LINEAR FUNCTIONAL ANALYSIS

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Chapter 2

CONNECTEDNESS, COMPLETENESS AND COMPACTNESS

2.1. Connected Metric Spaces

We begin by considering a very simple example.

EXAMPLE 2.1.1. Consider a metric space (X,ρ), where X is a non-empty set and ρ is the discrete metric. Suppose that $\mathbf{c}\in X$. Then it is easy to see that $B(\mathbf{c},\frac{1}{2})=\{\mathbf{c}\}$, so that $\{\mathbf{c}\}$ is open. It follows from Theorem 1B that every set $E\subseteq X$ is open, and from Theorem 1C that every set $E\subseteq X$ is closed. In other words, every set $E\subseteq X$ is both open and closed.

The above example illustrates the fact that there exist metric spaces (X,ρ) in which sets other than $\emptyset$ and X can be both open and closed. A metric space without such sets is said to be connected. Note that a set in X is both open and closed if its complement is both closed and open. It follows that X is not connected if and only if it is the union of two non-empty, disjoint open sets in X.

We would like to extend this definition of connectedness to an arbitrary set E in a metric space (X,ρ). One may attempt the following procedure. Take the metric ρ and restrict it to $E\times E$, and call this restriction ρ_E. We now consider the metric space (E,ρ_E). We could then say that E is not connected if it is the union of two non-empty, disjoint open sets in E. Note that this process will require the determination of all the open sets in the metric space (E,ρ_E). But then an open set in E may not be an open set in X.

We therefore proceed instead as follows. We first study open sets in a metric space.

DEFINITION. Suppose that (X,ρ) is a metric space. An open set $G\subseteq X$ is said to be connected if there do not exist two non-empty open sets G_1 and G_2 such that

$$G_1\cap G_2=\emptyset \qquad \text{and} \qquad G_1\cup G_2=G.$$

This definition can be generalized to arbitrary sets in a metric space.

DEFINITION. Suppose that (X, ρ) is a metric space. A set $E \subseteq X$ is said to be connected if there do not exist two non-empty open sets G_1 and G_2 such that

$$G_1 \cap E \neq \emptyset, \qquad G_2 \cap E \neq \emptyset, \qquad G_1 \cap G_2 \cap E = \emptyset \qquad \text{and} \qquad G_1 \cup G_2 \supseteq E.$$

REMARKS. (1) We say that the metric space (X, ρ) is connected if the set X is connected.
(2) Connectedness of a set $E \subseteq X$ depends on the metric ρ.

We state here without proof a result concerning connectedness in $\mathbb{R}$ and $\mathbb{C}$ with the euclidean metrics.

THEOREM 2A.
(a) In $\mathbb{R}$ with the euclidean metric, a non-empty set E is connected if and only if E is an interval.
(b) In $\mathbb{C}$ with the euclidean metric, a non-empty open set G is connected if and only if any two points of G can be joined by a polygon lying entirely in G.

EXAMPLE 2.1.2. Consider the interval $[0, 1)$ in $\mathbb{R}$. Then it follows from Theorem 2A that $[0, 1)$ is connected if $\mathbb{R}$ has the euclidean metric. Suppose now that we use the discrete metric instead. Then $[0, 1) = \{0\} \cup (0, 1)$, a union of two non-empty open sets, in view of Example 2.1.1. Hence $[0, 1)$ is not connected in this case.

DEFINITION. An open, connected subset of a metric space is called a domain.

DEFINITION. Suppose that (X, ρ) is a metric space, and that $E \subseteq X$. A set C is called a component of E if it is a maximal connected subset of E; in other words, for every connected set D satisfying $C \subseteq D \subseteq E$, we must have $D = C$.

THEOREM 2B. *Any set in a metric space can be decomposed uniquely into a union of components.*

The proof of Theorem 2B is based on the following intermediate result.

THEOREM 2C. *Suppose that $\mathcal{E}$ is a collection of connected sets in a metric space. Suppose further that the intersection $K = \bigcap_{E \in \mathcal{E}} E$ is non-empty. Then the union $H = \bigcup_{E \in \mathcal{E}} E$ is connected.*

PROOF. Suppose on the contrary that H is not connected. Then there exist open sets G_1 and G_2 such that

$$G_1 \cap H \neq \emptyset, \qquad G_2 \cap H \neq \emptyset, \qquad G_1 \cap G_2 \cap H = \emptyset \qquad \text{and} \qquad G_1 \cup G_2 \supseteq H.$$

Since the intersection K is non-empty, there exists $\mathbf{a} \in K \subseteq H \subseteq G_1 \cup G_2$. Without loss of generality, we assume that $\mathbf{a} \in G_1$. Note also that $\mathbf{a} \in E$ and $E \subseteq H$ for every $E \in \mathcal{E}$. Hence

$$G_1 \cap E \neq \emptyset, \qquad G_1 \cap G_2 \cap E = \emptyset \qquad \text{and} \qquad G_1 \cup G_2 \supseteq E.$$

Since E is connected, we must have $G_2 \cap E = \emptyset$. This holds for every $E \in \mathcal{E}$, and so

$$G_2 \cap H = G_2 \cap \left(\bigcup_{E \in \mathcal{E}} E \right) = \bigcup_{E \in \mathcal{E}} (G_2 \cap E) = \emptyset,$$

a contradiction. ♣

Proof of Theorem 2B. Suppose that E is a set in a metric space. If $E = \emptyset$, then the result is trivial, so we shall assume that E is non-empty. For any $\mathbf{x} \in E$, let $C_{\mathbf{x}}$ denote the union of all connected subsets of E which contain $\mathbf{x}$. It is easy to see that the set $\{\mathbf{x}\}$ is connected, so that $C_{\mathbf{x}}$ is non-empty. It follows from Theorem 2C that $C_{\mathbf{x}}$ is connected. Clearly $C_{\mathbf{x}}$ is a component, for if D is connected and $C_{\mathbf{x}} \subseteq D \subseteq E$, then D contains $\mathbf{x}$ and so $D \subseteq C_{\mathbf{x}}$, whence $D = C_{\mathbf{x}}$. The result now follows, since any component of E must be of the form $C_{\mathbf{x}}$ for some $\mathbf{x} \in E$. ♣

We conclude this section by studying the effect of a continuous function on a connected domain.

THEOREM 2D. *Suppose that (X, ρ) and (Y, σ) are metric spaces, and that X is connected. Suppose further that a function $f : X \to Y$ is continuous on X. Then the range $f(X)$ is connected. In other words, the range of a continuous function with connected domain is connected.*

Proof. Suppose on the contrary that $f(X)$ is not connected. Then there are open sets $G_1, G_2 \subseteq Y$ such that

$$G_1 \cap f(X) \neq \emptyset, \qquad G_2 \cap f(X) \neq \emptyset, \qquad G_1 \cap G_2 \cap f(X) = \emptyset \qquad \text{and} \qquad G_1 \cup G_2 \supseteq f(X).$$

Note that the first two conditions imply that the pre-images $f^{-1}(G_1)$ and $f^{-1}(G_2)$ are non-empty. Furthermore, they are open, in view of Theorem 1J. On the other hand, the last two conditions imply that

$$f^{-1}(G_1) \cap f^{-1}(G_2) = \emptyset \qquad \text{and} \qquad f^{-1}(G_1) \cup f^{-1}(G_2) = X,$$

so that X is not connected. ♣

The following special case should be familiar.

THEOREM 2E. *Under the hypotheses of Theorem 2D, suppose further that the metric space (Y, σ) is $\mathbb{R}$ with the euclidean metric. Then the range $f(X)$ is an interval. In other words, the range of a continuous real-valued function with connected domain is an interval.*

2.2. Complete Metric Spaces

Definition. Suppose that (X, ρ) is a metric space. A sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ of points in X is said to be a Cauchy sequence if, given any $\epsilon > 0$, there exists N such that

$$\rho(\mathbf{x}_m, \mathbf{x}_n) < \epsilon \qquad \text{whenever } m > n \geq N.$$

It is easy to prove the following result.

THEOREM 2F. *Suppose that (X, ρ) is a metric space. Suppose further that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a convergent sequence in X. Then $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.*

The converse of this result is in general not true, although it is true for the metric spaces $\mathbb{R}$ and $\mathbb{C}$ with respect to the euclidean metrics.

Example 2.2.1. Consider the metric space $(0, 1)$ with the euclidean metric for $\mathbb{R}$. Then the sequence $(x_n)_{n\in\mathbb{N}}$, given by $x_n = 1/n$ for every $n \in \mathbb{N}$, is a Cauchy sequence in $(0, 1)$, but $(x_n)_{n\in\mathbb{N}}$ does not converge in $(0, 1)$. Why?

Definition. Suppose that (X, ρ) is a metric space, and that $E \subseteq X$. We say that E is complete if every Cauchy sequence in E has a limit in E. If X is complete, then we say that the metric space (X, ρ) is complete.

EXAMPLE 2.2.2. In any metric space, the set $\emptyset$ is complete.

EXAMPLE 2.2.3. $\mathbb{R}$ and $\mathbb{C}$ with the euclidean metrics are complete.

EXAMPLE 2.2.4. In $\mathbb{R}$ with the euclidean metric, the set $(0,1)$ is not complete.

EXAMPLE 2.2.5. For every $r \in \mathbb{N}$, $\mathbb{R}^r$ with the euclidean metric is complete.

THEOREM 2G. *Suppose that (X, ρ) is a metric space, and that $E \subseteq X$.*
(a) If E is complete, then E is closed.
(b) If X is complete and E is closed, then E is complete.

PROOF. (a) Suppose that $\mathbf{x}$ is an accumulation point of E. Then there exists a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in E such that $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$. By Theorem 2F, $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since E is complete, we must have $\mathbf{x} \in E$. Hence E contains all its accumulation points and so is closed.

(b) Suppose that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E. Then it is a Cauchy sequence in X. Since X is complete, it follows that $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$ some some $\mathbf{x} \in X$. Note that $\mathbf{x}$ is an accumulation point of E, and so must belong to E since E is closed. It follows that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ has a limit in E. ♣

We conclude this section by establishing the following important consequence of completeness. Suppose that (X, ρ) is a metric space. For any bounded subset $E \subseteq X$, we denote the diameter of E by

$$\rho(E) = \sup_{\mathbf{x}_1,\mathbf{x}_2\in E} \rho(\mathbf{x}_1, \mathbf{x}_2).$$

THEOREM 2H. (CANTOR INTERSECTION THEOREM) *Suppose that (X, ρ) is a complete metric space. Suppose further that a sequence $(F_n)_{n\in\mathbb{N}}$ of sets in X satisfies the following conditions:*
(a) For every $n \in \mathbb{N}$, the set F_n is non-empty and closed.
(b) For every $n \in \mathbb{N}$, we have $F_{n+1} \subseteq F_n$.
(c) We have $\rho(F_n) \to 0$ as $n \to \infty$.

Then the intersection $F = \bigcap_{n=1}^{\infty} F_n$ consists of exactly one point.

PROOF. For every $n \in \mathbb{N}$, let $\mathbf{x}_n \in F_n$, since F_n is non-empty. Then in view of (b), we have

$$\rho(\mathbf{x}_m, \mathbf{x}_n) \le \rho(F_n) \le \rho(F_N) \qquad \text{whenever } m > n \ge N.$$

It follows from (c) that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is complete, there exists $\mathbf{x} \in X$ such that $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$. For every $n \in \mathbb{N}$, it is easy to see that $\mathbf{x}$ is an accumulation point of F_n, so that $\mathbf{x} \in F_n$ since F_n is closed. It follows that $\mathbf{x} \in F$ and so F contains at least one point. Suppose next that $\mathbf{y} \in F$. Then $\rho(\mathbf{x}, \mathbf{y}) \le \rho(F_n)$ for every $n \in \mathbb{N}$. We must therefore have $\rho(\mathbf{x}, \mathbf{y}) = 0$, whence $\mathbf{x} = \mathbf{y}$. It follows that F contains precisely one point. ♣

2.3. Compact Metric Spaces

DEFINITION. Suppose that (X, ρ) is a metric space, and that $E \subseteq X$. We say that E is compact if every sequence in E has a convergent subsequence with limit in E. If X is compact, then we say that the metric space (X, ρ) is compact.

REMARK. Compactness of a set $E \subseteq X$ depends on the metric ρ.

Example 2.3.1. In $\mathbb{R}$ with the euclidean metric, the set $[0, 1]$ is compact. However, note that with the discrete metric, the set $[0, 1]$ is no longer compact. To see this, recall that with the discrete metric, the only sequences that are convergent are those that remain constant for all large values of n. It follows that the sequence $(x_n)_{n\in\mathbb{N}}$, given by $x_n = 1/n$ for every $n \in \mathbb{N}$, is not convergent and does not have any convergent subsequence under the discrete metric.

Example 2.3.2. In any metric space, the set $\emptyset$ and any finite set of points are compact.

Example 2.3.3. In $\mathbb{R}$ with the euclidean metric, any closed interval is compact.

Example 2.3.4. For every $r \in \mathbb{N}$, $\mathbb{R}^r$ with the euclidean metric is not compact.

Combining Examples 2.2.5 and 2.3.4, we observe that a complete metric space need not be compact.

THEOREM 2J. *A compact set in a metric space is also complete.*

Sketch of Proof. Suppose that (X, ρ) is a metric space, and that E is a compact set in X. Let $(\mathbf{x}_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in E. Since E is compact, $(\mathbf{x}_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(\mathbf{x}_{n_p})_{p\in\mathbb{N}}$. Suppose that $\mathbf{x}_{n_p} \to \mathbf{x}$ as $p \to \infty$, where $\mathbf{x} \in E$. We need to show that $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$. Suppose not. Then there exists another subsequence $(\mathbf{x}_{n'_r})_{r\in\mathbb{N}}$ of $(\mathbf{x}_n)_{n\in\mathbb{N}}$ which converges to a different limit $\mathbf{x}'$, so that $\rho(\mathbf{x}, \mathbf{x}') > 0$. This will contradict the assumption that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E. ♣

THEOREM 2K. *Suppose that (X, ρ) is a metric space, and that $E \subseteq X$. Then E is compact if and only if every infinite subset of E has at least one accumulation point in E.*

Proof. ($\Rightarrow$) Suppose that E is compact. If E is finite, then there is nothing to prove. Suppose now that E is infinite. Then any infinite subset of E must contain a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ of distinct points in E. Since E is compact, $(\mathbf{x}_n)_{n\in\mathbb{N}}$ has a convergent subsequence with limit in E. Hence E contains at least one accumulation point.

($\Leftarrow$) Suppose that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a sequence in E. If the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ takes finitely many distinct values, then at least one value is taken infinitely often, so that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ has a constant subsequence which clearly converges with limit in E. If the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ takes infinitely many distinct values, then the set $\{\mathbf{x}_n : n \in \mathbb{N}\}$ is an infinite subset of E, and so must have an accumulation point in E. In other words, $(\mathbf{x}_n)_{n\in\mathbb{N}}$ must have a convergent subsequence with limit in E. ♣

THEOREM 2L. *Suppose that (X, ρ) is a metric space, and that $E \subseteq X$.*
(a) If E is compact, then E is bounded and closed.
(b) If X is compact and E is closed, then E is compact.

Proof. (a) If E is compact, then E is complete by Theorem 2J and closed by Theorem 2G. Suppose on the contrary that E is not bounded. Let $\mathbf{a} \in E$. Given any $n \in \mathbb{N}$, there exists $\mathbf{x}_n \in E$ such that $\rho(\mathbf{x}_n, \mathbf{a}) > n$. Then $\rho(\mathbf{x}_n, \mathbf{a}) \to \infty$ as $n \to \infty$. It is clear that the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ has no convergent subsequence, so that E is not compact.

(b) Suppose that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a sequence in E, and so also in X. Since X is compact, $(\mathbf{x}_n)_{n\in\mathbb{N}}$ has a convergent subsequence with limit $\mathbf{x}$, say. If $\mathbf{x} = \mathbf{x}_n$ for some $n \in \mathbb{N}$, then clearly $\mathbf{x} \in E$. If $\mathbf{x} \neq \mathbf{x}_n$ for any $n \in \mathbb{N}$, then $\mathbf{x}$ is an accumulation point of E, so that $\mathbf{x} \in E$ since E is closed. ♣

Remark. For every $r \in \mathbb{N}$, a set in $\mathbb{R}^r$ with the euclidean metric is compact if it is bounded and closed. However, in arbitrary metric spaces, boundedness and closedness do not necessarily imply compactness.

Example 2.3.5. In the metric space $(0, 1)$ with the euclidean metric for $\mathbb{R}$, the set $(0, 1)$ is bounded and closed in $(0, 1)$, but the sequence $(x_n)_{n\in\mathbb{N}}$, given by $x_n = 1/n$ for every $n \in \mathbb{N}$, has no convergent subsequence in $(0, 1)$.

2.4. Continuous Functions with Compact Domains

The purpose of this section is to generalize some of the ideas concerning continuous functions on closed intervals. Clearly such intervals are bounded as well, and therefore form compact sets in $\mathbb{R}$.

THEOREM 2M. *Suppose that (X,ρ) and (Y,σ) are metric spaces. Suppose further that X is compact. Then for any function $f : X \to Y$ which is continuous on X, the range $f(X)$ is also compact.*

PROOF. Suppose that $(\mathbf{y}_n)_{n\in\mathbb{N}}$ is a sequence in $f(X)$. For every $n \in \mathbb{N}$, let $\mathbf{x}_n \in X$ satisfy $f(\mathbf{x}_n) = \mathbf{y}_n$. Then $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a sequence in X. Since X is compact, $(\mathbf{x}_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(\mathbf{x}_{n_p})_{p\in\mathbb{N}}$. Suppose that $\mathbf{x}_{n_p} \to \mathbf{x}$ as $p \to \infty$. Since f is continuous at $\mathbf{x}$, it follows from Theorem 1H that $\mathbf{y}_{n_p} = f(\mathbf{x}_{n_p}) \to f(\mathbf{x})$ as $p \to \infty$. ♣

The following result is a special case of Theorem 2M.

THEOREM 2N. *Suppose that the domain of a continuous real-valued function is compact. Then the function is bounded and attains its supremum and infimum.*

We shall state without proof the following result.

THEOREM 2P. *Suppose that the domain of a bijective, continuous function is compact. Then the inverse function is also continuous.*

REMARK. A bijective, continuous function with a continuous inverse function is called a homeomorphism.

We conclude this chapter by discussing the idea of uniformity.

DEFINITION. Suppose that (X,ρ) and (Y,σ) are metric spaces. A function $f : X \to Y$ is said to be uniformly continuous on X if, given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sigma(f(\mathbf{x}_1), f(\mathbf{x}_2)) < \epsilon \qquad \text{whenever } \rho(\mathbf{x}_1,\mathbf{x}_2) < \delta.$$

Uniform continuity implies continuity. However, the converse is not true. Consider, for example, the function $f(x) = 1/x$ in $(0,1)$ with the euclidean metric in $\mathbb{R}$.

THEOREM 2Q. *Suppose that (X,ρ) and (Y,σ) are metric spaces. Suppose further that X is compact. Then for any function $f : X \to Y$ which is continuous on X, the function f is also uniformly continuous on X.*

PROOF. Suppose on the contrary that f is not uniformly continuous on X. Then there exists $\epsilon > 0$ such that for every $n \in \mathbb{N}$, there exist $\mathbf{x}_n, \mathbf{y}_n \in X$ such that

$$\rho(\mathbf{x}_n,\mathbf{y}_n) < \frac{1}{n} \qquad \text{and} \qquad \sigma(f(\mathbf{x}_n), f(\mathbf{y}_n)) \geq \epsilon.$$

Since X is compact, the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(\mathbf{x}_{n_p})_{p\in\mathbb{N}}$. Suppose that $\mathbf{x}_{n_p} \to \mathbf{c}$ as $p \to \infty$. Then

$$\rho(\mathbf{y}_{n_p},\mathbf{c}) \leq \rho(\mathbf{x}_{n_p},\mathbf{y}_{n_p}) + \rho(\mathbf{x}_{n_p},\mathbf{c}) \to 0 \qquad \text{as } p \to \infty,$$

so that $\mathbf{y}_{n_p} \to \mathbf{c}$ as $p \to \infty$. Since f is continuous on X, it is continuous at $\mathbf{c}$, and so $f(\mathbf{x}_{n_p}) \to f(\mathbf{c})$ and $f(\mathbf{y}_{n_p}) \to f(\mathbf{c})$ as $p \to \infty$. Note now that

$$\sigma(f(\mathbf{x}_{n_p}), f(\mathbf{y}_{n_p})) \leq \sigma(f(\mathbf{x}_{n_p}), f(\mathbf{c})) + \sigma(f(\mathbf{y}_{n_p}), f(\mathbf{c})).$$

This implies that $\sigma(f(\mathbf{x}_{n_p}), f(\mathbf{y}_{n_p})) \to 0$ as $p \to \infty$, clearly a contradiction. ♣

Problems for Chapter 2

1. Suppose that (X, ρ) is a metric space.
   a) Prove that any set in X consisting of precisely a single point is connected.
   b) Prove that any set in X which contains more than one point and contains an isolated point is not connected.

2. Show that in $\mathbb{R}$ with the euclidean metric, any subset of $\mathbb{R} \setminus \mathbb{Q}$ with more than one element is not connected.

3. Suppose that ρ is a metric defined on the set $X = \{x \in \mathbb{R} : 0 < x < 3\}$.
   a) Can the set $\{1, 2\}$ be connected in (X, ρ)? Justify your assertion.
   b) Is the set $\{x \in \mathbb{R} : 1 < x < 2\}$ necessarily connected in (X, ρ)? Justify your assertion.

4. Suppose that E and G are sets in a metric space (X, ρ), and that G is open.
   a) Prove that if $G \cap \overline{E} \neq \emptyset$, then $G \cap E \neq \emptyset$.
   b) Prove that if E is connected, then $\overline{E}$ is connected.
   c) Suppose that $\overline{E}$ is connected. Is E necessarily connected? Justify your assertion.

5. Consider the set $E = \{(x, y) : 0 < x \leq 1 \text{ and } y = \sin(1/x)\}$ in $\mathbb{R}^2$ with the euclidean metric.
   a) Use Theorem 2D to prove that E is connected.
   b) Prove that the set $E \cup \{(x, y) : x = 0 \text{ and } -1 \leq y \leq 1\}$ is connected.

6. For every $n \in \mathbb{N}$, let $s_n = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \ldots + \dfrac{1}{n!}$.
   a) Show that $(s_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$ under the euclidean metric for $\mathbb{R}$.
   b) Show that for every $q \in \mathbb{N}$, we have $q!s_q \in \mathbb{Z}$ and $0 < q!(e - s_q) < 1$, where $\mathrm{e} = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$.
   c) Deduce that e is irrational and that $\mathbb{Q}$ is not complete.

7. Suppose that (X, ρ) is a metric space.
   a) Prove that the intersection of any collection of complete subsets of X is complete.
   b) Prove that the union of a finite number of complete subsets of X is complete.

8. Consider the unit cube $S = [0, 1]^3$ in $\mathbb{R}^3$ with the euclidean metric.
   a) Prove that S is connected.
   b) Prove that S is compact.

9. Suppose that (X, ρ) is a metric space.
   a) By modifying the proof of Theorem 2L(b), prove that the intersection of a compact set and a closed set in X is compact.
   b) Using part (a) and Theorem 1D(a), prove that the intersection of any collection of compact sets in X is compact.
   c) Prove that the union of two compact sets in X is compact.
   d) Is the union of any collection of compact sets in X necessarily compact? Justify your assertion.

10. Suppose that E is a non-empty compact set in a metric space (X, ρ). Prove that there exist $\mathbf{x}, \mathbf{y} \in E$ such that $\rho(\mathbf{x}, \mathbf{y}) = \displaystyle\sup_{\mathbf{u}, \mathbf{v} \in E} \rho(\mathbf{u}, \mathbf{v})$.

    [Hint: The proof has some similarity with the proof of the Bolzano-Weierstrass theorem – picking a subsequence of a subsequence.]

11. Suppose that (X,ρ) is a metric space. For any two non-empty subsets A and B of X, we define their distance $\rho(A,B)$ by $\rho(A,B) = \inf_{\mathbf{x}\in A,\, \mathbf{y}\in B} \rho(\mathbf{x},\mathbf{y})$.
    a) Prove that if A and B are compact, then $\rho(A,B) = \rho(\mathbf{a},\mathbf{b})$ for some $\mathbf{a} \in A$ and $\mathbf{b} \in B$.
    b) Show that if A is compact and B is closed, then $\rho(A,B) > 0$, but there may not exist $\mathbf{a} \in A$ and $\mathbf{b} \in B$ such that $\rho(A,B) = \rho(\mathbf{a},\mathbf{b})$.
    c) Give an example of a metric space (X,ρ) with disjoint closed subsets A and B of X such that $\rho(A,B) = 0$.

12. The domain of a complex valued function f is the open set $G = \{(x,y) \in \mathbb{R}^2 : |x| < 1 \text{ and } |y| < 1\}$ in $\mathbb{R}^2$. Suppose that f is continuous on G.
    a) Is f necessarily bounded on the open disc $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < \frac{1}{2}\}$?
    b) Is f necessarily bounded on the open disc $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$?

13. Suppose that $f : [0,\infty) \to \mathbb{R}$ is continuous, and that $f(x) \to \ell$ as $x \to \infty$. Prove that f is uniformly continuous on $[0,\infty)$.

14. A real valued function g is uniformly continuous and differentiable on the open interval $(0,\infty)$. Is the derivative g' necessarily bounded on $(0,\infty)$? Justify your assertion.

15. For each of the following functions, determine whether the function is uniformly continuous on the interval $[0,\infty)$, and justify your assertion:
    a) $f(x) = x^2$ b) $f(x) = \sin x$ c) $f(x) = \sin(x^2)$ d) $f(x) = \dfrac{\sin(x^2)}{x+1}$

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