}} \quad / \int$$

$$g(r) = \frac{A}{2-n} \frac{1}{r^{n-2}} + B$$

Sea $a = \frac{A}{2-n} \qquad r = \|x\|$

$$f(x) = g(\|x\|) = \frac{a}{\|x\|^{n-2}} + b \qquad x \neq 0$$

$\blacksquare$

Control #2 ⑥

4.-

a)

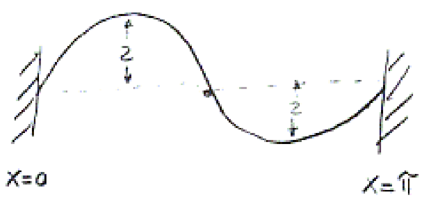
i) $\frac{\partial u}{\partial t}(x,t)$: velocidad vertical de un trozo de cuerda ubicado en x en el instante t.

$\frac{\partial^2 u}{\partial t^2}(x,t)$: aceleración vertical en la posición x en el instante t.

$\frac{\partial u}{\partial x}(x,t)$: pendiente de un trozo de cuerda en la posición x en el instante t.

ii) $u(x,t) = 2\,\text{sen}(2x)\cos(2\pi t)$

forma inicial $\Rightarrow t=0 \quad u(x,0) = 2\,\text{sen}(2x)$



velocidad $x=\pi/2$, $t=\frac{3}{4}$ $\Rightarrow \frac{\partial u}{\partial t} = -2\,\text{sen}(2x)\cdot 2\pi\,\text{sen}(2\pi t)$

$\Rightarrow \frac{\partial u}{\partial t}\left(\frac{\pi}{2},\frac{3}{4}\right) = -4\pi\,\text{sen}\left(2\cdot\frac{\pi}{2}\right)\text{sen}\left(2\pi\cdot\frac{3}{4}\right) = 0$

Pendiente $x=\pi/2$, $t=3$ $\Rightarrow \frac{\partial u}{\partial x} = 4\cos(2x)\cos(2\pi t)$

$\Rightarrow \frac{\partial u}{\partial x}\left(\frac{\pi}{2},3\right) = 4\cos\left(2\cdot\frac{\pi}{2}\right)\cos(2\pi\cdot 3) = -4$

Extremos : $u(0,t) = u(L,t) = 0$

iii) $\frac{\partial^2 u}{\partial t^2} = \pi^2 \frac{\partial^2 u}{\partial x^2}$

$\frac{\partial u}{\partial t} = -4\pi\,\text{sen}\,2x\,\text{sen}\,2\pi t \Rightarrow \frac{\partial^2 u}{\partial t^2} = -8\pi^2\,\text{sen}\,2x\cos 2\pi t$

$\frac{\partial u}{\partial x} = 4\cos 2x\cos 2\pi t \Rightarrow -8\,\text{sen}\,2x\cos 2\pi t = \frac{\partial^2 u}{\partial x^2}$

$\Bigg\} \Rightarrow \frac{\partial^2 u}{\partial x^2}\cdot\pi^2 = \frac{\partial^2 u}{\partial t^2}$ ▨

Control #2

4.-

b) $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ $\qquad \varphi, \psi : \mathbb{R} \to \mathbb{R}$

i) $u(x,t) = \varphi(x+ct) + \psi(x-ct)$ es solución de la ec. de ondas

$$\frac{\partial u}{\partial x} = \varphi' + \psi' \Rightarrow \frac{\partial^2 u}{\partial x^2} = \varphi'' + \psi''$$

$$\frac{\partial u}{\partial t} = \varphi' \cdot c - \psi' c = c(\varphi' - \psi') \Rightarrow \frac{\partial^2 u}{\partial t^2} = c(\varphi'' c + \psi'' c) = c^2(\varphi'' + \psi'')$$

Luego

$$c^2 \frac{\partial^2 u}{\partial x^2} = c^2 \cdot (\varphi'' + \psi'') = \frac{\partial^2 u}{\partial t^2}$$

ii) Si $u(x,t) = 2\,\mathrm{sen}(2x)\cos(2\pi t)$

Como $\mathrm{sen}\,\alpha + \mathrm{sen}\,\beta = 2\,\mathrm{sen}\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$$\begin{array}{l} \dfrac{\alpha+\beta}{2} = 2x \\ \dfrac{\alpha-\beta}{2} = 2\pi t \end{array} \Rightarrow \begin{array}{l} \alpha+\beta = 4x \\ \alpha-\beta = 4\pi t \end{array} \qquad \begin{array}{l} 2\alpha = 4x + 4\pi t \\ \boxed{\alpha = 2x + 2\pi t} \\ \boxed{\beta = 2x - 2\pi t} \end{array} \qquad \beta = \alpha - 4\pi t =$$

Luego

$$u(x,t) = \mathrm{sen}(2x + 2\pi t) + \mathrm{sen}(2x - 2\pi t)$$

$$= \mathrm{sen}(2(x+\pi t)) + \mathrm{sen}(2(x-\pi t))$$

$\therefore \ \varphi(x) = \mathrm{sen}\,2x \qquad \psi(x) = \mathrm{sen}\,2x$

Control #2

4.-

b)

iii) $\mathfrak{z} : \mathbb{R} \to \mathbb{R}$ $\quad u(x,0) = \mathfrak{z}(x)$ $\quad$ si $\dfrac{\partial u}{\partial t}(x,0) = 0$

PDQ $\quad u(x,t) = \dfrac{\mathfrak{z}(x+ct) + \mathfrak{z}(x-ct)}{2}$

Si $u(x,t) = \varphi(x+ct) + \psi(x-ct)$

$$\boxed{u(x,0) = \varphi(x) + \psi(x) = \mathfrak{z}(x)} \quad (1)$$

$$\frac{\partial u}{\partial t}(x,t) = c\,\varphi'(x+ct) - c\,\psi'(x-ct)$$

$$\frac{\partial u}{\partial t}(x,0) = c\left(\varphi'(x) - \psi'(x)\right) = 0 \implies \boxed{\varphi'(x) = \psi'(x)} \quad (2)$$

$(2) \Rightarrow \quad \varphi(x) = \psi(x) + A$

$\therefore \quad \psi(x) + A + \psi(x) = \mathfrak{z}(x) \implies 2\psi = \mathfrak{z} - A \Rightarrow \psi(x) = \dfrac{\mathfrak{z}(x) - A}{2}$

$\Rightarrow \quad \varphi(x) = \dfrac{\mathfrak{z}(x)}{2} + \dfrac{A}{2}$

$$\therefore \quad \boxed{u(x,t) = \frac{\mathfrak{z}(x+ct) + \mathfrak{z}(x-ct)}{2}}$$