Statistical distributions for fitting diameter and height data in even-aged stands[1]

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The beta, Johnson's S_B, Weibull, lognormal, gamma, and normal distributions are discussed in terms of their flexibility in the skewness squared (β_1) – kurtosis (β_2) plane. The S_B and the beta are clearly the most flexible distributions since they represent surfaces in the plane, whereas the Weibull, lognormal, and gamma are represented by lines, and the normal is represented by a single point.

The six distributions are fit to 21 data sets for which both diameters and heights are available. The log likelihood criterion is used to rank the six distributions in regard to their fit to each data set. Overall, Johnson's S_B distribution gave the best performance in terms of quality of fit to the variety of sample distributions.

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Les fonctions de distribution bêta, Johnson S_B, Weibull, lognormale, gamma et normale sont discutées en termes de leur flexibilité dans le plan des coefficients (β_1, β_2) mesurant respectivement le degré de symétrie et le degré d'aplatissement. Les fonctions S_B et bêta sont les distributions les plus flexibles parce qu'elles représentent des surfaces dans le plan alors que les fonctions Weibull, lognormale et gamma sont représentées par des lignes et la normale par un point.

Les six distributions ont été utilisées à partir d'un ensemble de 21 observations comprenant des mesures pour le diamètre et la hauteur. Le critère de vraisemblance maximum a été utilisé pour ordonner les six distributions en fonction de leur ajustement aux observations. La distribution Johnson S_B s'est avérée la meilleure en terme d'ajustement à la distribution échantillonnale.

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Introduction

For many years there has been activity and interest in describing the frequency distribution of diameter measurements in forest stands using probability density functions. In 1898 DeLiocourt applied the exponential distribution to frequency data from all-aged forests (Meyer and Stevenson 1943). Since then, researchers have used various distributions for both even-aged and mixed-aged stands with varying degrees of success (Bailey and Dell 1973 and references therein; Clutter and Allison 1974; Zöhrer 1972). Very little work has been done on height distributions although Assmann (1961) indicates that heights usually have negatively skewed distributions, whereas diameters usually have positively skewed distributions.

The main problem in fitting distributions has been the choice of statistical distribution function for describing the probabilities of interest. The criteria for choosing a distribution appear to be that the distribution be relatively simple to fit in terms of parameter estimation, be sufficiently flexible to fit a relatively broad spectrum of shapes, lend itself easily to simple integration techniques for estimating proportions in various size classes, and, finally, fit any given set of observations well.

In this paper we offer what we believe to be a new viewpoint, at least for foresters, of this

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TABLE 1. Summary of skewness and kurtosis for diameter, height, and number of trees per acre, species composition, origin, and age for each stand[a]

Stand	Species[b]	Age	No. trees per acre	Origin[c]	Diameter $\sqrt{b_1}$	Diameter b_2	Height $\sqrt{b_1}$	Height b_2
1	SL	20	105	N	0.08	1.95	−0.31	1.81
2	SL	25	230	P	0.68	3.87	−0.85	3.19
3	SL	30	200	N	−0.60	2.39	−1.13	3.10
4	SL	25	223	N	0.69	2.85	−0.18	1.63
5	SL	35	220	N	−0.07	2.75	−1.77	7.64
6	SL	30	169	N	−0.48	2.72	−1.60	6.07
7	LB	30	289	N	1.21	4.20	0.26	2.58
8	LL	35	229	N	−0.29	1.74	−0.71	2.06
9	LB	20	738	N	0.76	3.33	−0.27	2.74
10	LB	20	565	P	−0.12	3.25	−0.68	3.50
11	LB	30	113	N	−0.62	3.05	−1.69	5.24
12	SL	18	162	P	−0.65	3.40	−1.58	5.87
13	SL	18	566	N	0.47	2.58	−0.06	2.10
14	SL	30	286	N	1.06	4.29	−0.31	2.26
15	SL	30	161	N	0.23	2.11	−0.64	2.32
16	SL	18	183	P	−0.77	3.54	−1.86	6.33
17	LL	32	176	N	0.05	1.73	−0.66	2.43
18	SL	12	507	P	0.17	2.81	−0.84	3.51
19	SL	30	213	N	0.50	2.30	−0.37	2.04
20	LL	18	231	N	0.60	2.62	−0.16	2.38
21	LL	34	198	N	−0.11	2.50	−1.28	4.54

[a]Each stand is 1 acre in size (1 acre = 0.405 ha).
[b]LL = longleaf pine, SL = shortleaf pine, and LB = loblolly pine.
[c]N = natural stand and P = planted stand.

fitting activity and discuss some of the strengths and weaknesses of distributions that have been used for describing diameter distributions in even-aged stands. In particular we resort to a use of the skewness coefficient, $\sqrt{\beta_1}$, and kurtosis coefficient, β_2, of various statistical distributions as a measure of the flexibility of the distributions in regard to their changes in shape. Here $\sqrt{\beta_1} = \mu_3/\mu_2^{3/2}$ and $\beta_2 = \mu_4/\mu_2^2$, where $\mu_k = \int_{-\infty}^{\infty}[x - E(x)]^k f(x)dx$, and $f(x)$ is the probability density function of the random variable x. Skewness, or asymmetry, is defined as a departure from symmetry about the mean where negative values indicate a distribution with a long tail to the left and positive values a long tail to the right. Kurtosis is generally considered to be a relative measure of the flatness or peakedness of a distribution, the larger the value of β_2 the more peaked the distribution; however, Darlington (1970) and Hildebrand (1971) show that in several cases β_2 may better be considered a measure of bimodality. It must also be observed that knowing the values of β_1 and β_2 does not in itself uniquely define a distribution. It is, however, helpful in identifying distributions that should not be fit. We present a chart of the β_1–β_2 space and discuss the flexibility of various statistical distributions based on this chart. We also compare six distributions in terms of how well they fit diameter and height data obtained from 21 even-aged stands of southern pine species located in the Coastal Plains geographic region of the southeastern United States. The data were gathered from 1-acre plots (1 acre = 0.405 ha) and the stand ages span a range from 14 to 35 years. Five of the stands were plantations and the rest were natural stands. A summary of the stand data is presented in Table 1.

The β_1 and β_2 Space

In the statistical literature a graph of the β_1–β_2 space is commonly used to demonstrate the range of skewness and kurtosis covered by various statistical distributions (see Johnson and Kotz 1970). Such a graph is extremely informative in considering the strengths and weaknesses of the distributions. In Fig. 1, we present such a graph for statistical distributions that have been suggested for describing diameter distributions.

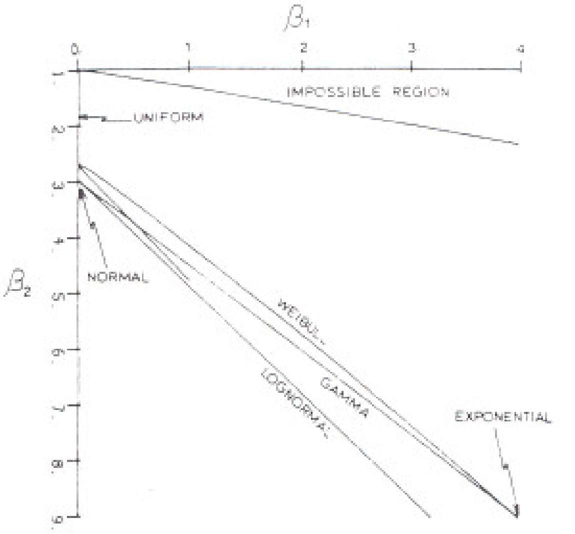

FIG. 1. The β_1–β_2 space showing the plots of possible values for selected distributions.

There is in the graph an 'impossible region' for which combinations of β_1 and β_2 are mathematically impossible. Further, the ordinate scale is presented upside down. This is by tradition, and while we can offer no plausible explanation for why it should be so, we have chosen to conform to that tradition. Finally, the β_1–β_2 space presented simply spans the segment of the space appropriate to our discussion. One use of such a graph is to suggest distributions which might fit a set of data based on sample estimates of β_1 and β_2.

The normal, exponential, and uniform are all represented by points in the space, a verification of the fact that they all have but one shape. Very limited regions of the space can be approximated by distributions represented by points.

The other three distributions shown in the figure are more flexible in terms of their ability to approximate a broader segment of the space. The three distributions gamma, lognormal, and Weibull are represented by lines in the β_1–β_2 space, demonstrating their capability to assume a variety of shapes. The fact that these lines fall rather close to each other helps to explain why sets of data can often be fitted equally well (or equally poorly) by either of these distributions. Their respective locations on the graph also offer an explanation as to why the Weibull distribution has often been found to give a 'better fit' to diameter data than either the gamma or lognormal. Our experience has shown that the skewness and kurtosis estimates from forest mensurational data generally fall in the region above the upper Weibull line of Fig. 1.

A further distinction between these three distributions is their ability to represent different types of skewness. Both the lognormal and gamma distributions are limited to shapes that have positive skewness, while the Weibull distribution has the ability to describe both positive and negative skewness. Since the graph of Fig. 1 presents β_1, the square of the skewness coefficient, the positive and negative skewness aspect of a set of data is not readily obvious from the graph. One must consider the sign of $\sqrt{\beta_1}$. For instance, the lower line of the Weibull plot in Fig. 1 is generated by negatively skewed shapes.

Not identified in Fig. 1 is the beta distribution. The beta distribution covers the entire region between the gamma distribution line, the impossible region, and the β_2 axis. Hence, the beta distribution covers a broad spectrum of shapes and is quite flexible, fitting both positively and negatively skewed data.

Another distribution not identified in Fig. 1, one that has not to our knowledge been suggested for use in forestry, is Johnson's S_B distribution. N.L. Johnson (1949*a*) proposed a system of distributions which span the β_1–β_2 space that are based on transformations of a standard normal variate. His system consists basically of three distributions identified as S_B, S_L, and S_U. (Sometimes the normal distribution, which is a special case of the three, is included and denoted by S_N.) The S_L distribution is a three-parameter lognormal distribution with one parameter being the lower limit, the S_B distribution covers the region above the lognormal line in Fig. 1, and the S_U distribution covers the region below the lognormal line. Hence, Johnson's S_B distribution provides somewhat more flexibility in skewness and kurtosis than the beta distribution.

One feature of both the beta and S_B distributions is the range of positive density from 0 to 1, thus it is necessary to identify upper and lower limits of any data set to which the distributions are to be fit and make the appropriate transformation of scale. In many instances this identification requires sample estimation of the limit parameters.

We believe that the S_B distribution has some important advantages over the beta distribution from the viewpoint of practical application. First, it spans a slightly broader range of the β_1–β_2 space than the beta distribution (i.e., it covers, in addition, the region between the lognormal and the gamma). Second, it is possible to obtain maximum likelihood estimators that have closed-form solutions once we identify upper and lower limits of the data set, where the beta distribution requires iterative solution for the maximum likelihood estimators. Further, since the distribution is obtained by a transformation on a standard normal variate, integration over specific classes can be accomplished by simple application of the well tabulated standard normal. This transformation on the normal distribution also facilitates the consideration of multivariate problems since the wealth of literature on the multivariate normal distribution becomes appropriate (Johnson 1949*b*). The S_B distribution has four parameters, two of which are the lower limit, ϵ, and range, λ, respectively.

The equation for Johnson's S_B distribution is

$$f(x) = \frac{\delta}{\sqrt{2\pi}} \frac{\lambda}{(x-\epsilon)(\epsilon+\lambda-x)} \times \exp\left\{-\tfrac{1}{2}\left[\gamma + \delta \ln\left(\frac{x-\epsilon}{\epsilon+\lambda-x}\right)\right]^2\right\},$$

$$\epsilon < x < \epsilon+\lambda,\ \delta > 0,\ -\infty < \gamma < \infty,\ \lambda > 0,\ -\infty < \epsilon < \infty = 0 \text{ elsewhere}$$

$$\text{where } \gamma + \delta \ln\left(\frac{x-\epsilon}{\epsilon+\lambda-x}\right) = Z_x \sim N(0,1).$$

Some Empirical Results

Moment estimators of $\sqrt{\beta_1}$ and β_2 are

$$\sqrt{b_1} = \frac{\sum(X_i - \bar{x})^3}{[\sum(X_i - \bar{x})^2]^{3/2}}$$

and

$$b_2 = \frac{(X_i - \bar{x})^4}{[\sum(X_i - \bar{x})^2]^2}.$$

Fisher (1930) gives asymptotic variances of $\sqrt{6/n}$ and $24/n$ for $\sqrt{b_1}$ and b_2.

Figure 2 presents the location in the β_1–β_2 space of the moment estimators, b_1 and b_2, from 21 data sets of diameter and height measurements obtained from even-aged stands located in the Coastal Plain of North and South Carolina and Georgia. Descriptive information regarding the data sets and estimates of $\sqrt{\beta_1}$ and β_2 are presented in Table 1. Since the graph presents b_1, negative skewness is not obvious and must be evaluated in conjunction with $\sqrt{b_1}$ shown in Table 1. About 40% of the diameter distributions exhibited negative skewness; while the height distributions, on the other hand, demonstrated negative skewness for all but 1 of the 21 data sets. This is not unexpected for even-aged stands, particularly once stand differentiation has begun. Hence, it should be unreasonable to attempt to fit lognormal or gamma distributions to height data. However, for purposes of demonstration we have fit the beta, S_B, Weibull, gamma, lognormal, and normal distributions to both the diameter and height measurements in our data sets.

In the fitting procedure we set the lower bound for diameter at 0 in. (1 in. = 25.4 mm) and for height at 4.5 ft (1 ft = 0.3048 m) for all distributions. We solved iteratively to locate the upper bounds of the S_B and beta distributions. While the likelihood values could be improved by iteratively solving for the lower bound of all distributions, the conclusions would not be substantially altered. In addition, we feel that the specified lower bounds are realistic values and appropriate to the ultimate use of the fitted distributions.

Tables 2 and 3 present summaries of the quality of fit based on the log of the likelihood for each data set. The likelihood is a measure of the probability of the particular sample arising given that it came from the distribution of interest. The log of the likelihood is used because it is much easier to compute than the likelihood and provides the same result for ranking purposes. The numbers in parentheses beside the log likelihood values are the relative rankings of the distributions for the data sets. No attempt has been made to test for 'goodness-of-fit' or to test differences between distributions. Our goal in ranking the distributions was to attempt to identify one or more distributions which would perform well over a variety of empirical data sets.

At the bottom of Tables 2 and 3 we present the rank sums across the 21 data sets. Clearly, the normal, lognormal, and gamma distributions are inferior to the other three distributions in terms of their general performance over the variety of stands represented. The S_B distribution seems to be the most consistent performer. It was never the poorest distribution to fit (which was also true for the beta distribution) and was at least the second best in all but four instances. The beta

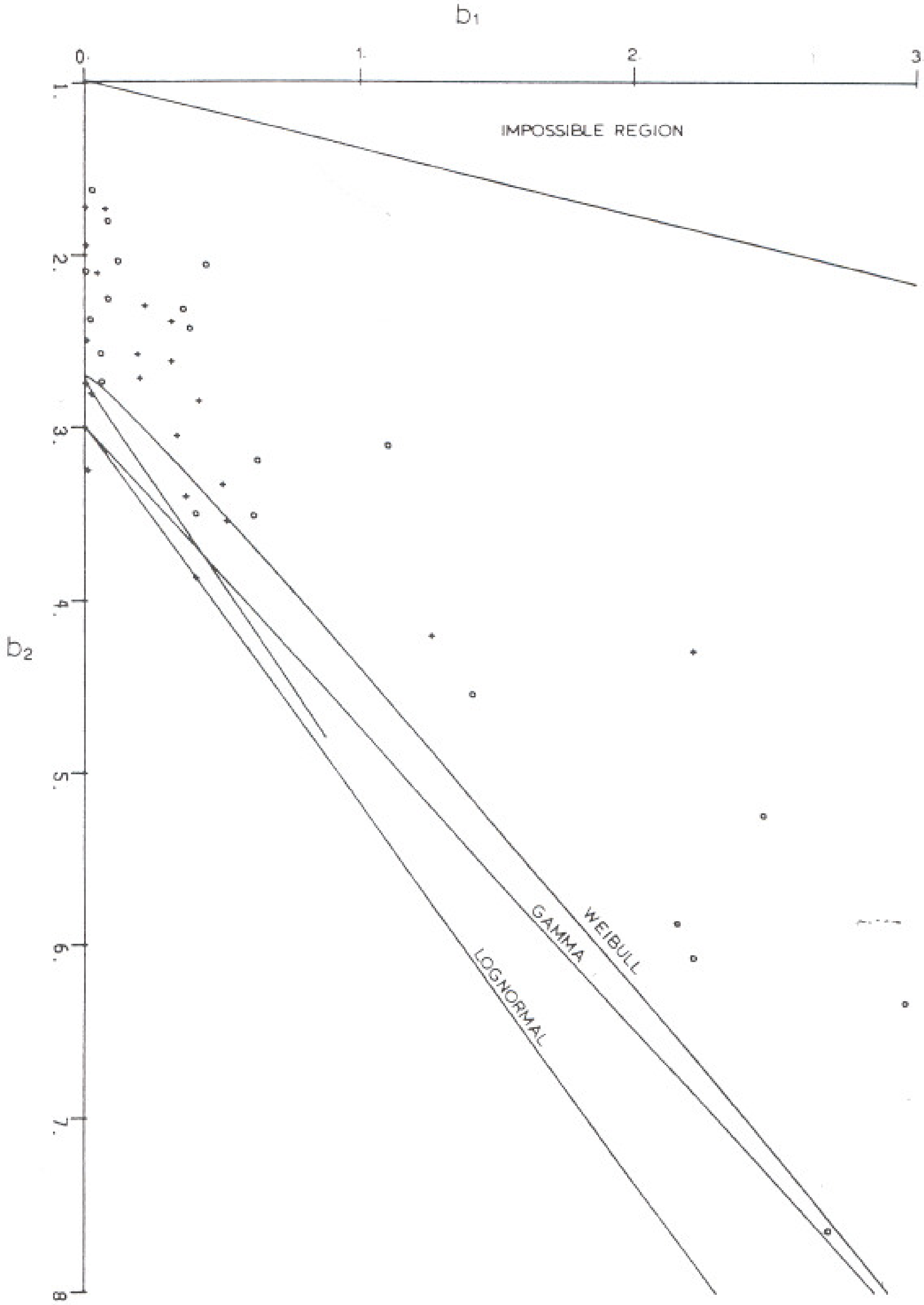

FIG. 2. The β_1–β_2 space showing the location of the estimates b_1 and b_2 for the diameter and height measurements from the 21 data sets of Table 1. The diameter points are indicated by + and the height points by ○.

TABLE 2. Goodness of fit and ranking of the beta (B), S_B, Weibull (W), gamma (G), lognormal (LN), and normal (N) distributions for diameters as measured by log likelihood criterion, with the rank sums for the Kolmogorov–Smirnov (K–S) statistics

Stand	N	LN	G	W	S_B	B
1	−245.87 (5)	−249.43 (6)	−245.55 (4)	−242.74 (3)	−239.11 (1)	−240.24 (2)
2	−477.09 (5)	−468.80 (3)	−468.28 (2)	−479.40 (6)	−468.01 (1)	−468.94 (4)
3	−481.28 (4)	−512.04 (6)	−498.62 (5)	−477.59 (3)	−466.23 (1)	−467.64 (2)
4	−591.46 (6)	−567.38 (2)	−570.39 (3)	−577.13 (5)	−566.66 (1)	−571.37 (4)
5	−447.12 (3)	−456.86 (6)	−451.70 (5)	−447.91 (4)	−446.60 (2)	−446.54 (1)
6	−393.57 (4)	−413.29 (6)	−404.63 (5)	−389.90 (1)	−391.63 (2)	−391.63 (3)
7	−715.34 (6)	−666.38 (1)	−674.48 (3)	−692.26 (5)	−668.60 (2)	−678.56 (4)
8	−589.71 (4)	−611.54 (6)	−599.09 (5)	−585.89 (3)	−564.62 (1)	−567.93 (2)
9	−1404.62 (6)	−1355.69 (2)	−1360.17 (3)	−1398.06 (5)	−1355.31 (1)	−1363.17 (4)
10	−1029.24 (1)	−1063.69 (6)	−1046.34 (5)	−1030.89 (2)	−1037.10 (4)	−1034.03 (3)
11	−244.51 (4)	−263.05 (6)	−255.03 (5)	−242.26 (3)	−242.21 (1)	−242.24 (2)
12	−321.49 (4)	−348.68 (6)	−336.80 (5)	−317.87 (3)	−317.03 (2)	−316.91 (1)
13	−1177.86 (6)	−1159.56 (4)	−1154.57 (3)	−1163.46 (5)	−1149.16 (1)	−1153.69 (2)
14	−723.76 (6)	−689.91 (1)	−693.66 (3)	−708.11 (5)	−690.36 (2)	−696.50 (4)
15	−400.35 (5)	−400.42 (6)	−367.14 (4)	−395.87 (3)	−392.17 (1)	−393.87 (2)
16	−384.56 (3)	−429.87 (6)	−410.06 (5)	−382.87 (1)	−385.68 (4)	−384.50 (2)
17	−415.12 (5)	−417.64 (6)	−414.23 (4)	−411.23 (3)	−403.32 (1)	−404.08 (2)
18	−983.59 (3)	−993.22 (6)	−984.82 (4)	−987.80 (5)	−980.83 (2)	−980.81 (1)
19	−538.81 (6)	−525.54 (2)	−525.86 (3)	−530.43 (5)	−523.07 (1)	−525.95 (4)
20	−526.46 (6)	−511.19 (2)	−512.93 (3)	−522.06 (5)	−510.91 (1)	−513.79 (4)
21	−430.57 (4)	−443.09 (6)	−436.38 (5)	−429.26 (3)	−427.98 (1)	−428.08 (2)
Rank sum	96	95	84	78	33	55
K–S rank sum	94	78	76	83	47	63

TABLE 3. Goodness of fit and ranking (rank in parentheses) of the beta (B), S_B, Weibull (W), gamma (G), lognormal (LN), and normal (N) distributions for heights as measured by likelihood criterion, with the rank sums for the Kolmogorov–Smirnov (K–S) statistics

Stand	N	LN	G	W	S_B	B
1	−421.47 (4)	−432.54 (6)	−426.65 (5)	−419.85 (3)	−410.51 (1)	−411.80 (2)
2	−759.98 (4)	−784.51 (6)	−774.94 (5)	−745.74 (3)	−741.20 (1)	−742.36 (2)
3	−816.90 (4)	−852.65 (6)	−839.03 (5)	−802.12 (3)	−761.23 (1)	−761.55 (2)
4	−933.21 (4)	−945.31 (6)	−938.31 (5)	−928.64 (3)	−910.09 (2)	−908.80 (1)
5	−820.02 (4)	−872.19 (6)	−851.77 (5)	−794.46 (2)	−791.01 (1)	−795.77 (3)
6	−650.24 (4)	−692.17 (6)	−675.26 (5)	−631.55 (3)	−621.83 (1)	−625.30 (2)
7	−1184.94 (5)	−1191.73 (6)	−1183.86 (4)	−1182.22 (3)	−1179.39 (1)	−1180.07 (2)
8	−951.94 (4)	−986.51 (6)	−971.85 (5)	−945.38 (3)	−913.46 (1)	−919.07 (2)
9	−2643.69 (4)	−2709.74 (6)	−2677.60 (5)	−2636.52 (1)	−2638.01 (3)	−2637.31 (2)
10	−1843.18 (4)	−1889.90 (6)	−1870.76 (5)	−1820.49 (1)	−1821.65 (2)	−1823.34 (3)
11	−424.09 (4)	−451.41 (6)	−441.31 (5)	−409.84 (3)	−392.78 (1)	−398.14 (2)
12	−583.73 (4)	−636.45 (6)	−614.95 (5)	−571.58 (3)	−556.49 (1)	−558.47 (2)
13	−2160.78 (4)	−2189.26 (6)	−2171.68 (5)	−2153.33 (3)	−2141.92 (1)	−2143.73 (2)
14	−1172.14 (4)	−1198.89 (6)	−1185.95 (5)	−1167.27 (3)	−1159.82 (1)	−1160.62 (2)
15	−670.86 (4)	−695.26 (6)	−684.65 (5)	−666.69 (3)	−650.22 (1)	−652.41 (2)
16	−666.01 (4)	−736.81 (6)	−708.86 (5)	−650.51 (3)	−626.59 (1)	−633.73 (2)
17	−677.51 (4)	−699.29 (6)	−690.17 (5)	−670.54 (3)	−657.69 (1)	−659.69 (2)
18	−1629.00 (4)	−1699.34 (6)	−1671.11 (5)	−1605.53 (1)	−1626.22 (3)	−1626.17 (2)
19	−845.92 (4)	−866.32 (6)	−856.61 (5)	−841.34 (3)	−827.12 (1)	−828.32 (2)
20	−891.03 (4)	−901.94 (6)	−896.43 (5)	−888.99 (3)	−887.06 (1)	−887.26 (2)
21	−735.61 (4)	−769.06 (6)	−756.28 (5)	−717.23 (3)	−705.26 (1)	−708.06 (2)
Rank sum	85	126	104	56	27	43
K–S rank sum	81	125	106	58	26	45

distribution was generally the second best fitting distribution. The Weibull was generally third best. No distribution seemed to perform any better for either natural stands or plantations.

We performed the same sort of ranking using the Kolmogorov–Smirnov statistic, and while we do not present the complete results for this statistic, the rank sums are given at the bottom of the tables. The same relative comments hold for the K–S statistic as above with the exception that the S_B failed to be at least second best in seven instances.

The observations appear to be consistent with the implications of the plot of the 42 b_1 and b_2 points shown in Fig. 2. With few exceptions these points are not close to the lines associated with the Weibull, gamma, and lognormal distributions. For many of the points which do fall close to these lines, the sign of b_1 is inappropriate. In all cases where $\sqrt{b_1}$ has a negative sign (29 out of 42) the gamma and lognormal are always the two poorest performing distributions. Note, however, that calculation of b_1 and b_2 is insufficient in itself for selecting the best distribution for a given data set. For example, because the region of the β_1–β_2 space spanned by the S_B and the beta distributions overlap, calculated values of b_1 and b_2 will not identify which of these two distributions will give the better fit to a given data set.

Conclusions

Each of the statistical distributions tranditionally considered for fitting forest mensuration data have strengths and weaknesses which can result in extremes of goodness-of-fit from one data set to another. Johnson's S_B distribution, which up to now has not been considered in forestry, demonstrated a relative stability across a variety of data sets.

This distribution has a considerable flexibility in terms of its ability to fit empirical data sets and is relatively simple to apply. When the upper and lower bounds of a data set are known, maximum likelihood estimators of the parameters of the distribution are easily computed. When only one bound, say the lower, is known, the fitting procedure is still not complicated. Since the distribution is a transformation on a normal variate, the table of the standard normal distribution is all that is necessary for calculating probabilities associated with the distribution.

The authors prefer the S_B distribution for another reason. The assumption of a bivariate S_B (S_{BB}) distribution for the distribution of height and diameter yields a regression relationship between height and diameter which appears to be consistent with accepted concepts of the height–diameter relationship and is a rather easy bivariate distribution to work with (Schreuder and Hafley 1977). The S_{BB} distribution is a bivariate distribution for which both marginal distributions are S_B distributions.

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