Control #2

1

a) $$g(x) = \frac{x^t A x}{x^t x}$$

Calculemos

$$\nabla(x^t A x) = \nabla\left([x_1 \dots x_n] \begin{bmatrix} a_{11} & & a_{1n} \\ & & \\ a_{n1} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \\ x_n \end{bmatrix} \right) = \begin{pmatrix} \sum_{i=1}^n a_{1i} x_i + a_{i1} x_i \\ \vdots \\ \sum_{i=1}^n a_{ni} x_i + a_{in} x_i \end{pmatrix} = (A + A^t) x$$

Si A es simétrica $\Rightarrow \nabla(x^t A x) = (A + A^t) x = 2 A x$

Luego

$$\nabla g(x) = \nabla\left(\frac{x^t A x}{x^t x} \right) = \frac{x^t x \cdot \nabla(x^t A x) - x^t A x \cdot \nabla(x^t x)}{(x^t \cdot x)^2} = \frac{x^t \cdot x (A + A^t) x - x^t A x \cdot 2x}{(x^t \cdot x)^2}$$

Pero $\nabla g(x_0) = 0 \Rightarrow x_0^t \cdot x_0 (A + A^t) x_0 - x_0^t A x_0 \cdot 2 x_0 = 0$

$$\Rightarrow (A + A^t) x_0 = \frac{2 \cdot x_0^t A x_0 \cdot x_0}{x_0^t x_0} = 2 \cdot g(x_0) \cdot x_0$$

- $2 g(x_0) . \in \mathbb{R}$ es un valor propio, asociado al vector propio $x_0 \in \mathbb{R}^n$ de la matriz $(A + A^t)$
- Si A es simétrica $\Rightarrow A + A^t = 2A$

$$\Rightarrow \cancel{2} A x_0 = \cancel{2} g(x_0) \cdot x_0 \Rightarrow A x_0 = g(x_0) \cdot x_0$$

$g(x_0)$ v.p y x_0 $\vec{v.p.}$

Control #2

1|

b) $f(x) = \| Ax - b \|_2^2 \qquad A \in M_{m \times n} \qquad b \in \mathbb{R}^m \qquad x \in \mathbb{R}^n$

$$f(x) = (Ax-b)^t \circ (Ax-b)$$

$$\Rightarrow \quad \nabla f(x) = 2(Ax-b)^t \cdot A$$

$\nabla f(x)$ es continua $\Rightarrow$ $f(x)$ es diferenciable.

Control #2

2]

a) $F(x,y) = (-y, x)$ no es campo gradiente vectorial

Si F es un campo entonces $\exists f /\ F = \nabla f$ con f de clase C^2

$$\Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Luego $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} \Rightarrow$

$$F_1 = \frac{\partial f}{\partial x} \Rightarrow \frac{\partial F_1}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$F_2 = \frac{\partial f}{\partial y} \Rightarrow \frac{\partial F_2}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$\frac{\partial F_1}{\partial y} = -1 \qquad \frac{\partial F_2}{\partial x} = 1$, Como $\frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}$, F no es campo gradiente vectorial.

b) $F(x) = \xi \frac{Qq}{\|x\|_2^3} \cdot x \qquad f(x) = -\xi \frac{Qq}{\|x\|_2}$

Basta verificar que $\nabla f = F$

$$\nabla\left(-\xi \frac{Qq}{\|x\|_2}\right) = -\xi Qq \cdot \nabla\left(\frac{1}{\|x\|_2}\right) = -\xi Qq\, \nabla\left(\frac{1}{\sqrt{x^t x}}\right)$$

$$= -\xi Qq \left[\frac{-\frac{1}{2\sqrt{x^t x}} \cdot 2x}{x^t \cdot x} \right] = \xi Qq \frac{\cdot\ x}{(x^t \cdot x)^{3/2}}$$

$$\nabla f = \frac{\xi Qq}{\|x\|_2^3} \cdot x = F(x)$$

$\blacksquare$

Control #2

2|

c) $F(x,y) = (x,-y)$

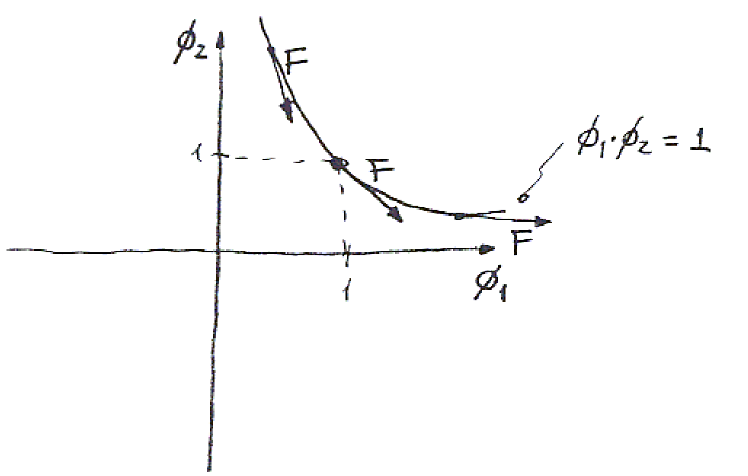
$$\phi(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix}$$ Si es linea de flujo de F

entonces $\phi'(t) = F(\phi(t))$

$$\Rightarrow \begin{bmatrix} \phi_1'(t) \\ \phi_2'(t) \end{bmatrix} = \begin{bmatrix} \phi_1(t) \\ -\phi_2(t) \end{bmatrix} \Rightarrow \begin{matrix} \phi_1' = \phi_1 & \Rightarrow & \phi_1(t) = Ae^{t} \\ \phi_2' = -\phi_2 & \Rightarrow & \phi_2(t) = Be^{-t} \end{matrix}$$

$$\therefore \phi(t) = \begin{pmatrix} Ae^{t} \\ Be^{-t} \end{pmatrix} \Rightarrow \left.\begin{matrix} \phi_1 = Ae^{t} \\ \phi_2 = \dfrac{B}{e^{t}} \end{matrix}\right\} \quad e^{t} = \frac{\phi_1}{A} \Rightarrow \phi_2 = \frac{B}{\frac{\phi_1}{A}}$$

$$\Rightarrow \phi_1 \cdot \phi_2 = AB \quad \text{(Hipérbola)}$$



Control #2

3] $f : \Omega \to \mathbb{R}$

a) f diferenciable en x_0 si $\exists\, G : \Omega \to \mathbb{R}^n$ continua en x_0 tal que

$$f(x) - f(x_0) = \langle G(x), x - x_0 \rangle$$

En efecto

Si f es diferenciable en x_0:

$$f(x_0 + h) = f(x_0) + \langle \nabla f(x_0), h \rangle + \varepsilon(h) \qquad \text{con } \lim_{h \to 0} \frac{\varepsilon(h)}{\|h\|_2} = 0$$

Sea $h = x - x_0$

$$\Rightarrow \quad f(x) = f(x_0) + \langle \nabla f(x_0), x - x_0 \rangle + \varepsilon(x - x_0) \qquad \text{con } \lim_{x \to x_0} \frac{\varepsilon(x - x_0)}{\|x - x_0\|_2} = 0$$

Sea $\varepsilon_1(x - x_0) = \dfrac{\varepsilon(x - x_0)}{\|x - x_0\|}$

$$\Rightarrow \quad f(x) - f(x_0) = \langle \nabla f(x_0), x - x_0 \rangle + \varepsilon_1(x - x_0) \cdot \|x - x_0\| \qquad \text{con } \lim_{x \to x_0} \varepsilon_1(x - x_0) = 0$$

$$= \left\langle \nabla f(x_0) + \frac{\varepsilon_1(x - x_0)}{\|x - x_0\|}(x - x_0) \,,\, x - x_0 \right\rangle$$

Sea

$$G(x) = \begin{cases} \nabla f(x_0) + \dfrac{\varepsilon_1(x - x_0)}{\|x - x_0\|}(x - x_0) & \text{si } x \neq x_0 \\ \nabla f(x_0) & \text{si } x = x_0 \end{cases}$$

$G(x)$ es continua en x_0.

$$\lim_{x \to x_0} \left\| G(x) - G(x_0) \right\| = \left\| \nabla f(x_0) + \frac{\varepsilon_1(x - x_0)}{\|x - x_0\|}(x - x_0) - \nabla f(x_0) \right\| = \left\| \frac{\varepsilon_1(x - x_0)}{\|x - x_0\|} \cdot (x - x_0) \right\|$$

$$\lim_{x \to x_0} \frac{\|\varepsilon_1(x - x_0)\| \cdot \cancel{\|x - x_0\|}}{\cancel{\|x - x_0\|}} = 0 \quad \Rightarrow \quad G(x) \text{ es continua en } x_0.$$

Control #2

3|

b) Sea $G: \Omega \to \mathbb{R}^n$ continua en x_0 tal que

$$f(x) - f(x_0) = \langle G(x), x - x_0 \rangle$$

Probemos que f es diferenciable en x_0.

Tenemos que:

$$f(x) - f(x_0) = \|x - x_0\| \left\langle G(x) - G(x_0), \frac{x - x_0}{\|x - x_0\|} \right\rangle + \langle G(x_0), x - x_0 \rangle$$

Queremos probar que

$$f(x) - f(x_0) = \langle \nabla f(x_0), x - x_0 \rangle + \varepsilon_1(x - x_0) \|x - x_0\| \quad \text{con } \lim_{x \to x_0} \varepsilon_1(x - x_0) = 0$$

Sea $\varepsilon_1(x - x_0) = \left\langle G(x) - G(x_0); \frac{x - x_0}{\|x - x_0\|} \right\rangle$.

Basta probar que $\lim_{x \to x_0} \varepsilon_1(x - x_0) = 0$

$$\lim_{x \to x_0} \varepsilon_1(x - x_0) = \lim_{x \to x_0} \left\langle G(x) - G(x_0); \frac{x - x_0}{\|x - x_0\|} \right\rangle$$

$$= \lim_{x \to x_0} \|G(x) - G(x_0)\| \cdot \frac{\cancel{\|x - x_0\|}}{\cancel{\|x - x_0\|}} \qquad G \text{ es continua en } x_0.$$

$$= 0$$

$\therefore$ f es diferenciable en x_0

con $\nabla f(x_0) = G(x_0)$