LINEAR FUNCTIONAL ANALYSIS

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Chapter 4

INNER PRODUCT SPACES

4.1. Introduction

The complex vector space $\mathbb{C}^r$ can be endowed with a euclidean inner product

$$\mathbf{x} \cdot \mathbf{y} = x_1\overline{y_1} + \ldots + x_r\overline{y_r} = \sum_{i=1}^{r} x_i\overline{y_i}.$$

For every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^r$ and $c \in \mathbb{C}$, we have
(EIP1) $\mathbf{x} \cdot \mathbf{y} = \overline{\mathbf{y} \cdot \mathbf{x}}$;
(EIP2) $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = (\mathbf{x} \cdot \mathbf{z}) + (\mathbf{y} \cdot \mathbf{z})$;
(EIP3) $c(\mathbf{x} \cdot \mathbf{y}) = (c\mathbf{x}) \cdot \mathbf{y}$;
(EIP4) $\mathbf{x} \cdot \mathbf{x} \geq 0$, and $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

Remark. Note that in the above we can impose the restriction $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^r$ and $c \in \mathbb{R}$, and ignore complex conjugation. We then obtain the familiar euclidean inner product in the real vector space $\mathbb{R}^r$.

We now wish to extend the above discussion from r-tuples $\mathbf{x} = (x_1, \ldots, x_r)$ of complex numbers to infinite sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ of complex numbers. Consider the set $\mathbb{C}^\infty$ of all infinite sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ of complex numbers. It is not difficult to show that $\mathbb{C}^\infty$ is a complex vector space, with vector addition $\mathbf{x} + \mathbf{y}$ and scalar multiplication $c\mathbf{x}$ defined respectively by

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots) \qquad \text{and} \qquad c\mathbf{x} = (cx_1, cx_2, cx_3, \ldots).$$

However, any attempt to extend the euclidean inner product on $\mathbb{C}^r$ to the vector space $\mathbb{C}^\infty$ raises immediately the question of the convergence of the series

$$\sum_{i=1}^{\infty} x_i\overline{y_i}. \tag{1}$$

For certain choices of infinite sequences $\mathbf{x}, \mathbf{y} \in \mathbb{C}^\infty$, this series clearly diverges. We therefore must impose certain convergence restrictions on the infinite sequences that we consider.

Let ℓ^2 denote the subset of $\mathbb{C}^\infty$ consisting of all square summable infinite sequences of complex numbers, so that

$$\ell^2 = \left\{ \mathbf{x} = (x_1, x_2, x_3, \ldots) \in \mathbb{C}^\infty : \sum_{i=1}^\infty |x_i|^2 < \infty \right\}.$$

We shall first show that ℓ^2 is a linear subspace of $\mathbb{C}^\infty$. To do this, we need to show that for every $\mathbf{x}, \mathbf{y} \in \ell^2$ and $c \in \mathbb{C}$, we have $\mathbf{x} + \mathbf{y} \in \ell^2$ and $c\mathbf{x} \in \ell^2$. For the first assertion, note that for every $n \in \mathbb{N}$, we have

$$\sum_{i=1}^n |x_i + y_i|^2 \le \sum_{i=1}^n \left(|x_i|^2 + 2|x_i||y_i| + |y_i|^2 \right) \le \sum_{i=1}^n \left(2|x_i|^2 + 2|y_i|^2 \right) \le 2\sum_{i=1}^\infty |x_i|^2 + 2\sum_{i=1}^\infty |y_i|^2.$$

For the second assertion, note that for every $n \in \mathbb{N}$, we have

$$\sum_{i=1}^n |cx_i|^2 = |c|^2 \sum_{i=1}^n |x_i|^2 \le |c|^2 \sum_{i=1}^\infty |x_i|^2.$$

The assertions follow on letting $n \to \infty$.

Next, note that for every $\mathbf{x}, \mathbf{y} \in \ell^2$, we have, for every $n \in \mathbb{N}$,

$$\sum_{i=1}^n |x_i \overline{y_i}| = \sum_{i=1}^n |x_i||y_i| \le \sum_{i=1}^n \left(\frac{1}{2}|x_i|^2 + \frac{1}{2}|y_i|^2 \right) \le \frac{1}{2} \sum_{i=1}^\infty |x_i|^2 + \frac{1}{2} \sum_{i=1}^\infty |y_i|^2.$$

Letting $n \to \infty$, we conclude that the series (1) converges absolutely. This enables us to meaningfully write

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^\infty x_i \overline{y_i}.$$

With this definition, it is then not difficult to show that for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \ell^2$ and $c \in \mathbb{C}$, we have
(L2IP1) $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$;
(L2IP2) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$;
(L2IP3) $\langle c\mathbf{x}, \mathbf{y} \rangle = c\langle \mathbf{x}, \mathbf{y} \rangle$;
(L2IP4) $\langle \mathbf{x}, \mathbf{x} \rangle \ge 0$, and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

Remark. We can restrict our discussion in this section to real sequences and therefore ignore complex conjugation.

4.2. Inner Product Spaces

We now generalize our discussion in the last section to arbitrary vector spaces over $\mathbb{F}$, where $\mathbb{F}$ denotes either the set $\mathbb{R}$ of all real numbers or the set $\mathbb{C}$ of all complex numbers.

Definition. By an inner product space, we mean a vector space V over $\mathbb{F}$, together with a function $\langle\ ,\ \rangle : V \times V \to \mathbb{F}$, called an inner product, satisfying the following conditions:

(IP1) For every $\mathbf{x}, \mathbf{y} \in V$, we have $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$.
(IP2) For every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, we have $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$.
(IP3) For every $\mathbf{x}, \mathbf{y} \in V$ and $c \in \mathbb{F}$, we have $\langle c\mathbf{x}, \mathbf{y} \rangle = c\langle \mathbf{x}, \mathbf{y} \rangle$.
(IP4) For every $\mathbf{x} \in V$, we have $\langle \mathbf{x}, \mathbf{x} \rangle \ge 0$, and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

Remark. An inner product space over $\mathbb{C}$ is also called a unitary space.

EXAMPLE 4.2.1. Suppose that $a, b \in \mathbb{R}$ and $a < b$. Consider the vector space $C[a, b]$ of all continuous complex valued functions on $[a, b]$, with the usual addition of functions and multiplication of functions by complex numbers. For every $f, g \in C[a, b]$, let

$$\langle f, g \rangle = \int_a^b f(t)\overline{g(t)}\,\mathrm{d}t.$$

It is not difficult to check conditions (IP1)–(IP3). To check condition (IP4), note first of all that for every function $f \in C[a, b]$, we have

$$\langle f, f \rangle = \int_a^b f(t)\overline{f(t)}\,\mathrm{d}t = \int_a^b |f(t)|^2\,\mathrm{d}t \geq 0.$$

On the other hand, if $f(t) = 0$ for every $t \in [a, b]$, then clearly $\langle f, f \rangle = 0$. Finally, note that the condition

$$\langle f, f \rangle = \int_a^b |f(t)|^2\,\mathrm{d}t = 0,$$

together with the continuity of f in $[a, b]$, ensures that $f(t) = 0$ for every $t \in [a, b]$.

EXAMPLE 4.2.2. Suppose that V is a finite dimensional vector space over $\mathbb{F}$, with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$. For any vectors $\mathbf{x}, \mathbf{y} \in V$, there exist unique $c_1, \ldots, c_r, a_1, \ldots, a_r \in \mathbb{F}$ such that $\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r$ and $\mathbf{y} = a_1\mathbf{v}_1 + \ldots + a_r\mathbf{v}_r$, and let

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^r c_i\overline{a_i}.$$

It is not difficult to check conditions (IP1)–(IP3). To check condition (IP4), note that for every vector $\mathbf{x} \in V$, we have

$$\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^r |c_i|^2 \geq 0.$$

On the other hand, note that $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $c_1 = \ldots = c_r = 0$, if and only if $\mathbf{x} = \mathbf{0}$.

EXAMPLE 4.2.3. Let T denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ on the complex plane, followed in the positive (anticlockwise) direction. We denote by $Q(T)$ the complex vector space of all rational functions that are analytic on T. For every $f, g \in Q(T)$, let

$$\langle f, g \rangle = \frac{1}{2\pi\mathrm{i}} \int_T f(z)\overline{g(z)}\,\frac{\mathrm{d}z}{z} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}t})\overline{g(\mathrm{e}^{\mathrm{i}t})}\,\mathrm{d}t,$$

if we parametrize T by writing $z = \mathrm{e}^{\mathrm{i}t}$, where $t \in [-\pi, \pi]$. It is not difficult to check that conditions (IP1)–(IP3) are satisfied. Note also that rational functions that are analytic on T have no singularities in T and are continuous in T, so condition (IP4) can be shown to be valid. Hence $Q(T)$ is a complex inner product space. One can study $Q(T)$ in some detail by using standard techniques in complex analysis. To illustrate this point, suppose that $\alpha, \beta \in \mathbb{C}$ satisfy $|\alpha| < 1$ and $|\beta| < 1$. Let $f, g \in Q(T)$ be defined by

$$f(z) = \frac{1}{z - \alpha} \qquad \text{and} \qquad g(z) = \frac{1}{z - \beta}.$$

Then

$$\langle f, g \rangle = \frac{1}{2\pi\mathrm{i}} \int_T \frac{\mathrm{d}z}{(z - \alpha)(\overline{z} - \overline{\beta})z} = \frac{1}{2\pi\mathrm{i}} \int_T \frac{\mathrm{d}z}{(z - \alpha)(z\overline{z} - \overline{\beta}z)} = \frac{1}{2\pi\mathrm{i}} \int_T \frac{\mathrm{d}z}{(z - \alpha)(1 - \overline{\beta}z)},$$

since $z\overline{z} = |z|^2 = 1$ on T. Cauchy's integral formula applied to the function $k(z) = (1 - \overline{\beta}z)^{-1}$ then gives

$$\langle f, g \rangle = k(\alpha) = \frac{1}{1 - \overline{\beta}\alpha}.$$

EXAMPLE 4.2.4. Let $\overline{D}$ denote the closed unit disc $\{z \in \mathbb{C} : |z| \le 1\}$ on the complex plane. We denote by $Q(\overline{D})$ the complex vector space of all rational functions that are analytic on $\overline{D}$. Clearly $Q(\overline{D})$ is a linear subspace of $Q(T)$. We can also make $Q(\overline{D})$ a complex inner product space by endowing it with the complex inner product for $Q(T)$.

We conclude this section by establishing a few elementary results concerning inner products.

THEOREM 4A. *Suppose that V is an inner product space over $\mathbb{F}$. Then the following assertions hold:*
(a) For every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, we have $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$.
(b) For every $\mathbf{x}, \mathbf{y} \in V$ and $c \in \mathbb{F}$, we have $\langle \mathbf{x}, c\mathbf{y} \rangle = \overline{c}\langle \mathbf{x}, \mathbf{y} \rangle$.
(c) For every $\mathbf{x} \in V$, we have $\langle \mathbf{x}, \mathbf{0} \rangle = 0 = \langle \mathbf{0}, \mathbf{x} \rangle$.

PROOF. For every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, we have, using (IP1) and (IP2),

$$\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \overline{\langle \mathbf{y} + \mathbf{z}, \mathbf{x} \rangle} = \overline{\langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{z}, \mathbf{x} \rangle} = \overline{\langle \mathbf{y}, \mathbf{x} \rangle} + \overline{\langle \mathbf{z}, \mathbf{x} \rangle} = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle.$$

This gives part (a). For every $c \in \mathbb{F}$, we have, using (IP1) and (IP3),

$$\langle \mathbf{x}, c\mathbf{y} \rangle = \overline{\langle c\mathbf{y}, \mathbf{x} \rangle} = \overline{c\langle \mathbf{y}, \mathbf{x} \rangle} = \overline{c}\,\overline{\langle \mathbf{y}, \mathbf{x} \rangle} = \overline{c}\langle \mathbf{x}, \mathbf{y} \rangle.$$

This gives part (b). Finally, part (c) follows from putting $c = 0$ in part (b) and observing (IP1). ♣

THEOREM 4B. *Suppose that V is an inner product space over $\mathbb{F}$, and that $\mathbf{x}, \mathbf{y} \in V$. Suppose further that $\langle \mathbf{x}, \mathbf{z} \rangle = \langle \mathbf{y}, \mathbf{z} \rangle$ for every $\mathbf{z} \in V$. Then $\mathbf{x} = \mathbf{y}$.*

PROOF. If $\langle \mathbf{x}, \mathbf{z} \rangle = \langle \mathbf{y}, \mathbf{z} \rangle$, then using (IP2) and (IP3), we have

$$0 = \langle \mathbf{x}, \mathbf{z} \rangle - \langle \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + (-1)\langle \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle -\mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x} - \mathbf{y}, \mathbf{z} \rangle.$$

Since this holds for every $\mathbf{z} \in V$, it holds in particular for $\mathbf{z} = \mathbf{x} - \mathbf{y}$, so that $\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle = 0$. It then follows from (IP4) that $\mathbf{x} - \mathbf{y} = \mathbf{0}$. ♣

4.3. Norm in an Inner Product Space

Our task in this section is to show that an inner product space over $\mathbb{F}$ can be viewed as a normed vector space over $\mathbb{F}$. This will be achieved by defining a suitable norm from the given inner product, and showing that this norm satisfies conditions (NS1)–(NS4). Recall that any normed vector space over $\mathbb{F}$ is a metric space. It follows that any inner product space over $\mathbb{F}$ can also be viewed as a metric space.

DEFINITION. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are vectors in an inner product space V over $\mathbb{F}$. Then the norm of $\mathbf{x}$ is defined by $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$.

EXAMPLE 4.3.1. Consider the sequence $\mathbf{x} = (1/i)_{i=1}^\infty$ in ℓ^2. We have

$$\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^\infty \frac{1}{i^2} = \frac{\pi^2}{6}, \qquad \text{so that} \qquad \|\mathbf{x}\| = \frac{\pi}{\sqrt{6}}.$$

EXAMPLE 4.3.2. Consider the function $f \in C[0,1]$, where $f(t) = t + \mathrm{i}t^2$ for every $t \in [0,1]$. We have

$$\langle f, f \rangle = \int_0^1 (t+\mathrm{i}t^2)\overline{(t+\mathrm{i}t^2)}\,\mathrm{d}t = \int_0^1 (t+\mathrm{i}t^2)(t-\mathrm{i}t^2)\,\mathrm{d}t = \int_0^1 (t^2+t^4)\,\mathrm{d}t = \frac{8}{15}, \qquad \text{so that} \qquad \|f\| = \sqrt{\frac{8}{15}}.$$

THEOREM 4C. *Suppose that V is an inner product space over $\mathbb{F}$. Then the following assertions hold:*
(a) For every $\mathbf{x} \in V$, we have $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.
(b) For every $\mathbf{x} \in V$ and every $c \in \mathbb{F}$, we have $\|c\mathbf{x}\| = |c|\|\mathbf{x}\|$.

PROOF. Part (a) follows immediately from the definition of norm and (IP4). On the other hand, for every $\mathbf{x} \in V$ and every $c \in \mathbb{F}$, we have, using (IP3) and Theorem 4A(b),

$$\|c\mathbf{x}\|^2 = \langle c\mathbf{x}, c\mathbf{x}\rangle = c\langle \mathbf{x}, c\mathbf{x}\rangle = c\bar{c}\langle \mathbf{x}, \mathbf{x}\rangle = |c|^2\|\mathbf{x}\|^2.$$

Part (b) follows on taking square roots. ♣

THEOREM 4D. (TRIANGLE INEQUALITY) *For any vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space V over $\mathbb{F}$, we have $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.*

Recall that to establish the Triangle inequality in the normed vector spaces in Examples 3.2.2 and 3.2.5, we make use of the Cauchy-Schwarz inequality for sums and integrals respectively. It then comes as no surprise that to prove Theorem 4D, we need a corresponding Cauchy-Schwarz inequality.

THEOREM 4E. (CAUCHY-SCHWARZ INEQUALITY) *For any vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space V over $\mathbb{F}$, we have $|\langle \mathbf{x}, \mathbf{y}\rangle| \leq \|\mathbf{x}\|\|\mathbf{y}\|$.*

PROOF OF THEOREM 4D. For every $\mathbf{x}, \mathbf{y} \in V$, we have

$$\|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle \leq \|\mathbf{x}\|^2 + 2|\langle \mathbf{x}, \mathbf{y}\rangle| + \|\mathbf{y}\|^2.$$

On the other hand, it follows from the Cauchy-Schwarz inequality that

$$\|\mathbf{x}\|^2 + 2|\langle \mathbf{x}, \mathbf{y}\rangle| + \|\mathbf{y}\|^2 \leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.$$

The result now follows on taking square roots. ♣

PROOF OF THEOREM 4E. Suppose first of all that $\mathbf{x} = \mathbf{0}$. Clearly we have $0\mathbf{x} = \mathbf{0}$. It follows that $\langle \mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{0}, \mathbf{y}\rangle = \langle 0\mathbf{x}, \mathbf{y}\rangle = 0\langle \mathbf{x}, \mathbf{y}\rangle = 0$, and so the assertion is clearly satisfied. Suppose now that $\mathbf{x} \neq \mathbf{0}$, so that $\langle \mathbf{x}, \mathbf{x}\rangle \neq 0$. For every $t \in \mathbb{F}$, it follows from (IP4) that $\langle t\mathbf{x}+\mathbf{y}, t\mathbf{x}+\mathbf{y}\rangle \geq 0$. Hence

$$\begin{aligned}0 &\leq \langle t\mathbf{x}+\mathbf{y}, t\mathbf{x}+\mathbf{y}\rangle = \langle t\mathbf{x}, t\mathbf{x}+\mathbf{y}\rangle + \langle \mathbf{y}, t\mathbf{x}+\mathbf{y}\rangle = \langle t\mathbf{x}, t\mathbf{x}\rangle + \langle t\mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, t\mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle \\ &= |t|^2\|\mathbf{x}\|^2 + t\langle \mathbf{x}, \mathbf{y}\rangle + \bar{t}\langle \mathbf{y}, \mathbf{x}\rangle + \|\mathbf{y}\|^2 = |t|^2\|\mathbf{x}\|^2 + t\langle \mathbf{x}, \mathbf{y}\rangle + \overline{t\langle \mathbf{x}, \mathbf{y}\rangle} + \|\mathbf{y}\|^2.\end{aligned}$$

Let $u \in \mathbb{F}$ satisfy $|u| = 1$ and $u\langle \mathbf{x}, \mathbf{y}\rangle = |\langle \mathbf{x}, \mathbf{y}\rangle|$. For any real number $\lambda \in \mathbb{R}$, the above with $t = \lambda u$ gives

$$0 \leq \lambda^2\|\mathbf{x}\|^2 + 2\lambda|\langle \mathbf{x}, \mathbf{y}\rangle| + \|\mathbf{y}\|^2.$$

Since $\langle \mathbf{x}, \mathbf{x}\rangle \neq 0$, the right hand side is a quadratic polynomial in λ. Since the inequality holds for every real number λ, it follows that the quadratic polynomial

$$\lambda^2\|\mathbf{x}\|^2 + 2\lambda|\langle \mathbf{x}, \mathbf{y}\rangle| + \|\mathbf{y}\|^2$$

has either repeated roots or no real root, and so the discriminant is non-positive. In other words, we must have

$$0 \geq (2|\langle \mathbf{x}, \mathbf{y}\rangle|)^2 - 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2 = 4|\langle \mathbf{x}, \mathbf{y}\rangle|^2 - 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2.$$

The assertion follows once again. ♣

REMARK. It can be shown that equality holds in the conclusion of Theorem 4E if and only if the two vectors $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent over $\mathbb{F}$.

EXAMPLE 4.3.3. Suppose that $f \in C[0,1]$. Let $g \in C[0,1]$ be given by $g(t) = \cos \pi t$ for every $t \in [0,1]$. Then

$$\langle f, g \rangle = \int_0^1 f(t) \cos \pi t \, dt.$$

Furthermore,

$$\|f\|^2 = \int_0^1 |f(t)|^2 \, dt \qquad \text{and} \qquad \|g\|^2 = \int_0^1 \cos^2 \pi t \, dt = \frac{1}{2}.$$

The Cauchy-Schwarz inequality applied to f and g now gives the inequality

$$\left| \int_0^1 f(t) \cos \pi t \, dt \right|^2 \le \frac{1}{2} \int_0^1 |f(t)|^2 \, dt.$$

The proof of the next result is rather simple.

THEOREM 4F. (PARALLELOGRAM LAW) *For any vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *in an inner product space* V *over* $\mathbb{F}$, *we have* $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$.

REMARK. The most important observation of the Parallelogram law is that the norm is induced by an inner product. The identity may fail to hold in a normed vector space. In this case, the failure of the Parallelogram law is useful in showing that the given norm cannot possibly arise from an inner product.

EXAMPLE 4.3.4. Consider the normed vector space $C[0,1]$ of all continuous complex valued functions on $[0,1]$, with supremum norm

$$\|f\|_\infty = \sup_{t \in [0,1]} |f(t)|.$$

Let $f, g \in C[0,1]$ be defined by $f(t) = 1$ and $g(t) = t$ for every $t \in [0,1]$. Then we have

$$\|f\|_\infty^2 = 1, \qquad \|g\|_\infty^2 = 1, \qquad \|f+g\|_\infty^2 = 4, \qquad \|f-g\|_\infty^2 = 1,$$

so that clearly the Parallelogram law does not hold.

EXAMPLE 4.3.5. Consider the normed vector space ℓ^∞ of all bounded infinite sequences of complex numbers, with supremum norm

$$\|\mathbf{x}\|_\infty = \sup_{i \in \mathbb{N}} |x_i|.$$

Let $\mathbf{x} = (0,1,0,1,0,1,\ldots)$ and $\mathbf{y} = (1,0,1,0,1,0,\ldots)$. Then we have $\mathbf{x}+\mathbf{y} = (1,1,1,1,1,1,\ldots)$ and $\mathbf{x}-\mathbf{y} = (-1,1,-1,1,-1,1,\ldots)$, and

$$\|\mathbf{x}\|_\infty^2 = 1, \qquad \|\mathbf{y}\|_\infty^2 = 1, \qquad \|\mathbf{x}+\mathbf{y}\|_\infty^2 = 1, \qquad \|\mathbf{x}-\mathbf{y}\|_\infty^2 = 1,$$

so that clearly the Parallelogram law does not hold.

4.4. Hilbert Spaces

Recall that any inner product space $(V, \langle \ , \ \rangle)$ over $\mathbb{F}$ is a metric space, with induced metric

$$\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y} \rangle^{1/2}.$$

DEFINITION. Suppose that V is an inner product space over $\mathbb{F}$. Suppose further that V is a complete metric space under the metric induced by its inner product. Then we say that V is a Hilbert space.

Remark. Note that every Hilbert space over $\mathbb{F}$ is a Banach space over $\mathbb{F}$.

Example 4.4.1. Since any inner product induces a norm, it follows from Theorem 3F that every finite dimensional inner product space over $\mathbb{F}$ is a Hilbert space. These include the real enclidean space $\mathbb{R}^r$ and the complex euclidean space $\mathbb{C}^r$ for every $r \in \mathbb{N}$.

Example 4.4.2. The inner product space ℓ^2 of all square summable infinite sequences of complex numbers is complete, so that ℓ^2 is a Hilbert space. Recall that ℓ^2 has inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i},$$

and norm

$$\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2} = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2},$$

giving rise to the metric

$$\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \left(\sum_{i=1}^{\infty} |x_i - y_i|^2 \right)^{1/2}.$$

Suppose that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in ℓ^2. For every $n \in \mathbb{N}$, let

$$\mathbf{x}_n = (x_{n1}, x_{n2}, x_{n3}, \ldots, x_{ni}, \ldots).$$

Then

$$\begin{aligned} \mathbf{x}_1 &= (x_{11}, x_{12}, x_{13}, \ldots, x_{1i}, \ldots), \\ \mathbf{x}_2 &= (x_{21}, x_{22}, x_{23}, \ldots, x_{2i}, \ldots), \\ \mathbf{x}_3 &= (x_{31}, x_{32}, x_{33}, \ldots, x_{3i}, \ldots), \\ &\vdots \\ \mathbf{x}_n &= (x_{n1}, x_{n2}, x_{n3}, \ldots, x_{ni}, \ldots), \\ &\vdots \end{aligned}$$

For any fixed $i \in \mathbb{N}$, let us consider the sequence $x_{1i}, x_{2i}, x_{3i}, \ldots, x_{ni}, \ldots$. It is clear that for every $m, n \in \mathbb{N}$, we have

$$|x_{mi} - x_{ni}| \le \|\mathbf{x}_m - \mathbf{x}_n\| = \rho(\mathbf{x}_m, \mathbf{x}_n).$$

Since $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in ℓ^2, it follows that $(x_{ni})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, it follows that there exists $a_i \in \mathbb{C}$ such that $x_{ni} \to a_i$ as $n \to \infty$. Let

$$\mathbf{a} = (a_1, a_2, a_3, \ldots, a_i, \ldots).$$

We shall show that $\mathbf{a} \in \ell^2$, and that $\mathbf{x}_n \to \mathbf{a}$ as $n \to \infty$ in ℓ^2. Given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\rho(\mathbf{x}_m, \mathbf{x}_n) < \epsilon \qquad \text{for every } m, n \in \mathbb{N} \text{ satisfying } m > n \ge N. \tag{2}$$

Taking $n = N$, we have

$$\rho(\mathbf{x}_m, \mathbf{x}_N) < \epsilon \qquad \text{for every } m \in \mathbb{N} \text{ satisfying } m > N.$$

It follows that for every $I \in \mathbb{N}$, we have

$$\sum_{i=1}^{I} |x_{mi} - x_{Ni}|^2 < \epsilon^2 \qquad \text{for every } m \in \mathbb{N} \text{ satisfying } m > N.$$

Keeping I fixed and letting $m \to \infty$, we obtain

$$\sum_{i=1}^{I} |a_i - x_{Ni}|^2 \le \epsilon^2.$$

Since $I \in \mathbb{N}$ is arbitrary, we conclude that

$$\rho(\mathbf{a}, \mathbf{x}_N) = \left(\sum_{i=1}^{\infty} |a_i - x_{Ni}|^2 \right)^{1/2} \le \epsilon.$$

Note next that

$$\|\mathbf{a}\| = \|\mathbf{x}_N + \mathbf{a} - \mathbf{x}_N\| \le \|\mathbf{x}_N\| + \|\mathbf{a} - \mathbf{x}_N\| = \|\mathbf{x}_N\| + \rho(\mathbf{a}, \mathbf{x}_N) \le \|\mathbf{x}_N\| + \epsilon,$$

so that $\mathbf{a} \in \ell^2$. To show that $\mathbf{x}_n \to \mathbf{a}$ as $n \to \infty$ in ℓ^2, we observe that it follows from (2) that for every $n \ge N$, we have

$$\rho(\mathbf{x}_m, \mathbf{x}_n) < \epsilon \qquad \text{for every } m \in \mathbb{N} \text{ satisfying } m > n.$$

It follows that for every $I \in \mathbb{N}$, we have

$$\sum_{i=1}^{I} |x_{mi} - x_{ni}|^2 < \epsilon^2 \qquad \text{for every } m \in \mathbb{N} \text{ satisfying } m > n.$$

Keeping I fixed and letting $m \to \infty$, we obtain

$$\sum_{i=1}^{I} |a_i - x_{ni}|^2 \le \epsilon^2.$$

Since $I \in \mathbb{N}$ is arbitrary, we conclude that

$$\rho(\mathbf{a}, \mathbf{x}_n) = \left(\sum_{i=1}^{\infty} |a_i - x_{ni}|^2 \right)^{1/2} \le \epsilon.$$

To summarize, we have shown that given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\rho(\mathbf{a}, \mathbf{x}_n) \le \epsilon$ for every $n \in \mathbb{N}$ satisfying $n \ge N$. Hence $\mathbf{x}_n \to \mathbf{a}$ as $n \to \infty$ in ℓ^2.

Remark. A Banach space is not necessarily a Hilbert space. For example, the space ℓ^∞ is a Banach space, as shown in Example 3.6.2. On the other hand, in view of Example 4.3.5 and Theorem 4F, the space ℓ^∞ does not have an inner product that gives rise to the supremum norm, and so cannot be a Hilbert space under the same induced metric.

Example 4.4.3. The space $C[-1,1]$ of all continuous complex valued functions on $[-1,1]$, with inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(t)\overline{g(t)} \,\mathrm{d}t,$$

is not a Hilbert space. Note that the inner product induces the metric

$$\rho(f, g) = \int_{-1}^{1} |f(t) - g(t)|^2 \,\mathrm{d}t.$$

Consider the sequence $(f_n)_{n\in\mathbb{N}}$, where for every $n \in \mathbb{N}$, the function $f_n : [-1,1] \to \mathbb{C}$ is defined by

$$f_n(t) = \begin{cases} -1 & \text{if } -1 \le t \le -n^{-1}, \\ nt & \text{if } -n^{-1} \le t \le n^{-1}, \\ 1 & \text{if } n^{-1} \le t \le 1. \end{cases}$$

Clearly $f_n \in C[-1,1]$ for every $n \in \mathbb{N}$. Note that for every $m,n \in \mathbb{N}$ satisfying $m > n$, we have

$$\begin{aligned}\rho(f_m, f_n) &= \int_{-1}^{1} |f_m(t) - f_n(t)|^2 \, \mathrm{d}t = 2\int_0^{m^{-1}} (mt - nt)^2 \, \mathrm{d}t + 2\int_{m^{-1}}^{n^{-1}} (1-nt)^2 \, \mathrm{d}t \\ &= \frac{2(m-n)^2}{3m^3} + \frac{2}{3n} - \frac{2}{m} + \frac{2n}{m^2} - \frac{2n^2}{3m^3} < \frac{6}{m} + \frac{1}{n} \to 0\end{aligned}$$

as $m,n \to \infty$. Hence $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C[-1,1]$. On the other hand, note that the pointwise limit of the sequence is the function $f : [-1,1] \to \mathbb{C}$, where

$$f(t) = \begin{cases} -1 & \text{if } -1 \le t < 0, \\ 0 & \text{if } t = 0, \\ 1 & \text{if } 0 < t \le 1. \end{cases}$$

This does not belong to $C[-1,1]$.

Example 4.4.4. Similarly, for any $a,b \in \mathbb{R}$ satisfying $a < b$, the space $C[a,b]$ of all continuous complex valued functions on $[a,b]$ is not complete under the metric induced by the inner product

$$\langle f, g \rangle = \int_a^b f(t)\overline{g(t)} \, \mathrm{d}t.$$

In an attempt to complete $C[a,b]$, one may include functions that are not continuous on $[a,b]$ but are nevertheless square integrable on $[a,b]$, so that the integral

$$\int_a^b |f(t)|^2 \, \mathrm{d}t$$

exists. However, the Riemann integral turns out to be insufficient for this task. It was not until Lebesgue introduced his theory of measure that a satisfactory answer to this question was obtained. Nowadays, we consider the space $L^2[a,b]$ of all Lebesgue measurable functions that are square integrable on $[a,b]$, with the inner product above, where the integral is understood to be in the sense of Lebesgue. The inner product space $L^2[a,b]$ is then a Hilbert space with respect to the metric induced by this inner product, except for the fact that

$$\langle f, f \rangle = \|f\|^2 = \int_a^b |f(t)|^2 \, \mathrm{d}t = 0$$

does not necessarily imply that $f = 0$ in $[a,b]$, so that condition (IP4) for a complex inner product space does not hold.

To overcome the difficulty highlighted in the last example, we have to introduce the idea of equality almost everywhere. A set E of real numbers is said to be a null set if, for every $\epsilon > 0$, there exists a sequence of open intervals (a_k, b_k) such that

$$E \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k) \qquad \text{and} \qquad \sum_{k=1}^{\infty} (b_b - a_k) < \epsilon.$$

In other words, E is a null set if it can be covered by a union of open intervals of arbitrarily small total length. We then say that two complex valued functions f and g defined on the interval $[a,b]$ are equal almost everywhere if the exceptional set

$$E = \{t \in [a,b] : f(t) \neq g(t)\}$$

is a null set. Then in Lebesgue's theory of integration, two such functions f and g must satisfy

$$\int_a^b |f(t) - g(t)|^2 \, \mathrm{d}t = 0.$$

In particular, a complex valued function f defined on the interval $[a,b]$ is zero almost everywhere if the exceptional set $E=\{t\in[a,b]:f(t)\neq 0\}$ is a null set. Under equality almost everywhere, we can then modify Example 4.2.1 to show that $L^2[a,b]$ is a complex inner product space, noting that if the integrals

$$\int_a^b |f(t)|^2\,\mathrm{d}t \qquad \text{and} \qquad \int_a^b |g(t)|^2\,\mathrm{d}t$$

exist, then the inequality

$$2|f(t)\overline{g(t)}|\le |f(t)|^2+|g(t)|^2$$

ensures that $\langle f,g\rangle$ is well defined. We shall not show here that $L^2[a,b]$ is complete, since this will involve some discussion on measure theory which is outside the scope of our present investigation.

4.5. The Closest Point Property

Recall that a Hilbert space has a metric induced by an inner product, so that the Parallelogram law for norms holds. This leads to a very beautiful result concerning the distance of a point in a Hilbert space to a non-empty closed convex subset.

A set A in a vector space V over $\mathbb{F}$ is said to be convex if, given any two points $\mathbf{a},\mathbf{b}\in A$, the line segment $\{\lambda\mathbf{a}+(1-\lambda)\mathbf{b}:\lambda\in[0,1]\}$ is also contained in A.

THEOREM 4G. *Suppose that A is a non-empty closed convex set in a Hilbert space V over $\mathbb{F}$. Then for every $\mathbf{x}\in V$, there exists a unique $\mathbf{a}\in A$ such that*

$$\|\mathbf{x}-\mathbf{a}\|=\inf_{\mathbf{y}\in A}\|\mathbf{x}-\mathbf{y}\|.$$

Proof. We shall first of all establish the existence of such an element $\mathbf{a}\in A$. Write

$$M=\inf_{\mathbf{y}\in A}\|\mathbf{x}-\mathbf{y}\|, \qquad \text{so that} \qquad M^2=\inf_{\mathbf{y}\in A}\|\mathbf{x}-\mathbf{y}\|^2.$$

Given any $n\in\mathbb{N}$, there exists $\mathbf{y}_n\in A$ such that

$$\|\mathbf{x}-\mathbf{y}_n\|^2<M^2+\frac{1}{n}. \tag{3}$$

We shall show that $(\mathbf{y}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in A. For every $m,n\in\mathbb{N}$, we have

$$\|\mathbf{x}-\mathbf{y}_m\|^2<M^2+\frac{1}{m} \qquad \text{and} \qquad \|\mathbf{x}-\mathbf{y}_n\|^2<M^2+\frac{1}{n}. \tag{4}$$

Applying the Parallelogram law, we have

$$\begin{aligned}2\|\mathbf{x}-\mathbf{y}_m\|^2+2\|\mathbf{x}-\mathbf{y}_n\|^2&=\|(\mathbf{x}-\mathbf{y}_m)+(\mathbf{x}-\mathbf{y}_n)\|^2+\|(\mathbf{x}-\mathbf{y}_m)-(\mathbf{x}-\mathbf{y}_n)\|^2\\&=4\|\mathbf{x}-\tfrac{1}{2}(\mathbf{y}_m+\mathbf{y}_n)\|^2+\|\mathbf{y}_m-\mathbf{y}_n\|^2,\end{aligned}$$

so that

$$\|\mathbf{y}_m-\mathbf{y}_n\|^2=2\|\mathbf{x}-\mathbf{y}_m\|^2+2\|\mathbf{x}-\mathbf{y}_n\|^2-4\|\mathbf{x}-\tfrac{1}{2}(\mathbf{y}_m+\mathbf{y}_n)\|^2. \tag{5}$$

Since A is convex, we have $\frac{1}{2}(\mathbf{y}_m+\mathbf{y}_n)\in A$, and so

$$\|\mathbf{x}-\tfrac{1}{2}(\mathbf{y}_m+\mathbf{y}_n)\|^2\ge M^2. \tag{6}$$

Combining (4)–(6), we have

$$\|\mathbf{y}_m-\mathbf{y}_n\|^2<2\left(\frac{1}{m}+\frac{1}{n}\right).$$

Hence $(\mathbf{y}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in A, and so a Cauchy sequence in V. Since V is complete, there exists $\mathbf{a} \in V$ such that $\mathbf{y}_n \to \mathbf{a}$ as $n \to \infty$. Furthermore, since A is closed, it follows that $\mathbf{a} \in A$, and so $\|\mathbf{x}-\mathbf{a}\| \geq M$. On the other hand, letting $n \to \infty$ in (3), we obtain $\|\mathbf{x}-\mathbf{a}\| \leq M$. Hence $\|\mathbf{x}-\mathbf{a}\| = M$. To establish uniqueness, suppose that $\mathbf{b} \in A$ satisfies $\|\mathbf{x}-\mathbf{b}\| = M$. Then since A is convex, we have $\frac{1}{2}(\mathbf{a}+\mathbf{b}) \in A$, and so $\|\mathbf{x}-\frac{1}{2}(\mathbf{a}+\mathbf{b})\| \geq M$. Applying the Parallelogram law, we have, analogous to (5),

$$\|\mathbf{a}-\mathbf{b}\|^2 = 2\|\mathbf{x}-\mathbf{a}\|^2 + 2\|\mathbf{x}-\mathbf{b}\|^2 - 4\|\mathbf{x}-\tfrac{1}{2}(\mathbf{a}+\mathbf{b})\|^2 \leq 0.$$

Hence $\|\mathbf{a}-\mathbf{b}\| = 0$, and so $\mathbf{a}-\mathbf{b} = 0$, whence $\mathbf{a} = \mathbf{b}$. ♣

Problems for Chapter 4

1. Prove the properties (L2IP1)–(L2IP4) for the vector space ℓ^2 discussed in Section 4.1.

2. Consider the set $C^1[0,1]$ of all complex valued functions defined on $[0,1]$ and with continuous derivatives in $[0,1]$.
   a) Show that $C^1[0,1]$ is a vector space over $\mathbb{C}$.
   b) Show that
   $$\langle f,g\rangle = \int_0^1 \left(f(t)\overline{g(t)} + f'(t)\overline{g'(t)}\right) \mathrm{d}t$$
   defines an inner product on $C^1[0,1]$.

3. Consider the set Ψ of all trigonometric polynomials of the form
   $$f(t) = \sum_{j=1}^{k} a_i \mathrm{e}^{\mathrm{i}\lambda_j t}, \qquad \text{where } k \in \mathbb{N},\ a_1,\ldots,a_k \in \mathbb{C} \text{ and } \lambda_1,\ldots,\lambda_k \in \mathbb{R}.$$
   a) Show that Ψ is a vector space over $\mathbb{C}$.
   b) Show that
   $$\langle f,g\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)\overline{g(t)}\,\mathrm{d}t$$
   defines an inner product on Ψ.

4. For every $n \in \mathbb{Z}$, consider the function $e_n : [0,1] \to \mathbb{C}$ where $e_n(t) = \mathrm{e}^{2\pi \mathrm{i} nt}$ for every $t \in [0,1]$.
   a) Show that for distinct $n,m \in \mathbb{Z}$, we have $\langle e_n, e_m\rangle = 0$ under the inner product of $C[0,1]$ in Example 4.2.1 with $a=0$ and $b=1$, and $\|e_n - e_m\|^2 = 2$.
   b) Show that for distinct $n,m \in \mathbb{Z}$, we have $\langle e_n, e_m\rangle = 0$ under the inner product of $C^1[0,1]$ in Problem 2, and $\|e_n - e_m\|^2 = 2 + 4\pi^2(n^2+m^2)$.

5. Let $\mathbb{C}[x]$ denote the set of all polynomials in a complex variable x and with coefficients in $\mathbb{C}$.
   a) Show that $\mathbb{C}[x]$ is a vector space over $\mathbb{C}$.
   b) Find an inner product on $\mathbb{C}[x]$ such that for any $f \in \mathbb{C}[x]$, we have
   $$\|f\| = \left(\int_{-1}^{1} \left(|x||f(x)|^2 + 3|f'(x)|^2\right) \mathrm{d}x\right)^{1/2}.$$
   c) Show that for any $f \in \mathbb{C}[x]$, we have
   $$\left|\int_{-1}^{1} \left(|x|^3 f(x) + 6xf'(x)\right) \mathrm{d}x\right|^2 \leq \frac{25}{3}\int_{-1}^{1} \left(|x||f(x)|^2 + 3|f'(x)|^2\right) \mathrm{d}x.$$

6. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are vectors in an inner product space over $\mathbb{F}$ such that $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\|$. Show that one of vectors $\mathbf{x}$ and $\mathbf{y}$ must be a scalar multiple of the other.

7. Prove Theorem 4F.

8. Suppose that V is a real inner product space, and that $\mathbf{u},\mathbf{v},\mathbf{x},\mathbf{y} \in V$.
   a) Show that $\langle \mathbf{u}+\mathbf{v}, \mathbf{x}+\mathbf{y}\rangle - \langle \mathbf{u}-\mathbf{v}, \mathbf{x}-\mathbf{y}\rangle = 2\langle \mathbf{u},\mathbf{y}\rangle + 2\langle \mathbf{v},\mathbf{x}\rangle$.
   b) Deduce that $4\langle \mathbf{x},\mathbf{y}\rangle = \|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2$.

9. Suppose that V is a complex inner product space, and that $\mathbf{u},\mathbf{v},\mathbf{x},\mathbf{y} \in V$.
   a) Show that $\langle \mathbf{u}+\mathbf{v}, \mathbf{x}+\mathbf{y}\rangle - \langle \mathbf{u}-\mathbf{v}, \mathbf{x}-\mathbf{y}\rangle + \mathrm{i}\langle \mathbf{u}+\mathrm{i}\mathbf{v}, \mathbf{x}+\mathrm{i}\mathbf{y}\rangle - \mathrm{i}\langle \mathbf{u}-\mathrm{i}\mathbf{v}, \mathbf{x}-\mathrm{i}\mathbf{y}\rangle = 4\langle \mathbf{u},\mathbf{y}\rangle$.
   b) Deduce the Polarization identity, that $4\langle \mathbf{x},\mathbf{y}\rangle = \|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2 + \mathrm{i}\|\mathbf{x}+\mathrm{i}\mathbf{y}\|^2 - \mathrm{i}\|\mathbf{x}-\mathrm{i}\mathbf{y}\|^2$.

10. For every $\alpha \in \mathbb{C}$ such that $|\alpha| \neq 1$, consider the complex valued rational function $g_\alpha \in Q(T)$ given by $g_\alpha(z) = (1-\overline{\alpha}z)^{-1}$.
   a) Show that with respect to the inner product of $Q(T)$ discussed in Example 4.2.3, we have, for every $f \in Q(\overline{D})$ discussed in Example 4.2.4,
   $$\langle f, g_\alpha\rangle = \begin{cases} f(\alpha) & \text{if } |\alpha| < 1, \\ 0 & \text{if } |\alpha| > 1. \end{cases}$$
   b) Hence, or otherwise, show that for every $\alpha \in \mathbb{C}$ satisfying $|\alpha| < 1$ and for every $f \in Q(\overline{D})$, we have
   $$|f(\alpha)|^2 \le \frac{1}{2\pi(1-|\alpha|^2)} \int_{-\pi}^{\pi} |f(\mathrm{e}^{\mathrm{i}t})|^2 \,\mathrm{d}t.$$

11. Suppose that $\alpha,\beta \in \mathbb{C}$ satisfy $|\alpha| < 1$, $|\beta| < 1$ and $\alpha \neq \beta$. Show that for the rational function $f \in Q(T)$ given by $f(z) = (z-\alpha)^{-1}(z-\beta)^{-1}$, we have
   $$\|f\|^2 = \frac{1-|\alpha\beta|^2}{(1-|\alpha|^2)(1-|\beta|^2)|1-\overline{\alpha}\beta|^2}$$
   with respect to the inner product on $Q(T)$ discussed in Example 4.2.3.
   [Hint: Use Cauchy's residue theorem.]

12. Use one of the following methods to prove that any inner product in a vector space V over $\mathbb{F}$ is continuous on $V \times V$:
   a) Use the ϵ-δ method: Suppose that $\mathbf{x}_0,\mathbf{y}_0 \in V$ are fixed. Given any $\epsilon > 0$, find $\delta > 0$ such that $|\langle \mathbf{x},\mathbf{y}\rangle - \langle \mathbf{x}_0,\mathbf{y}_0\rangle| < \epsilon$ for every $\mathbf{x},\mathbf{y} \in V$ satisfying $\|\mathbf{x}-\mathbf{x}_0\| < \delta$ and $\|\mathbf{y}-\mathbf{y}_0\| < \delta$.
   b) If $\mathbb{F} = \mathbb{R}$, use the identity in Problem 8(b) as well as Theorems 3A and 3B. If $\mathbb{F} = \mathbb{C}$, use the Polarization identity in Problem 9(b) as well as Theorem 3A and 3B.

13. Let $Q(\overline{D})$ be the inner product space defined in Example 4.2.4. For any $\alpha \in \mathbb{C}$ satisfying $|\alpha| < 1$, consider the set $S_\alpha = \{f \in Q(\overline{D}) : f(\alpha) = 0\}$.
   a) Show that S_α is a linear subspace of $Q(\overline{D})$.
   b) Show that S_α is closed in $Q(\overline{D})$ with respect to the norm from the inner product.
   [Hint: Use Problem 10(a).]

14. Let $Q(T)$ and $Q(\overline{D})$ be the inner product spaces defined in Examples 4.2.3 and 4.2.4. Show that the linear subspace $Q(\overline{D})$ is closed in $Q(T)$ with respect to the norm from the inner product.

15. Consider the real vector space $\mathbb{R}^2$.
    a) Show that the function $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$, defined for every $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ by

    $$\|\mathbf{x}\| = \max\{|x_1|, |x_2|\},$$

    is a norm on $\mathbb{R}^2$. Show also that $\mathbb{R}^2$ forms a Banach space under the metric induced by this norm.
    b) For the set $A = \{\mathbf{x} \in \mathbb{R}^2 : x_1 \geq 1\}$, determine the quantity

    $$\inf_{\mathbf{y} \in A} \|\mathbf{y}\|.$$

    c) Is the infimum in part (b) attained? If so, determine the set of all those vectors $\mathbf{y} \in A$ that attains this infimum.
    d) Is there an inner product in $\mathbb{R}^2$ that induces the norm in part (a)? Justify your assertion, quoting all relevant results.

16. Show that the inner product space $C^1[-1,1]$ of all complex valued functions defined on $[-1,1]$ and with continuous derivative in $[-1,1]$, with inner product

    $$\langle f, g \rangle = \int_{-1}^{1} \left(f(t)\overline{g(t)} + f'(t)\overline{g'(t)} \right) \mathrm{d}t,$$

    is not a Hilbert space.
    [Hint: Try using the indefinite integrals of the functions in Example 4.4.3.]

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