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1 Introduction

In 1906, Henry S. Graves wrote: "Forest mensuration deals with the determination of the volume of logs, trees, and stands, and with the study of increment and yield." Since that definition was written, however, the scope of forestry has widened. Although some foresters feel that Grave's definition is still adequate, others feel that mensuration should embrace the new measurement problems that have been created as the horizons of forestry have expanded.

If we accept the challenge of a broader field of study, we must ask: To what degree should mensuration be concerned with the measurement problems of wildlife management, recreation, watershed management, and the other aspects of multiple-use forestry? Furthermore, one might argue that it is unrealistic to imagine that forest mensuration can take as its domain such a diverse group of subjects.

The question becomes irrelevant and the objection allayed if we recognize forest mensuration as a subject that provides principles applicable to all measurement problems. Consequently, this book will include principles that provide a foundation for solving measurement problems in all aspects of forestry. However, the traditional measurement problems of forestry will not be neglected.

Through the years a number of textbooks on forest mensuration have been prepared in the United States and in foreign lands. In the United States the most recent are Bruce and Schumacher (1950); Spurr (1952); Meyer (1953); Husch (1963); Husch, Miller, and Beers (1972); and Avery (1975). In other countries the most recent are Prodan (1965), Germany; Seip (1964), Norway; Giurgiu (1968), Romania; and Carron (1968), Australia. Also, Loetsch, Zöhrer, and Haller (1973) give a comprehensive and imaginative treatment of many aspects of forest mensuration.

Since the end of World War II the application of statistical theory and the use of computers and programmable calculators in forest mensuration have wrought a revolution in the solution of forest measurement problems. Consequently, the mensurationist must be competent in these areas as well as in basic mathematics. A knowledge of calculus is also desirable. In addition, a knowledge of systems analysis and operations research, approaches to problem solving that depend on model building and techniques that include simulation and mathematical programming, is also helpful.

1-1 IMPORTANCE OF FOREST MENSURATION

Forest mensuration is one of the keystones in the foundation of forestry. Forestry, in the broadest sense, is a management activity involving forest land, the plants and animals on the land, and humans as they use the land. Thus, the forester is faced with many decisions in the management of a forest. The following questions convey a general idea of the problems that must be solved for a particular forest.

1. What silvicultural treatment will result in the best regeneration and growth?
2. What species is most suitable for reforestation?
3. Is there sufficient timber for an economical harvesting operation?
4. What is the value of the timber and land?
5. What is the recreational potential?
6. What is the wildlife potential?

A forester needs information to make intelligent decisions for these and countless other questions. Whenever possible, this information should be in quantifiable terms. As it has aptly been said, "You can't efficiently make, manage, or study anything you don't locate and measure." In this sense, forest mensuration is the application of measurement principles to obtain quantifiable information for decision making.

1-2 PRINCIPLES OF MEASUREMENT

Knowledge is to a large extent the result of the acquisition and systematic accumulation of observations, or measurements, of concrete objects and natural phenomena. Thus, measurement is a basic requirement for the extension of knowledge.

In its broadest sense, measurement consists of the assignment of numbers to measurable properties. Ellis (1966) gives this definition: "Measurement is the assignment of numerals to things according to any determinative, non-degenerate rule." ("Determinative" means that the same numerals, or range of numerals, are always assigned to the same things under the same conditions. "Non-degenerate" allows for the possibility of assigning different numerals to different things, or to the same thing under different conditions.) This definition implies that we have a scale that allows us to use a rule, and that each scale inherently has a different rule that must be adhered to in representing a property by a numerical quantity. Stevens (1946) summarized the problem and formulated a classification for different kinds of scales. In spite of some shortcomings, Stevens classification is useful in understanding the measurement process (Table 1-1).

1-2.1 Scales of Measurement

The four scales of measurement are nominal, ordinal, interval, and ratio (Table 1-1).

The *nominal scale* is used for numbering objects for identification (e.g., numbering of forest types in a stand map), and for numbering a class when each member of the class is assigned the same numeral (e.g., the assignment of code numbers to species).

Table 1-1
Classification of Scales of Measurements[1]

Scale	Basic Operation	Mathematical Group Structure	Permissible Statistics	Examples
Nominal	Determination of equality (numbering and counting)	Permutation group $X' = f(X)$ where $f(X)$ means any one-to-one substitution	Number of cases Mode Contingency correlation	Numbering of forest types on a stand map Assignment of code numbers to tree species in studying stand composition
Ordinal	Determination of greater or less (ranking)	Isotonic group $X' = f(X)$ where $f(X)$ means any increasing monotonic function	Median Percentiles Order correlation	Lumber grading Tree and log grading Site class estimation
Interval	Determination of the equality of intervals or of differences (numerical magnitude of quantity, arbitrary origin)	Linear group $X' = \alpha X + b$, $\alpha > 0$	Mean Standard deviation Correlation coefficient	Fahrenheit temperature Calendar time Available soil moisture Relative humidity
Ratio	Determination of the equality of ratios (numerical magnitude of quantity, absolute origin)	Similarity group $X' = cX$, $c > 0$	Geometric mean Harmonic mean Coefficient of variation	Length of objects Frequency of items Time intervals Volumes Weights Absolute temperature Absolute humidity

[1] Columns 2, 3, 4, and 5 are cumulative in that all characteristics listed opposite a particular scale are additive to those above it. In the column which records the group structure of each scale are listed the mathematical transformations which leave the scale invariant. Thus any numeral X on a scale can be replaced by another numeral X', where X' is the function of X listed in this column. The criterion for the appropriateness of a statistic is invariance under the transformations in column 3. Thus the case that stands at the median of a distribution maintains its position under all transformations which preserve order (isotonic group), but an item located at the mean remains at the mean only under transformations as restricted as those of the linear group. The ratio expressed by the coefficient of variation remains invariant only under the similarity transformation (multiplication by a constant). The rank-order correlation coefficient is usually considered appropriate to the ordinal scale, although the lack of a requirement for equal intervals between successive ranks really invalidates this statistic.

SOURCE: Adapted from S. S. Stevens, "On Theory of Scales of Measurement," *Science* 103(2684): 677–680.

The *ordinal scale* is used to express degree, quality, or position in a series, such as first, second, and third. In a scale of this type, the successive intervals on the scale are not necessarily equal. This scale is used for lumber grades, log grades, tree grades, and site classifications.

The *interval scale* includes a series of graduations marked off at uniform intervals from a reference point of fixed magnitude. There is no absolute reference point or true origin for the scale. The origin is arbitrarily chosen. The Celsius temperature scale is a good example of an interval scale. Equal intervals of temperature are scaled off by noting equal volumes of expansion referenced to an arbitrary zero.

The *ratio scale* is similar to the interval scale in that there is equality of intervals between successive points on the scale. However, an absolute zero of origin is always present or implied. Ratio scales are the most commonly employed and the most versatile in that all types of statistical measures are applicable. It is convenient to consider ratio scales as fundamental and derived.

- *Fundamental* scales are represented by such things as frequency, length, weight, and time intervals.
- *Derived* scales are represented by such things as stand volume per acre, stand density, and stand growth per unit of time. (These are derived scales in that the values on the scale are functions of two or more fundamental values.)

1-3 UNITS OF MEASUREMENT

To describe a physical quantity, one must establish a unit of measure and determine the number of times the unit occurs in the quantity. Thus, if an object has a length of 3 meters, the meter has been taken as the unit of length, and the length dimension of the object contains three of these standard units.

The *fundamental units* in mechanics are measures of length, mass, and time. These are regarded as independent and fundamental variables of nature, although they have been chosen arbitrarily by scientists. Other fundamental units have been established for thermal, electrical, and illumination quantity measurement.

Derived units are expressed in terms of fundamental units or in units derived from fundamental units. Derived units include ones for the measurement of volume (cubic feet or meters), area (acres or hectares), velocity (miles per hour, meters per second), force (kilogram-force), etc. Derived units are often expressed in formula form. For example, the area of a rectangle is defined by the equation

$$\text{Area} = WL$$

where W and L are fundamental units of length.

Physical quantities such as length, mass, and time are called scalar quantities or scalars. Physical quantities that require an additional specification of direction for their complete definition are called *vector* quantities or vectors.

1-4 SYSTEM OF UNITS

There are two methods of establishing measurement units. We may select an arbitrary unit for each type of quantity to be measured, or we may select fundamental units and formulate from them a consistent system of derived units. The first method was employed extensively in our early history. For example, units for measuring the length of cloth, the height of a horse, or land distances were all different. Reference units were objects such as the width of a barleycorn, the length of a man's foot, the length of a man's forearm (a cubit), and so on. However, these primitive units lacked uniformity. Vestiges of this system still exist, particularly in English-speaking countries (foot, yard, pound, etc.), although the units are now uniform.

The second method of establishing a system of units is illustrated by the metric system. In this system, an arbitrary set of units has been chosen that is uniformly applicable to the measurement of any object. Moreover, there is a logical, consistent, and uniform relationship between the basic units and their subdivisions.

1-4.1 Metric System

This system of weights and measures was formulated by the French Academy of Sciences in 1790. The system was adopted in France in 1799 and made compulsory in 1840. In 1875, the International Metric Convention, which was established by treaty, furnished physical standards of length and mass to the 17 member nations. The General Conference on Weights and Measures (referred to as CGPM from the French "Conférence Générale des Poids et Mesures") is an international organization established under the Convention. This organization meets periodically. The CGPM controls the International Bureau of Weights and Measures (BIPM), which is headquartered at Sèvres, near Paris, and maintains the physical standards of units. The United States Bureau of Standards represents the United States on the CGPM and maintains our standards of measure.

The metric system has been adopted by most of the technologically developed countries of the world. Although conversion in Great Britain and in the United States has met with some resistance, a gradual changeover is taking place. In 1975, the President signed the Metric Conversion Act in which the United States adopted a policy of actively encouraging a voluntary changeover to the metric system of weights and measures.

Discussions of the problems affecting forestry in converting to the metric system in the United States and Great Britain are presented by White (1971), Bruce (1974, 1976), and Hamilton (1974). Furthermore, an excellent summary of the system as now internationally accepted is given in the National Bureau of Standards Publication 330 (1977). The following information is abstracted from this publication.

The 11th meeting of the CGPM in 1960 adopted the name "International System of Units" with the international abbreviation SI (from the French "Le Système International d'Unités"). This is now the accepted form of the metric system. Other adaptations of metric units such as in the CGS, MTS, and MKS systems are discouraged.

The SI considers three classes of units: (1) base units, (2) derived units, and (3) supplementary units. There are seven *base units*, which by convention are considered dimensionally independent. These are the meter, kilogram, second, ampere, kelvin, mole, and candela. The *derived units* are formed by combining base units according to algebraic statements that relate the corresponding quantities. The *supplementary units* are those that the CGPM established without stating whether they are base or derived units.

Here is a list of the dimensions that are measured by these base units, along with the definitions of the units. The conventional symbol for each unit is shown in parentheses.

1. Length—meter (m). The meter is equal to 1,650,763.73 wavelengths in a vacuum of the orange-red light given off by krypton-86.
2. Mass—kilogram (kg).[1] The kilogram is equal to the mass of the international prototype standard, a cylinder of platinum-iridium alloy kept at the BIPM.
3. Time—second (s). The second was originally defined as 1/86,400 part of a mean solar day. In 1956, it was redefined as 1/31,556,925.9747 of the tropical year for 1900. More recently, it has been calculated by atomic standards to be 9,192,631,770 periods of vibration of the radiation emitted at a specific wavelength by an atom of cesium-133.
4. Electric current—ampere (A). The ampere is the current in a pair of equally long, parallel, straight wires (in vacuum and 1 meter apart) that produces a force of 2×10^{-7} newtons between the wires for each meter of their length.
5. Temperature—kelvin (K). The kelvin is 1/273.15 of the thermodynamic temperature of the triple point of water. The temperature 0 K is called absolute zero. The kelvin degree is the same size as the Celsius degree (also called centigrade). The freezing point of water (0°C) and the boiling point of water (100°C) correspond to 273.15 K and 373.15 K, respectively. On the Fahrenheit scale, 1.8 degrees are equal to 1.0°C or 1.0 K. The freezing point of water on the Fahrenheit scale is 32°F.
6. Amount of substance—mole (mol). The mole is a base unit used to specify the quantity of chemical elements or compounds. It is the amount of substance of a system that contains as many elementary entities as there are atoms in 0.012 kilogram of carbon-12. When the mole is used, the elementary entities must be qualified. They may be atoms, molecules, ions, electrons, or other particles, or specified groups of such particles.

[1] The term "weight" is commonly used for mass although this is, strictly speaking, incorrect. The weight of a body means the force caused by gravity, acting on a mass, which varies in time and space and which differs according to the location on earth. Since it is important to know whether mass or force is being measured, the SI has established two units: the kilogram for mass and the newton for force.

7. Luminous intensity—candela (cd). The candela is 1/600,000 of the intensity, in the perpendicular direction, of one square meter of a black body radiator at the temperature at which platinum solidifies (2045 K) under a pressure of 101,325 newtons per square meter.

Derived units are expressed algebraically in terms of the base units by means of mathematical symbols of multiplication and division. Some examples of derived units are given here.

Quantity	**SI Unit for the Quantity**	**Symbol**
Area	square meter	m^2
Volume	cubic meter (the liter, 0.001 cubic meter, is not an SI unit although commonly used to measure fluid volume)	m^3
Specific volume	cubic meter per kilogram	m^3/kg
Force	newton ($1\ N = 1\ kg \cdot m/s^2$)	N
Pressure	pascal ($1\ Pa = 1\ N/m^2$)	Pa
Work	joule ($1\ J = 1\ N \cdot m$)	J
Power	watt ($1\ W = 1\ J/s$)	W
Speed	meter per second	m/s
Acceleration	(meter per second) per second	m/s^2
Voltage	volt ($1\ V = 1\ W/A$)	V
Electric resistance	ohm ($1\ \Omega = 1\ V/A$)	Ω
Concentration (amount of substance)	mole per cubic meter	mol/m^3

At present, there are only two *supplementary units*, the radian and the steradian. They are defined as follows.

- The radian (rad) is the plane angle between two radii of a circle that cuts off on the circumference an arc equal to the radius. It is 57.29578 degrees for every circle.
- The steradian (sr) is the solid angle at the center of a sphere subtending a section on the surface equal in area to the square of the radius of the sphere.

There is a number of widely used units that are not part of SI. These units, which the International Committee on Weights and Measures (CIPM) recognized in 1969, are shown below.

Unit	**Symbol**	**Equivalence in SI Units**
minute	min	1 min = 60 s
hour	h	1 h = 60 min = 3600 s

Unit	Symbol	Equivalence in SI Units
day	d	$1\ d = 24\ h = 86{,}400\ s$
degree (angular)	°	$1° = (\pi/180)$ rad
minute (angular)	′	$1' = (1/60)° = (\pi/10{,}800)$ rad
second (angular)	″	$1'' = (1/60)' = (\pi/648{,}000)$ rad
liter	L	$1\ L = 1\ dm^3 = 10^{-3}\ m^3$
metric ton	t	$1\ t = 10^3$ kg

To form decimal multiples or fractions of SI units, the following prefixes or abbreviations are used.

Prefix	Factor	Abbreviation
tera	10^{12}	T
giga	10^9	G
mega	10^6	M
kilo	10^3	k
hecto	10^2	h
deka	10^1	da
deci	10^{-1}	d
centi	10^{-2}	c
milli	10^{-3}	m
micro	10^{-6}	μ
nano	10^{-9}	n
pico	10^{-12}	p

1-4.2 English System

The English system of weights and measures is still widely used in the United States but it should be supplanted gradually by the metric system (SI). This may be a long process since custom and tradition are strong, and the conversion is currently voluntary. Great Britain has taken a more vigorous stance and conversion to the metric system is proceeding rapidly (Hamilton, 1974).

The units of the English system still commonly used in the United States are practically the same as those employed in the American colonies prior to 1776. The names of these units are generally the same as those of the British Imperial System. However, their values differ slightly.

In the English system, the fundamental length is the yard. The British yard is the distance between two lines on a bronze bar kept in the Standards Office, Westminster, London. Originally, the United States yard was based on a prototype bar, but in 1893 the United States yard was redefined as 3600/3937 meter = 0.9144018 meter. The

British yard, on the other hand, was 3600/3937.0113 meter = 0.914399 meter. The foot is defined as the third part of a yard, and the inch as a twelfth part of a foot. In 1959, it was agreed by Canada, United States, New Zealand, United Kingdom, South Africa, and Australia that they would adopt the value of 1 yard = 0.9144 meter. In the United Kingdom, these new values were used only in scientific work, the older, slightly different values being used for other measurements. And in the United States, the results of geodetic surveys are still expressed in feet based on the former definition of the yard (3600/3937 meter).

In the British system, the unit of mass, the avoirdupois pound, is the mass of a certain cylinder of platinum in the possession of the British government (1 pound = 0.45359243 kilogram). In the United States, the unit of mass is also the pound, but in 1893 it was defined in terms of the mass unit of the metric system (1 pound = 0.4535924277 kilogram). Thus, the United States pound and the British pound were not exactly equal. The 1959 agreement of English-speaking countries established a new value of 1 pound = 0.45359237 kilogram.

In the British system, the second is the base unit of time, as in the metric system.

Secondary units in the English system also have different values in the United States and the United Kingdom. For example, the U.S. gallon is defined as 231 cubic inches. The British gallon is defined as 277.42 cubic inches. There are other secondary units used in English-speaking countries that are as arbitrary as the gallon, but some are derived from the fundamental units. For example, the unit of work in English and American engineering practice has been the foot-pound.

Very commonly, conversions from one system of measurement to another are required. Appendix Table A-1 shows unit conversions for length, area, volume, and mass in the English and the metric systems.

1-5 NUMBERING SYSTEMS

The numbering system in general use throughout the world is the *decimal system*. This can probably be attributed to the fact that human beings have ten fingers. But the decimal system is merely one of many possible numbering systems that could be utilized. In fact, there are examples of other numbering systems used by earlier civilizations (e.g., the vigesimal system, based on twenty, utilized by the Mayas, and the sexagesimal system of the Babylonians, based on sixty). Our own system of measuring time and angles in minutes and seconds comes from the sexagesimal system. Systems to other bases, such as the duodecimal system, based on twelve (which seems to have lingered on in the use of dozen and gross), may also have been used. For a discussion of the history of number theory, the student is referred to Ore (1948).

With the development of the electronic computer, interest has been revived in numbering systems using bases other than ten. Of primary interest is the binary system, because electronic digital computers that use two basic states have been found most practical.

1-6 CONTINUOUS AND DISCRETE VARIABLES

A *variable* is a characteristic that may assume any given value or set of values. Some variables are continuous in that they are capable of exhibiting every possible value within a given range. For example, height, weight, and volume are continuous variables. Other variables are discontinuous or discrete in that they only have values which jump from one number or position to the next. The number of employees in a company, of trees in a stand, or of deer per unit area are examples of discrete variables.

Data pertaining to continuous variables are obtained by measuring. Data pertaining to discrete variables are obtained by counting. The problems of measuring continuous variables are dealt with in Chapters 2, 3, 4, 5, 6, and 7 and will not be discussed here. But we will consider the measurement of discrete variables at this time.

The process of measuring according to nominal and ordinal scales, as shown in Table 1-1, consists of counting the frequencies of occurrences of specified events. Discrete variables describe these events. The general term "event" can refer to a discrete physical standard, such as a tree, which exists as a tangible object occupying space, or to an occurrence that cannot be thought of as spatial, such as a timber sale. In either case, the measurement consists of defining the variable and then counting the number of its occurrences. There is no choice for the unit of measurement—the frequency is the only permissible numerical value.

It is important not to confuse a class established for convenience in continuous-type measurement with a discrete variable. Classes for continuous variables are often established to facilitate the handling of data in computations. Frequencies may then be assigned to these classes. These frequencies represent the occurrence or recurrence of certain measurements of a continuous variable that have been placed in a group or class of defined limits for convenience.

A discrete variable can thus be characterized as a class or series of classes of defined characteristics with no possible intermediate classes or values. A few examples of discrete variables used in forestry are species, lumber grades, and forest-fire danger classes.

At times, it may not be clear as to whether a discrete or continuous variable is being measured. For example, in counting the number of trees per acre or per hectare, the interval, one tree, is so small and the number of trees so large that analyses are made of the frequencies as though they described a continuous variable. This has become customary and may be considered permissible if the true nature of the variable is understood.

1-6.1 Forest Measurements on a Nominal Scale

Species names or forest types are examples of the use of discrete variables on nominal scales. Tree species, for example, can be the classes for the variable. The order in which the classes are recorded does not affect the discrete variable, the species. Indeed, the order could be changed without changing the meaning.

It is frequently convenient to assign a code number or letter to each class of a discrete variable. Code numbers are especially useful if data are to be entered on punch cards for machine computation. It is important to be aware that the code numbers have no intrinsic meaning but are merely identifying labels. No meaningful mathematical operations can be performed on such code numbers.

1-6.2 Forest Measurement on an Ordinal Scale

Ordinal scales for discrete variables abound in forestry. Examples are lumber grades, log grades, piece products grades, Christmas tree grades, nursery stock grades, and site quality classes.

The order in which the classes of a discrete variable are arranged on an ordinal scale has an intrinsic meaning. The classes are arranged in order of increasing or decreasing qualitative rank, so the position on the scale affords an idea of comparative rank. The continuum of the variable consists of the range between the limits of the established ranks or grades. As many ranks or grades can be established as are deemed suitable. An attempt may be made to have each grade or rank occupy an equal interval of the continuum. However, this will rarely be achieved, since the ranks are subjectively defined with no assurance of equal increments between ranks.

Chapter 10 discusses in more detail the principles of quality measurement and its applications in forestry.

1-7 PRECISION, ACCURACY, AND BIAS

The terms "precision" and "accuracy" are frequently used interchangeably in nontechnical parlance and often with varying meaning in technical usage. In this text they will have two distinct meanings. *Precision* as used here (and generally accepted in forest mensuration) means the degree of agreement in a series of measurements.[2] *Accuracy*, on the other hand, is the closeness of a measurement to the true value. Of course, the ultimate objective is to obtain accurate measurements.

Bias refers to systematic errors that may result from faulty measurement procedures, instrumental errors, flaws in the sampling procedure, errors in the computations, mistakes in recording, and so forth.

In sampling, accuracy refers to the size of the deviation of a sample estimate from the true population value. Precision, expressed as a standard deviation, refers to the deviation of sample values about their own mean, which, if biased, does not correspond to the true population value. It is possible to have a very precise estimate in that the deviations from the sample mean are very small; yet, at the same time, the estimate may not be accurate if it differs from the true value due to bias. For example, one

[2] The term is also used to describe the resolving power of a measuring instrument or the smallest unit in observing a measurement. In this sense, the more decimal places used in the measurement the more precise the measurement.

might carefully measure a tree diameter repeatedly to the nearest millimeter with a caliper that reads about 5 mm low (Section 2-1.3). The results of this series of measurements are precise because there is little variation between readings, but they are biased and inaccurate because of improper adjustment of the instrument.

Bruce (1975) has shown that bias B and precision P can be equated with accuracy A as follows.

$$A^2 = B^2 + P^2$$

This indicates that, if we reduce B^2 to zero, accuracy equals precision.

1-8 SIGNIFICANT DIGITS

A significant digit is any digit denoting the true size of the unit at its specific location in the overall number. The term significant as used here should not be confused with its use in reference to statistical significance. The significant figures in a number are the digits reading from left to right beginning with the first nonzero digit and ending with the last digit written, which may be a zero. The numbers 24, 2.5, 0.25, and 0.025 all have two significant figures, the 2 and the 5. The numbers 25.0, 0.250, and 0.0250 all have three significant figures, the 2, 5, and 0. When one or more zeros occur immediately to the left of the decimal position and there is no digit to the right of the point, the number of significant digits may be in doubt. Thus, the number 2500 may have two, three, or four significant digits, depending on whether one or both zeros denote an actual measurement or have been used to round off a number and indicate the position of the decimal point. Thus, zero can be a significant figure if used to show the quantity in the position it occupies and not merely to denote a decimal place. A convention sometimes used to indicate the last significant digit is to place a dot above it. Thus, $5{,}12\dot{1}{,}000$ indicates four significant digits and $5{,}121{,}\dot{0}00$ indicates five significant digits. Another method is to first divide a number into two factors, one of them being a power of ten. A number such as 150,000,000 could be written as 1.5×10^8 or in some other form such as 15×10^7. A convention frequently used is to show the significant figures in the first factor and to use one nonzero digit to the left of the decimal point. Thus, the numbers 156,000,000 (with three significant figures), 31.53, and 0.005301 would be written as 1.56×10^8, 3.153×10, and 5.301×10^{-3}.

If a number has a significant zero to the right of the decimal place following a nonzero number, it should not be omitted. For example, 1.05010 indicates six significant digits including the last zero to the right. To drop it would reduce the precision of the number. Zeros when used to locate a decimal place are not significant. In the number 0.00530, only the last three digits 5, 3, and 0 are significant; the first two to the right of the decimal place are not.

When the units used for a measurement are changed, it may change the number of decimal places but not the number of significant digits. Thus, a weight of 355.62

grams has five significant figures, as does the same weight expressed as 0.35562 kilograms, although the number of decimal places has increased. This emphasizes the importance of specifying the number of significant digits in a measurement rather than simply the number of decimal places.

1-9 SIGNIFICANT DIGITS IN MEASUREMENTS

The numbers used in mensuration can be considered as arising from pure numbers, from direct measurements, and from computations involving pure numbers and values from direct measurements.

Pure numbers can be the result of a count in which a number is exact, or they can be the result of some definition. Examples of pure numbers are the number of sides on a square, the value of π, or the number of meters in a kilometer.

Values of direct measurements are obtained by reading a measuring instrument (e.g., measuring a length with a ruler). The numerical values obtained in this way are approximations in contrast to pure numbers. The precision of the approximation is indicated by the number of significant digits used. For example, measurement of a length could be taken to the nearest one, tenth, or hundredth of a foot, and recorded as 8, 7.6, or 7.60. Each of these measurements implies an increasing standard of precision. A length of 8 feet means a length closer to 8 than to 7 or 9 feet. The value of 8 can be considered to lie between 7.5 and 8.5. Similarly, a length of 7.6 means a measurement whose value is closer to 7.6 feet than to 7.5 or 7.7. The value of 7.6 lies between 7.5500 . . . 01 and 7.6499 . . . 99, or, conventionally, 7.55 and 7.65. In the measurement 7.60, the last digit is significant and the measurement implies a greater precision. The value 7.60 means the actual value lies anywhere between 7.59500 . . . 01 and 7.60499 . . . 99, or, conventionally, 7.595 and 7.605.

It is incorrect to record more significant digits than were observed. Thus, a length measurement of 8 feet taken to the nearest foot should not be recorded as 8.0 feet since this may mislead the reader into thinking the measurement is more precise than it actually is. On the other hand, one should not omit significant zeros in decimals. For example, one should write 112.0 instead of 112 if the zero is significant.

Since the precision of the final results is limited by the precision of the original data, it is necessary to consider the numbers of significant figures to take and record in original measurements. It is well to keep in mind that using greater precision than needed is a waste of time and money. A few suggestions follow.

1. Do not try to make measurements to a greater precision (more significant digits) than can be reliably indicated by the measuring process or instrument. For example, it would be illogical to try to measure the height of a standing tree to the nearest tenth of a foot with an Abney level.
2. The precision needed in original data may be influenced by how large a difference is important in comparing results. Thus, if the results of a series of silvicultural treatments are to be compared in terms of volume growth

response to the nearest tenth of a cubic meter, then there would be no need to estimate volumes more exactly than the nearest tenth of a cubic meter.

3. The variation in a population sampled and the size of the sample influence the precision chosen for original measurement. If the population varies greatly, or if there are few observations in the sample, then high measurement precision is not worthwhile.

1-10 ROUNDING OFF

When dealing with the numerical value of a measurement in the usual decimal notation, it is often necessary to round off to fewer significant figures than originally shown. Rounding off can be done by deleting the unwanted digits to the right of the decimal point (the fractional part of a number) and by substituting zeros for those to the left of the decimal place (the integer part). Three cases can arise: (1) If the deleted or replaced digits represent less than one-half unit in the last required place, no further change is required. (2) If the deleted or replaced digits represent more than one-half unit in the last required place, then this significant figure is raised by one. (Note that, if the significant figure in the last required place is 9, it changes to zero and the digit preceding it is increased by one.) (3) If the deleted or replaced digits represent exactly one-half unit in the last required place, a recommended convention is to raise this last digit by unity if it is odd but let it stand if it is even. Thus, 31.45 would be rounded to 31.4 but 31.55 would be 31.6. Here are a few examples.

	Number Rounded To:		
Number	**4 Significant Figures**	**3 Significant Figures**	**2 Significant Figures**
4.6495	4.650	4.65	4.6
93.65001	93.65	93.7	94
567851	567900	568000	570000
0.99687	0.9969	0.997	1.0

1-11 SIGNIFICANT DIGITS IN ARITHMETIC OPERATIONS

In arithmetic operations involving measurements, where figures are only approximations, the question of how many significant digits there are in the result becomes important.

In multiplication and division, the factor with the fewest significant figures limits the number of significant digits in the product or quotient. Thus, in multiplying a numerical measurement with five significant figures by another with three significant figures, only the first three figures of the product will be trustworthy, although there

may be up to eight digits in the product. For example, if the measurement 895.67 and 35.9 are multiplied, the product is 32,154.553. Only the first three figures in the product—3, 2, and 1—are significant. The number 895.67 represents a measurement between 898.665 and 895.675. The number 35.9 represents a measurement between 35.85 and 35.95. The products of the four possible limiting combinations will differ in all except the first three figures; thus,

$$(895.665)(35.85) = 32109.59025$$

$$(895.665)(35.95) = 32199.15675$$

$$(895.675)(35.85) = 32109.94875$$

$$(895.675)(35.95) = 32199.51625$$

Therefore, the first three figures are the only reliable ones in the product. Similarly, in dividing a measurement with eight significant digits by a measurement with three significant figures, the quotient will have only three significant figures. But, if a measurement is to be multiplied or divided by an exact number or a factor that is known to any desired number of significant digits, a slightly different situation occurs. For example, a total weight could be estimated as the product of a mean weight having five significant digits times 55. However, the 55 is an exact number and could also be validly written as 55.000. The product would thus still have five significant digits. It may be helpful to remember that multiplication is merely repetitive addition and the 55, in this case, means that a measurement is added exactly 55 times. Similarly, if the 55 objects had been weighed, as a group, to five significant digits, dividing by 55 would give a mean weight to five significant figures. In these cases, the significant digits are controlled by the number in the measurement. Another case occurs if a measurement is to be multiplied or divided by a factor such as π or e (base of Naperian logarithms), which are known to any number of significant figures. The number of significant digits in π or e should be made to agree with the number in the measurement before the operation of multiplication or division so that there is no loss in precision.

A good rule in multiplication or division is to keep one more digit in the product or quotient than occurs in the shorter of the two factors. This minimizes rounding-off errors in calculations involving a series of operations. At the end of the calculations, the final answer can be rounded off to the proper number of significant figures.

In addition and subtraction, the position of the decimal points will affect the number of significant digits in the result. It is necessary to align numbers according to their decimal places in order to carry out these operations. The statement that measurements can be added or subtracted when significant digits coincide at some place to the left or right of the decimal point can be used as a primary guide. Also, the number of significant digits in an answer can never be greater than those in the largest of the numbers, but may be fewer. As one example, measurements of 134.023 and 1.5 can be added or subtracted as shown below, since significant digits coincide at some place.

$$\begin{array}{r} 134.023 \\ +\ \underline{\quad 1.5} \\ 135.523 \end{array}$$

The sum has only four significant figures and should be expressed as 135.5. The last two significant figures of 134.023 cannot be used, since there is no information in the smaller measurement for coinciding positions. It is desirable to take measurements to uniform standards of significant figures or decimal places to avoid discarding a portion of a measurement, as we did in the case of the last two digits of 134.023.

Another example to consider is the addition of a series of measurements, the final total of which may have more figures than any of the individual measurements. The number of significant digits in the total will not exceed the number in the largest measurement. Consider the eleven measurements shown here.

Measurement	Range	
845.6	845.55	845.65
805.8	805.75	805.85
999.6	999.55	999.65
963.4	963.35	963.45
897.6	897.55	897.65
903.1	903.05	903.15
986.9	986.85	986.95
876.3	876.25	876.35
863.2	863.15	863.25
931.2	931.15	931.25
998.1	998.05	998.15
10,070.8	10,070.25	10,071.35

The total, 10,070.8, contains six digits, but only the first four are significant. Each measurement can be thought of as an estimate within a range, as shown in the two right-hand columns. Ths sum of the lesser values is 10,070.25 and that of the larger is 10,071.35. The total value of the sum of the eleven measurements can fall anywhere within these limits. The significant figures are 1, 0, 0, and 7. Beyond this, the digits are unreliable.