The Trigonometric Functions

5

APPLICATION

In Comiskey Park, the home of the Chicago White Sox baseball team, the angle of depression from a seat in the last row of the upper deck directly behind the batter to home plate is 29.9°, and the angle of depression to the pitcher's mound is 24.2°. Find (a) the viewing distance to home plate and (b) the viewing distance to the pitcher's mound.

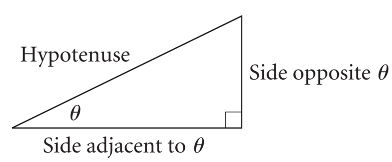
This problem appears as Example 7 in Section 5.2.

5.1 Trigonometric Functions of Acute Angles

- *Determine the six trigonometric ratios for a given acute angle of a right triangle.*
- *Determine the trigonometric function values of 30°, 45°, and 60°.*
- *Using a calculator, find function values for any acute angle, and given a function value of an acute angle, find the angle.*
- *Given the function values of an acute angle, find the function values of its complement.*

The Trigonometric Ratios

We begin our study of trigonometry by considering right triangles and acute angles measured in degrees. An **acute angle** is an angle with measure greater than 0° and less than 90°. Greek letters such as α (alpha), β (beta), γ (gamma), θ (theta), and ϕ (phi) are often used to denote an angle. Consider a right triangle with one of its acute angles labeled θ. The side opposite the right angle is called the **hypotenuse.** The other sides of the triangle are referenced by their position relative to the acute angle θ. One side is opposite θ and one is adjacent to θ.



The *lengths* of the sides of the triangle are used to define the six trigonometric ratios:

sine (sin),	cosecant (csc),
cosine (cos),	secant (sec),
tangent (tan),	cotangent (cot).

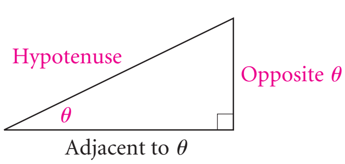


FIGURE 1

The **sine of θ** is the *length* of the side opposite θ divided by the *length* of the hypotenuse (see Fig. 1):

$$\sin\theta = \frac{\text{length of side opposite } \theta}{\text{length of hypotenuse}}.$$

The ratio depends on the measure of angle θ and thus is a function of θ. The notation $\sin\theta$ actually means $\sin(\theta)$, where sin, or sine, is the name of the function.

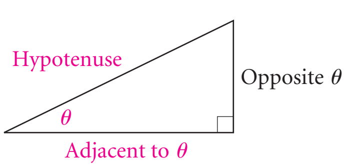
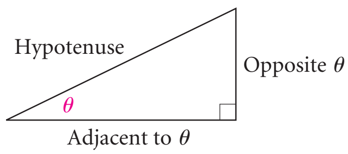


FIGURE 2

The **cosine of θ** is the *length* of the side adjacent to θ divided by the *length* of the hypotenuse (see Fig. 2):

$$\cos\theta = \frac{\text{length of side adjacent to } \theta}{\text{length of hypotenuse}}.$$

The six trigonometric ratios, or trigonometric functions, are defined as follows.



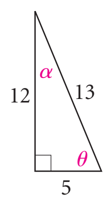
Trigonometric Function Values of an Acute Angle θ

Let θ be an acute angle of a right triangle. Then the six trigonometric functions of θ are as follows:

$$\sin\theta = \frac{\text{side opposite }\theta}{\text{hypotenuse}}, \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{side opposite }\theta},$$

$$\cos\theta = \frac{\text{side adjacent to }\theta}{\text{hypotenuse}}, \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{side adjacent to }\theta},$$

$$\tan\theta = \frac{\text{side opposite }\theta}{\text{side adjacent to }\theta}, \qquad \cot\theta = \frac{\text{side adjacent to }\theta}{\text{side opposite }\theta}.$$



EXAMPLE 1 In the right triangle shown at left, find the six trigonometric function values of (**a**) θ and (**b**) α.

Solution We use the definitions.

a)
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{12}{13}, \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{13}{12},$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{5}{13}, \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{13}{5},$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{12}{5}, \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{5}{12}$$

The references to opposite, adjacent, and hypotenuse are relative to $\boldsymbol{\theta}$.

b)
$$\sin\alpha = \frac{\text{opp}}{\text{hyp}} = \frac{5}{13}, \qquad \csc\alpha = \frac{\text{hyp}}{\text{opp}} = \frac{13}{5},$$

$$\cos\alpha = \frac{\text{adj}}{\text{hyp}} = \frac{12}{13}, \qquad \sec\alpha = \frac{\text{hyp}}{\text{adj}} = \frac{13}{12},$$

$$\tan\alpha = \frac{\text{opp}}{\text{adj}} = \frac{5}{12}, \qquad \cot\alpha = \frac{\text{adj}}{\text{opp}} = \frac{12}{5}$$

The references to opposite, adjacent, and hypotenuse are relative to $\boldsymbol{\alpha}$.

In Example 1(a), we note that the value of $\sin\theta$, $\frac{12}{13}$, is the reciprocal of $\frac{13}{12}$, the value of $\csc\theta$. Likewise, we see the same reciprocal relationship between the values of $\cos\theta$ and $\sec\theta$ and between the values of $\tan\theta$ and $\cot\theta$. For any angle, the cosecant, secant, and cotangent values are the reciprocals of the sine, cosine, and tangent function values, respectively.

Reciprocal Functions

$$\csc\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}$$

If we know the values of the sine, cosine, and tangent functions of an angle, we can use these reciprocal relationships to find the values of the cosecant, secant, and cotangent functions of that angle.

Study Tip

Success on the next exam can be planned. Include study time (even if only 30 minutes a day) in your daily schedule and commit to making this time a *priority*. Choose a time when you are most alert and a setting in which you can concentrate. You will be surprised how much more you can learn and retain if study time is included each day rather than in one long session before the exam.

EXAMPLE 2 Given that $\sin\phi = \frac{4}{5}$, $\cos\phi = \frac{3}{5}$, and $\tan\phi = \frac{4}{3}$, find $\csc\phi$, $\sec\phi$, and $\cot\phi$.

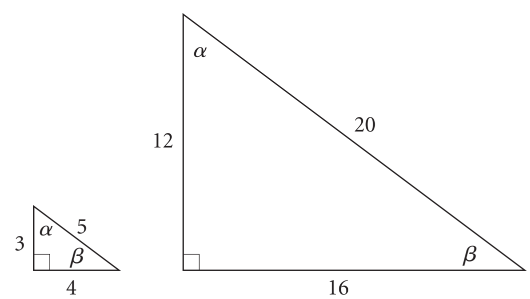
Solution Using the reciprocal relationships, we have

$$\csc\phi = \frac{1}{\sin\phi} = \frac{1}{\frac{4}{5}} = \frac{5}{4}, \qquad \sec\phi = \frac{1}{\cos\phi} = \frac{1}{\frac{3}{5}} = \frac{5}{3},$$

and

$$\cot\phi = \frac{1}{\tan\phi} = \frac{1}{\frac{4}{3}} = \frac{3}{4}.$$

Triangles are said to be **similar** if their corresponding angles have the *same* measure. In similar triangles, the lengths of corresponding sides are in the same ratio. The right triangles shown below are similar. Note that the corresponding angles are equal and the length of each side of the second triangle is four times the length of the corresponding side of the first triangle.



Let's observe the sine, cosine, and tangent values of β in each triangle. Can we expect corresponding function values to be the same?

FIRST TRIANGLE	SECOND TRIANGLE
$\sin\beta = \frac{3}{5}$	$\sin\beta = \frac{12}{20} = \frac{3}{5}$
$\cos\beta = \frac{4}{5}$	$\cos\beta = \frac{16}{20} = \frac{4}{5}$
$\tan\beta = \frac{3}{4}$	$\tan\beta = \frac{12}{16} = \frac{3}{4}$

For the two triangles, the corresponding values of $\sin\beta$, $\cos\beta$, and $\tan\beta$ are the same. The lengths of the sides are proportional—thus the *ratios* are the same. This must be the case because in order for the sine, cosine, and tangent to be functions, there must be only one output (the ratio) for each input (the angle β).

> The trigonometric function values of θ depend only on the measure of the angle, *not* on the size of the triangle.

The Six Functions Related

We can find the other five trigonometric function values of an acute angle when one of the function-value ratios is known.

EXAMPLE 3 If $\sin \beta = \frac{6}{7}$ and β is an acute angle, find the other five trigonometric function values of β.

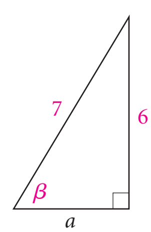
Solution We know from the definition of the sine function that the ratio

$$\frac{6}{7} \quad \text{is} \quad \frac{\text{opp}}{\text{hyp}}.$$

PYTHAGOREAN THEOREM

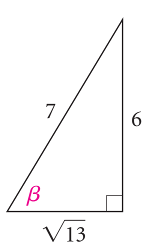
REVIEW SECTION 1.1.

Using this information, let's consider a right triangle in which the hypotenuse has length 7 and the side opposite β has length 6. To find the length of the side adjacent to β, we recall the *Pythagorean theorem*:



$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 6^2 &= 7^2 \\ a^2 + 36 &= 49 \\ a^2 &= 49 - 36 = 13 \\ a &= \sqrt{13}. \end{aligned}$$

We now use the lengths of the three sides to find the other five ratios:



$$\sin \beta = \frac{6}{7}, \qquad \csc \beta = \frac{7}{6},$$

$$\cos \beta = \frac{\sqrt{13}}{7}, \qquad \sec \beta = \frac{7}{\sqrt{13}}, \quad \text{or} \quad \frac{7\sqrt{13}}{13},$$

$$\tan \beta = \frac{6}{\sqrt{13}}, \quad \text{or} \quad \frac{6\sqrt{13}}{13}, \qquad \cot \beta = \frac{\sqrt{13}}{6}.$$

Function Values of 30°, 45°, and 60°

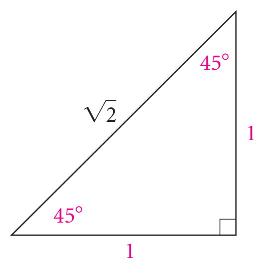
In Examples 1 and 3, we found the trigonometric function values of an acute angle of a right triangle when the lengths of the three sides were known. In most situations, we are asked to find the function values when the measure of the acute angle is given. For certain special angles such as

30°, 45°, and 60°, which are frequently seen in applications, we can use geometry to determine the function values.

A right triangle with a 45° angle actually has two 45° angles. Thus the triangle is *isosceles*, and the legs are the same length. Let's consider such a triangle whose legs have length 1. Then we can find the length of its hypotenuse, c, using the Pythagorean theorem as follows:

$$1^2 + 1^2 = c^2, \quad \text{or} \quad c^2 = 2, \quad \text{or} \quad c = \sqrt{2}.$$

Such a triangle is shown below. From this diagram, we can easily determine the trigonometric function values of 45°.



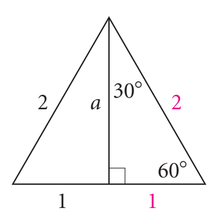
$$\sin 45^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.7071,$$

$$\cos 45^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.7071,$$

$$\tan 45^\circ = \frac{\text{opp}}{\text{adj}} = \frac{1}{1} = 1$$

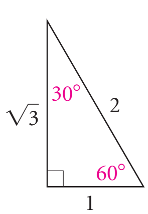
It is sufficient to find only the function values of the sine, cosine, and tangent, since the others are their reciprocals.

It is also possible to determine the function values of 30° and 60°. A right triangle with 30° and 60° acute angles is half of an equilateral triangle, as shown in the following figure. Thus if we choose an equilateral triangle whose sides have length 2 and take half of it, we obtain a right triangle that has a hypotenuse of length 2 and a leg of length 1. The other leg has length a, which can be found as follows:



$$\begin{aligned} a^2 + 1^2 &= 2^2 \\ a^2 + 1 &= 4 \\ a^2 &= 3 \\ a &= \sqrt{3}. \end{aligned}$$

We can now determine the function values of 30° and 60°:



$$\sin 30^\circ = \frac{1}{2} = 0.5, \qquad \sin 60^\circ = \frac{\sqrt{3}}{2} \approx 0.8660,$$

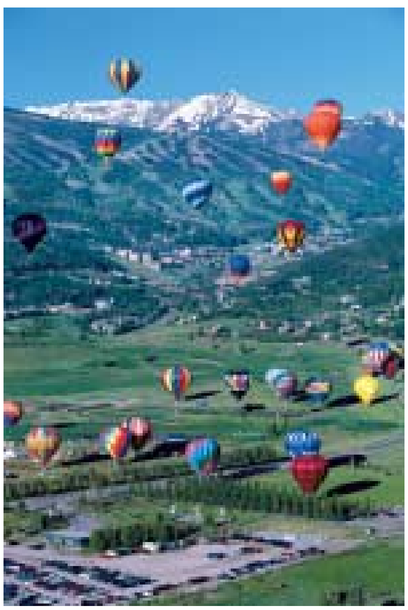
$$\cos 30^\circ = \frac{\sqrt{3}}{2} \approx 0.8660, \qquad \cos 60^\circ = \frac{1}{2} = 0.5,$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \approx 0.5774, \qquad \tan 60^\circ = \frac{\sqrt{3}}{1} = \sqrt{3} \approx 1.7321.$$

Since we will often use the function values of 30°, 45°, and 60°, either the triangles that yield them or the values themselves should be memorized.

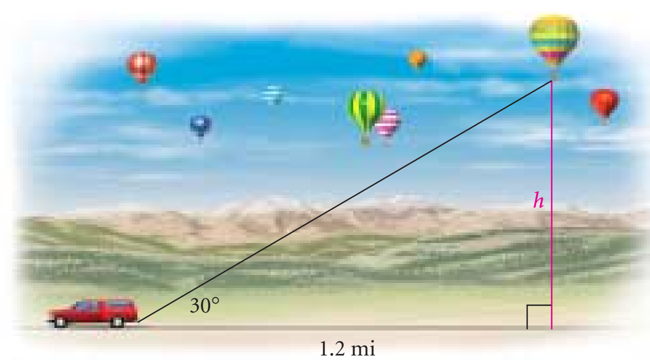
	30°	45°	60°
sin	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$
cos	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$
tan	$\sqrt{3}/3$	1	$\sqrt{3}$

Let's now use what we have learned about trigonometric functions of special angles to solve problems. We will consider such applications in greater detail in Section 5.2.



EXAMPLE 4 *Height of a Hot-air Balloon.* As a hot-air balloon began to rise, the ground crew drove 1.2 mi to an observation station. The initial observation from the station estimated the angle between the ground and the line of sight to the balloon to be 30°. Approximately how high was the balloon at that point? (We are assuming that the wind velocity was low and that the balloon rose vertically for the first few minutes.)

Solution We begin with a drawing of the situation. We know the measure of an acute angle and the length of its adjacent side.



Since we want to determine the length of the opposite side, we can use the tangent ratio, or the cotangent ratio. Here we use the tangent ratio:

$$\tan 30^\circ = \frac{\text{opp}}{\text{adj}} = \frac{h}{1.2}$$

$$1.2 \tan 30^\circ = h$$

$$1.2\left(\frac{\sqrt{3}}{3}\right) = h \qquad \textbf{Substituting; } \tan 30^\circ = \frac{\sqrt{3}}{3}$$

$$0.7 \approx h.$$

The balloon is approximately 0.7 mi, or 3696 ft, high.

Function Values of Any Acute Angle

Historically, the measure of an angle has been expressed in degrees, minutes, and seconds. One minute, denoted $1'$, is such that $60' = 1°$, or $1' = \frac{1}{60} \cdot (1°)$. One second, denoted $1''$, is such that $60'' = 1'$, or $1'' = \frac{1}{60} \cdot (1')$. Then 61 degrees, 27 minutes, 4 seconds could be written as $61°27'4''$. This **D°M′S″ form** was common before the widespread use of scientific calculators. Now the preferred notation is to express fractional parts of degrees in **decimal degree form.** Although the D°M′S″ notation is still widely used in navigation, we will most often use the decimal form in this text.

Most scientific calculators can convert D°M′S″ notation to decimal degree notation and vice versa. Procedures among calculators vary.

EXAMPLE 5 Convert $5°42'30''$ to decimal degree notation.

Solution We enter $5°42'30''$. The calculator gives us

$$5°42'30'' \approx 5.71°,$$

rounded to the nearest hundredth of a degree.

Without a calculator, we can convert as follows:

$$\begin{aligned}
5°42'30'' &= 5° + 42' + 30'' \\
&= 5° + 42' + \frac{30}{60}' && 1'' = \frac{1}{60}';\ 30'' = \frac{30}{60}' \\
&= 5° + 42.5' && \frac{30}{60}' = 0.5' \\
&= 5° + \frac{42.5}{60}° && 1' = \frac{1}{60}°;\ 42.5' = \frac{42.5}{60}° \\
&\approx 5.71°. && \frac{42.5}{60}° \approx 0.71°
\end{aligned}$$

EXAMPLE 6 Convert $72.18°$ to D°M′S″ notation.

Solution On a calculator, we enter 72.18. The result is

$$72.18° = 72°10'48''.$$

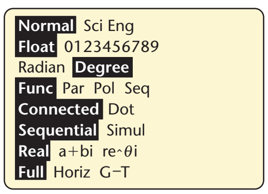
Without a calculator, we can convert as follows:

$$\begin{aligned}
72.18° &= 72° + 0.18 \times 1° \\
&= 72° + 0.18 \times 60' && 1° = 60' \\
&= 72° + 10.8' \\
&= 72° + 10' + 0.8 \times 1' \\
&= 72° + 10' + 0.8 \times 60'' && 1' = 60'' \\
&= 72° + 10' + 48'' \\
&= 72°10'48''.
\end{aligned}$$

So far we have measured angles using degrees. Another useful unit for angle measure is the radian, which we will study in Section 5.4.

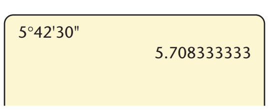
Technology Connection

We can use a graphing calculator set in DEGREE mode to convert between D°M′S″ form and decimal degree form.

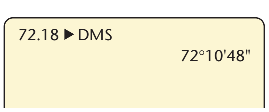


To convert D°M′S″ form to decimal degree form in Example 5, we enter $5°42'30''$ using the ANGLE menu for the degree and minute symbols and ALPHA + for the third symbol. Pressing ENTER gives us

$$5°42'30'' \approx 5.71°.$$



To convert decimal degree form to D°M′S″ form in Example 6, we enter 72.18 and access the ▶DMS feature in the ANGLE menu.



Calculators work with either degrees or radians. Be sure to use whichever mode is appropriate. In this section, we use the degree mode.

Keep in mind the difference between an exact answer and an approximation. For example,

$$\sin 60° = \frac{\sqrt{3}}{2}. \qquad \textbf{This is exact!}$$

But using a calculator, you get an answer like

$$\sin 60° \approx 0.8660254038. \qquad \textbf{This is an approximation!}$$

Calculators generally provide values only of the sine, cosine, and tangent functions. You can find values of the cosecant, secant, and cotangent by taking reciprocals of the sine, cosine, and tangent functions, respectively.

EXAMPLE 7 Find the trigonometric function value, rounded to four decimal places, of each of the following.

a) $\tan 29.7°$ **b)** $\sec 48°$ **c)** $\sin 84°10'39''$

Solution

a) We check to be sure that the calculator is in DEGREE mode. The function value is

$$\tan 29.7° \approx 0.5703899297$$
$$\approx 0.5704. \qquad \textbf{Rounded to four decimal places}$$

b) The secant function value can be found by taking the reciprocal of the cosine function value:

$$\sec 48° = \frac{1}{\cos 48°} \approx 1.49447655 \approx 1.4945.$$

c) We enter $\sin 84°10'39''$. The result is

$$\sin 84°10'39'' \approx 0.9948409474 \approx 0.9948.$$

We can use a calculator to find an angle for which we know a trigonometric function value.

EXAMPLE 8 Find the acute angle, to the nearest tenth of a degree, whose sine value is approximately 0.20113.

Solution The quickest way to find the angle with a calculator is to use an inverse function key. (We first studied inverse functions in Section 4.1 and will consider inverse *trigonometric* functions in Section 6.4.) First check to be sure that your calculator is in DEGREE mode. Usually two keys must be pressed in sequence. For this example, if we press

.20113 [SHIFT] [SIN], or

[2ND] [SIN] .20113 [ENTER],

we find that the acute angle whose sine is 0.20113 is approximately 11.60304613°, or 11.6°.

EXAMPLE 9 *Ladder Safety.* A paint crew has purchased new 30-ft extension ladders. The manufacturer states that the safest placement on a wall is to extend the ladder to 25 ft and to position the base 6.5 ft from the wall. (*Source*: R. D. Werner Co., Inc.) What angle does the ladder make with the ground in this position?

Solution We make a drawing and then use the most convenient trigonometric function. Because we know the length of the side adjacent to θ and the length of the hypotenuse, we choose the cosine function.

From the definition of the cosine function, we have

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{6.5\text{ ft}}{25\text{ ft}} = 0.26.$$

Using a calculator, we find the acute angle whose cosine is 0.26:

$$\theta \approx 74.92993786°.$$

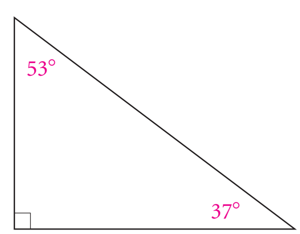
Pressing 0.26 SHIFT COS **or** 2ND COS **0.26** ENTER

Thus when the ladder is in its safest position, it makes an angle of about 75° with the ground.

Cofunctions and Complements

We recall that two angles are **complementary** whenever the sum of their measures is 90°. Each is the complement of the other. In a right triangle, the acute angles are complementary, since the sum of all three angle measures is 180° and the right angle accounts for 90° of this total. Thus if one acute angle of a right triangle is θ, the other is $90° - \theta$.

The six trigonometric function values of each of the acute angles in the triangle below are listed at the right. Note that 53° and 37° are complementary angles since $53° + 37° = 90°$.

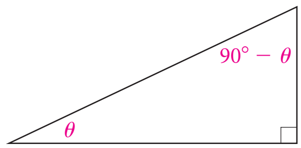


$\sin 37° \approx 0.6018$ $\csc 37° \approx 1.6616$
$\cos 37° \approx 0.7986$ $\sec 37° \approx 1.2521$
$\tan 37° \approx 0.7536$ $\cot 37° \approx 1.3270$

$\sin 53° \approx 0.7986$ $\csc 53° \approx 1.2521$
$\cos 53° \approx 0.6018$ $\sec 53° \approx 1.6616$
$\tan 53° \approx 1.3270$ $\cot 53° \approx 0.7536$

Try this with the acute, complementary angles 20.3° and 69.7° as well. What pattern do you observe? Look for this same pattern in Example 1 earlier in this section.

Note that the sine of an angle is also the cosine of the angle's complement. Similarly, the tangent of an angle is the cotangent of the angle's complement, and the secant of an angle is the cosecant of the angle's complement. These pairs of functions are called **cofunctions.** A list of cofunction identities follows.



Cofunction Identities

$\sin\theta = \cos(90° - \theta)$, $\quad\cos\theta = \sin(90° - \theta)$,
$\tan\theta = \cot(90° - \theta)$, $\quad\cot\theta = \tan(90° - \theta)$,
$\sec\theta = \csc(90° - \theta)$, $\quad\csc\theta = \sec(90° - \theta)$

EXAMPLE 10 Given that $\sin 18° \approx 0.3090$, $\cos 18° \approx 0.9511$, and $\tan 18° \approx 0.3249$, find the six trigonometric function values of 72°.

Solution Using reciprocal relationships, we know that

$$\csc 18° = \frac{1}{\sin 18°} \approx 3.2361,$$

$$\sec 18° = \frac{1}{\cos 18°} \approx 1.0515,$$

and $$\cot 18° = \frac{1}{\tan 18°} \approx 3.0777.$$

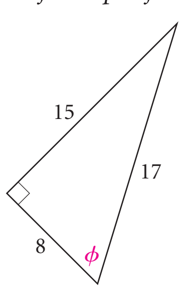
Since 72° and 18° are complementary, we have

$$\sin 72° = \cos 18° \approx 0.9511, \qquad \cos 72° = \sin 18° \approx 0.3090,$$
$$\tan 72° = \cot 18° \approx 3.0777, \qquad \cot 72° = \tan 18° \approx 0.3249,$$
$$\sec 72° = \csc 18° \approx 3.2361, \qquad \csc 72° = \sec 18° \approx 1.0515.$$

5.1 Exercise Set

In Exercises 1–6, find the six trigonometric function values of the specified angle.

1.



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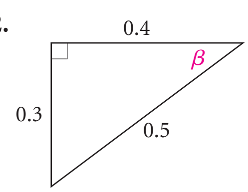

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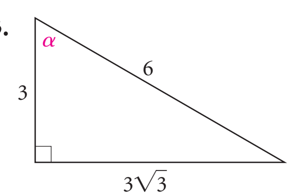

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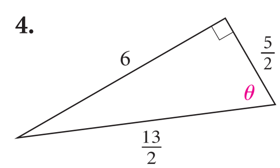

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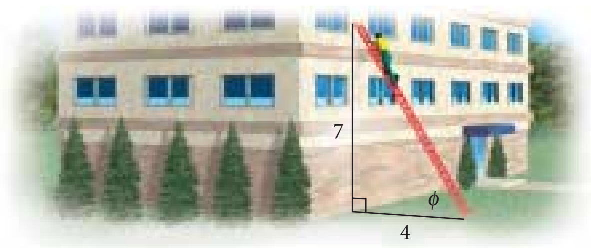

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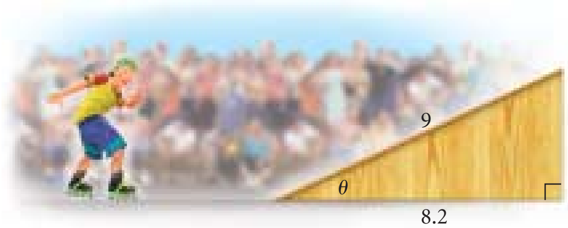


7. Given that $\sin \alpha = \frac{\sqrt{5}}{3}$, $\cos \alpha = \frac{2}{3}$, and $\tan \alpha = \frac{\sqrt{5}}{2}$, find $\csc \alpha$, $\sec \alpha$, and $\cot \alpha$.

8. Given that $\sin \beta = \frac{2\sqrt{2}}{3}$, $\cos \beta = \frac{1}{3}$, and $\tan \beta = 2\sqrt{2}$, find $\csc \beta$, $\sec \beta$, and $\cot \beta$.

Given a function value of an acute angle, find the other five trigonometric function values.

9. $\sin \theta = \frac{24}{25}$

10. $\cos \sigma = 0.7$

11. $\tan \phi = 2$

12. $\cot \theta = \frac{1}{3}$

13. $\csc \theta = 1.5$

14. $\sec \beta = \sqrt{17}$

15. $\cos \beta = \frac{\sqrt{5}}{5}$

16. $\sin \sigma = \frac{10}{11}$

Find the exact function value.

17. $\cos 45°$

18. $\tan 30°$

19. $\sec 60°$

20. $\sin 45°$

21. $\cot 60°$

22. $\csc 45°$

23. $\sin 30°$

24. $\cos 60°$

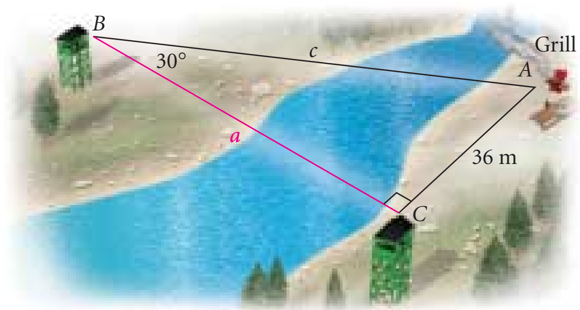
25. $\tan 45°$

26. $\sec 30°$

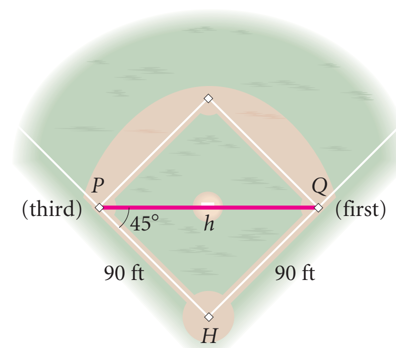
27. $\csc 30°$

28. $\cot 60°$

29. *Distance Across a River.* Find the distance a across the river.



30. *Distance Between Bases.* A baseball diamond is actually a square 90 ft on a side. If a line is drawn from third base to first base, then a right triangle QPH is formed, where $\angle QPH$ is 45°. Using a trigonometric function, find the distance from third base to first base.



Convert to decimal degree notation. Round to two decimal places.

31. $9°43'$

32. $52°15'$

33. $35°50''$

34. $64°53'$

35. $3°2'$

36. $19°47'23''$

37. $49°38'46''$

38. $76°11'34''$

39. $15'5''$

40. $68°2''$

41. $5°53''$

42. $44'10''$

Convert to degrees, minutes, and seconds. Round to the nearest second.

43. $17.6°$

44. $20.14°$

45. $83.025°$

46. $67.84°$

47. $11.75°$

48. $29.8°$

49. $47.8268°$

50. $0.253°$

51. $0.9°$

52. $30.2505°$

53. $39.45°$

54. $2.4°$

Find the function value. Round to four decimal places.

55. $\cos 51°$

56. $\cot 17°$

57. $\tan 4°13'$

58. $\sin 26.1°$

59. $\sec 38.43°$

60. $\cos 74°10'40''$

61. $\cos 40.35°$

62. $\csc 45.2°$

63. $\sin 69°$

64. $\tan 63°48'$

65. $\tan 85.4°$

66. $\cos 4°$

67. $\csc 89.5°$

68. $\sec 35.28°$

69. $\cot 30°25'6''$

70. $\sin 59.2°$

Find the acute angle θ, to the nearest tenth of a degree, for the given function value.

71. $\sin \theta = 0.5125$

72. $\tan \theta = 2.032$

73. $\tan \theta = 0.2226$

74. $\cos \theta = 0.3842$

75. $\sin \theta = 0.9022$

76. $\tan \theta = 3.056$

77. $\cos \theta = 0.6879$

78. $\sin \theta = 0.4005$

79. $\cot \theta = 2.127$
$\left(\textit{Hint}: \tan \theta = \dfrac{1}{\cot \theta}.\right)$

80. $\csc \theta = 1.147$

81. $\sec \theta = 1.279$

82. $\cot \theta = 1.351$

Find the exact acute angle θ for the given function value.

83. $\sin \theta = \dfrac{\sqrt{2}}{2}$

84. $\cot \theta = \dfrac{\sqrt{3}}{3}$

85. $\cos \theta = \dfrac{1}{2}$

86. $\sin \theta = \dfrac{1}{2}$

87. $\tan \theta = 1$

88. $\cos \theta = \dfrac{\sqrt{3}}{2}$

89. $\csc \theta = \dfrac{2\sqrt{3}}{3}$

90. $\tan \theta = \sqrt{3}$

91. $\cot \theta = \sqrt{3}$

92. $\sec \theta = \sqrt{2}$

Use the cofunction and reciprocal identities to complete each of the following.

93. $\cos 20° = \underline{\qquad}\ 70° = \dfrac{1}{\underline{\qquad}\ 20°}$

94. $\sin 64° = \underline{\qquad}\ 26° = \dfrac{1}{\underline{\qquad}\ 64°}$

95. $\tan 52° = \cot \underline{\qquad} = \dfrac{1}{\underline{\qquad}\ 52°}$

96. $\sec 13° = \csc \underline{\qquad} = \dfrac{1}{\underline{\qquad}\ 13°}$

97. Given that

$\sin 65° \approx 0.9063$, $\cos 65° \approx 0.4226$,
$\tan 65° \approx 2.1445$, $\cot 65° \approx 0.4663$,
$\sec 65° \approx 2.3662$, $\csc 65° \approx 1.1034$,

find the six function values of $25°$.

98. Given that

$\sin 8° \approx 0.1392$, $\cos 8° \approx 0.9903$,
$\tan 8° \approx 0.1405$, $\cot 8° \approx 7.1154$,
$\sec 8° \approx 1.0098$, $\csc 8° \approx 7.1853$,

find the six function values of $82°$.

99. Given that $\sin 71°10'5'' \approx 0.9465$, $\cos 71°10'5'' \approx 0.3228$, and $\tan 71°10'5'' \approx 2.9321$, find the six function values of $18°49'55''$.

100. Given that $\sin 38.7° \approx 0.6252$, $\cos 38.7° \approx 0.7804$, and $\tan 38.7° \approx 0.8012$, find the six function values of $51.3°$.

101. Given that $\sin 82° = p$, $\cos 82° = q$, and $\tan 82° = r$, find the six function values of $8°$ in terms of p, q, and r.

Technology Connection

102. Using the TABLE feature, scroll through a table of values to find the acute angle θ in each of Exercises 71–80, to the nearest tenth of a degree, for the given function value.

Collaborative Discussion and Writing

103. Explain why it is not necessary to memorize the function values for both $30°$ and $60°$.

104. Explain the difference between reciprocal functions and cofunctions.

Skill Maintenance

Graph the function.

105. $f(x) = e^{x/2}$

106. $f(x) = 2^{-x}$

107. $h(x) = \ln x$

108. $g(x) = \log_2 x$

Solve.

109. $5^x = 625$

110. $e^t = 10{,}000$

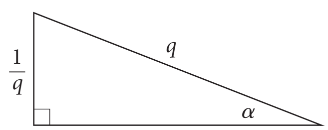
111. $\log_7 x = 3$

112. $\log(3x + 1) - \log(x - 1) = 2$

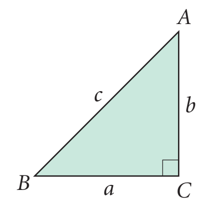
Synthesis

113. Given that $\sec \beta = 1.5304$, find $\sin(90° - \beta)$.

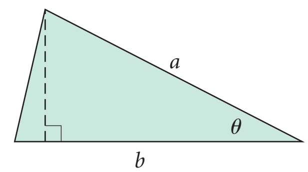
114. Find the six trigonometric function values of α.



115. Show that the area of this right triangle is $\frac{1}{2}bc \sin A$.



116. Show that the area of this triangle is $\frac{1}{2}ab \sin \theta$.

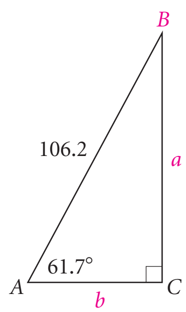


5.2 Applications of Right Triangles

- *Solve right triangles.*
- *Solve applied problems involving right triangles and trigonometric functions.*

Solving Right Triangles

Now that we can find function values for any acute angle, it is possible to *solve* right triangles. To **solve** a triangle means to find the lengths of *all* sides and the measures of *all* angles.



EXAMPLE 1 In $\triangle ABC$ (shown at left), find a, b, and B, where a and b represent lengths of sides and B represents the measure of $\angle B$. Here we use standard lettering for naming the sides and angles of a right triangle: Side a is opposite angle A, side b is opposite angle B, where a and b are the legs, and side c, the hypotenuse, is opposite angle C, the right angle.

Solution In $\triangle ABC$, we know three of the measures:

$$\begin{array}{ll} A = 61.7°, & a = ?, \\ B = ?, & b = ?, \\ C = 90°, & c = 106.2. \end{array}$$

Since the sum of the angle measures of any triangle is 180° and $C = 90°$, the sum of A and B is 90°. Thus,

$$B = 90° - A = 90° - 61.7° = 28.3°.$$

We are given an acute angle and the hypotenuse. This suggests that we can use the sine and cosine ratios to find a and b, respectively:

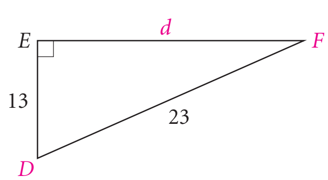
$$\sin 61.7° = \frac{\text{opp}}{\text{hyp}} = \frac{a}{106.2} \quad \text{and} \quad \cos 61.7° = \frac{\text{adj}}{\text{hyp}} = \frac{b}{106.2}.$$

Solving for a and b, we get

$$\begin{array}{lll} a = 106.2 \sin 61.7° & \text{and} & b = 106.2 \cos 61.7° \\ a \approx 93.5 & & b \approx 50.3. \end{array}$$

Thus,

$$\begin{array}{ll} A = 61.7°, & a \approx 93.5, \\ B = 28.3°, & b \approx 50.3, \\ C = 90°, & c = 106.2. \end{array}$$



EXAMPLE 2 In $\triangle DEF$ (shown at left), find D and F. Then find d.

Solution In $\triangle DEF$, we know three of the measures:

$$\begin{array}{ll} D = ?, & d = ?, \\ E = 90°, & e = 23, \\ F = ?, & f = 13. \end{array}$$

We know the side adjacent to D and the hypotenuse. This suggests the use of the cosine ratio:

$$\cos D = \frac{\text{adj}}{\text{hyp}} = \frac{13}{23}.$$

We now find the angle whose cosine is $\frac{13}{23}$. To the nearest hundredth of a degree,

$D \approx 55.58°$. **Pressing (13/23)** SHIFT COS
or 2ND COS **(13/23)** ENTER

Since the sum of D and F is 90°, we can find F by subtracting:

$$F = 90° - D \approx 90° - 55.58° \approx 34.42°.$$

We could use the Pythagorean theorem to find d, but we will use a trigonometric function here. We could use cos F, sin D, or the tangent or cotangent ratios for either D or F. Let's use tan D:

$$\tan D = \frac{\text{opp}}{\text{adj}} = \frac{d}{13}, \quad \text{or} \quad \tan 55.58^\circ \approx \frac{d}{13}.$$

Then

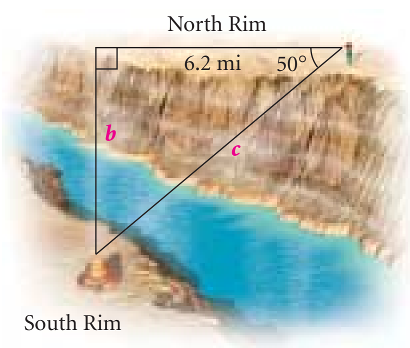
$$d \approx 13 \tan 55.58^\circ \approx 19.$$

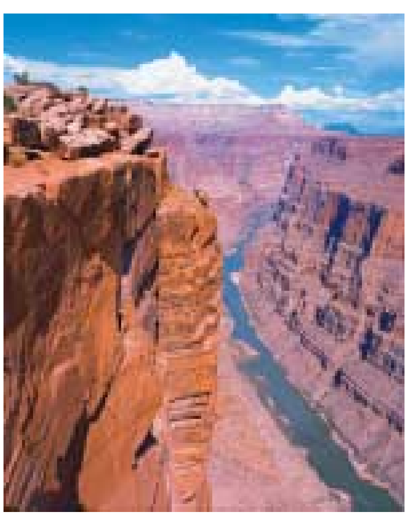
The six measures are

$$\begin{aligned} D &\approx 55.58^\circ, & d &\approx 19, \\ E &= 90^\circ, & e &= 23, \\ F &\approx 34.42^\circ, & f &= 13. \end{aligned}$$

Applications

Right triangles can be used to model and solve many applied problems in the real world.





EXAMPLE 3 *Hiking at the Grand Canyon.* A backpacker hiking east along the North Rim of the Grand Canyon notices an unusual rock formation directly across the canyon. She decides to continue watching the landmark while hiking along the rim. In 2 hr, she has gone 6.2 mi due east and the landmark is still visible but at approximately a 50° angle to the North Rim. (See the figure at left.)

a) How many miles is she from the rock formation?

b) How far is it across the canyon from her starting point?

Solution

a) We know the side adjacent to the 50° angle and want to find the hypotenuse. We can use the cosine function:

$$\cos 50^\circ = \frac{6.2 \text{ mi}}{c}$$

$$c = \frac{6.2 \text{ mi}}{\cos 50^\circ} \approx 9.6 \text{ mi}.$$

After hiking 6.2 mi, she is approximately 9.6 mi from the rock formation.

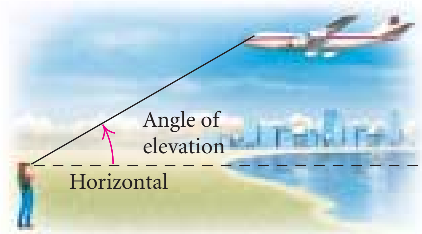
b) We know the side adjacent to the 50° angle and want to find the opposite side. We can use the tangent function:

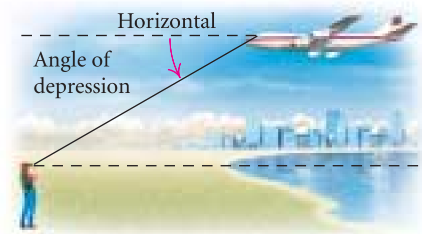
$$\tan 50^\circ = \frac{b}{6.2 \text{ mi}}$$

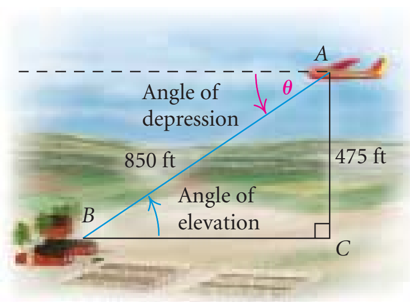
$$b = 6.2 \text{ mi} \cdot \tan 50^\circ \approx 7.4 \text{ mi}.$$

Thus it is approximately 7.4 mi across the canyon from her starting point.

Many applications with right triangles involve an *angle of elevation* or an *angle of depression*. The angle between the horizontal and a line of sight above the horizontal is called an **angle of elevation.** The angle between the horizontal and a line of sight below the horizontal is called an **angle of depression.** For example, suppose that you are looking straight ahead and then you move your eyes up to look at an approaching airplane. The angle that your eyes pass through is an angle of elevation. If the pilot of the plane is looking forward and then looks down, the pilot's eyes pass through an angle of depression.







EXAMPLE 4 *Aerial Photography.* An aerial photographer who photographs farm properties for a real estate company has determined from experience that the best photo is taken at a height of approximately 475 ft and a distance of 850 ft from the farmhouse. What is the angle of depression from the plane to the house?

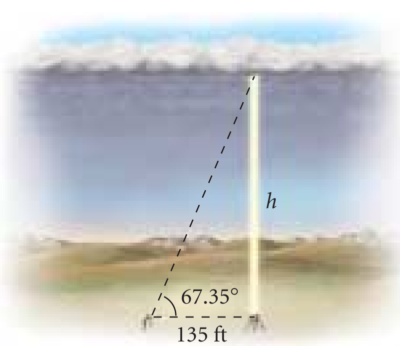
Solution When parallel lines are cut by a transversal, alternate interior angles are equal. Thus the angle of depression from the plane to the house, θ, is equal to the angle of elevation from the house to the plane, so we can use the right triangle shown in the figure. Since we know the side opposite $\angle B$ and the hypotenuse, we can find θ by using the sine function:

$$\sin \theta = \sin B = \frac{475 \text{ ft}}{850 \text{ ft}}.$$

Using a calculator, we find that

$$\theta \approx 34°.$$

Thus the angle of depression is approximately 34°.



EXAMPLE 5 *Cloud Height.* To measure cloud height at night, a vertical beam of light is directed on a spot on the cloud. From a point 135 ft away from the light source, the angle of elevation to the spot is found to be 67.35°. Find the height of the cloud.

Solution From the figure, we have

$$\tan 67.35° = \frac{h}{135 \text{ ft}}$$

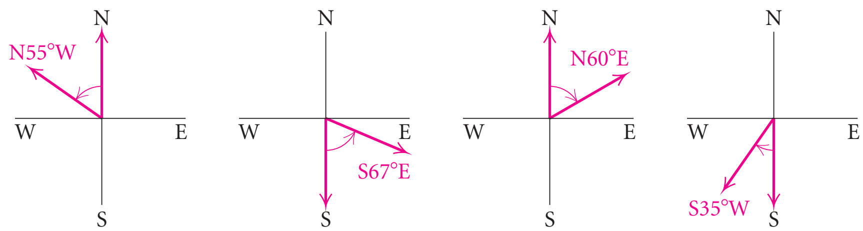
$$h = 135 \text{ ft} \cdot \tan 67.35° \approx 324 \text{ ft}.$$

The height of the cloud is about 324 ft.

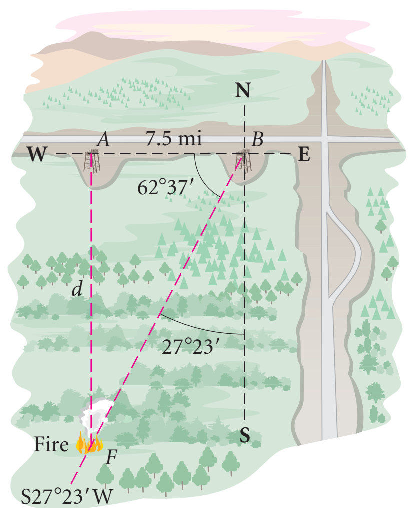
Some applications of trigonometry involve the concept of direction, or bearing. In this text we present two ways of giving direction, the first below and the second in Exercise Set 5.3.

Bearing: First-Type

One method of giving direction, or bearing, involves reference to a north–south line using an acute angle. For example, N55°W means 55° west of north and S67°E means 67° east of south.



EXAMPLE 6 *Distance to a Forest Fire.* A forest ranger at point A sights a fire directly south. A second ranger at point B, 7.5 mi east, sights the same fire at a bearing of S27°23′W. How far from A is the fire?



Solution We first find the complement of 27°23′:

$$B = 90° - 27°23'$$ **Angle B is opposite side d in the right triangle.**

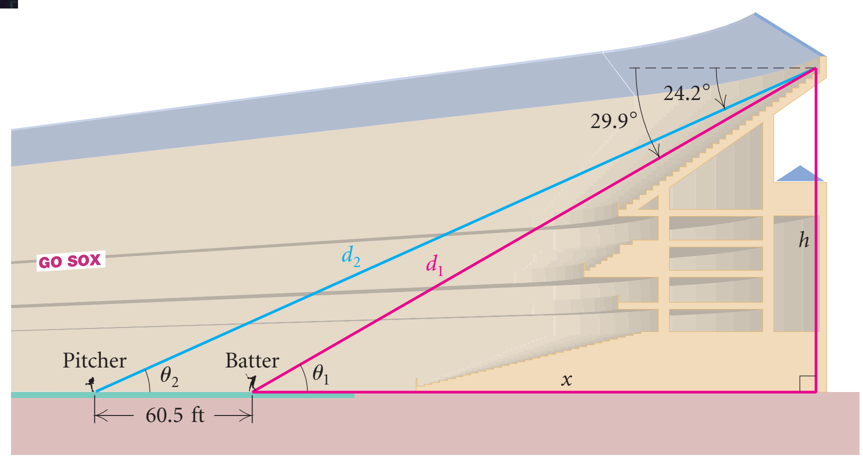
$$= 62°37'$$
$$\approx 62.62°.$$

From the figure shown above, we see that the desired distance d is part of a right triangle. We have

$$\frac{d}{7.5 \text{ mi}} \approx \tan 62.62°$$
$$d \approx 7.5 \text{ mi} \tan 62.62° \approx 14.5 \text{ mi}.$$

The forest ranger at point A is about 14.5 mi from the fire.

EXAMPLE 7 *Comiskey Park.* In the new Comiskey Park, the home of the Chicago White Sox baseball team, the first row of seats in the upper deck is farther away from home plate than the last row of seats in the old Comiskey Park. Although there is no obstructed view in the new park, some of the fans still complain about the present distance from home plate to the upper deck of seats. (*Source*: *Chicago Tribune*, September 19, 1993) From a seat in the last row of the upper deck directly behind the batter, the angle of depression to home plate is 29.9°, and the angle of depression to the pitcher's mound is 24.2°. Find (**a**) the viewing distance to home plate and (**b**) the viewing distance to the pitcher's mound.



Solution From geometry we know that $\theta_1 = 29.9°$ and $\theta_2 = 24.2°$. The standard distance from home plate to the pitcher's mound is 60.5 ft. In the drawing, we let d_1 be the viewing distance to home plate, d_2 the viewing distance to the pitcher's mound, h the elevation of the last row, and x the horizontal distance from the batter to a point directly below the seat in the last row of the upper deck.

We begin by determining the distance x. We use the tangent function with $\theta_1 = 29.9°$ and $\theta_2 = 24.2°$:

$$\tan 29.9° = \frac{h}{x} \qquad \text{and} \qquad \tan 24.2° = \frac{h}{x + 60.5}$$

or

$$h = x \tan 29.9° \qquad \text{and} \qquad h = (x + 60.5) \tan 24.2°.$$

Then substituting $x \tan 29.9°$ for h in the second equation, we obtain

$$x \tan 29.9° = (x + 60.5) \tan 24.2°.$$

Study Tip

Tutoring is available to students using this text. The AW Math Tutor Center, staffed by mathematics instructors, can be reached by telephone, fax, or email. When you are having difficulty with an exercise, this *live* tutoring can be a valuable resource. These instructors have a copy of your text and are familiar with the content objectives in this course.

Solving for x, we get

$$\begin{aligned}
x\tan 29.9° &= x\tan 24.2° + 60.5\tan 24.2° \\
x\tan 29.9° - x\tan 24.2° &= x\tan 24.2° + 60.5\tan 24.2° - x\tan 24.2° \\
x(\tan 29.9° - \tan 24.2°) &= 60.5\tan 24.2° \\
x &= \frac{60.5\tan 24.2°}{\tan 29.9° - \tan 24.2°} \\
x &\approx 216.5.
\end{aligned}$$

We can then find d_1 and d_2 using the cosine function:

$$\cos 29.9° = \frac{216.5}{d_1} \quad \text{and} \quad \cos 24.2° = \frac{216.5 + 60.5}{d_2}$$

or

$$d_1 = \frac{216.5}{\cos 29.9°} \quad \text{and} \quad d_2 = \frac{277}{\cos 24.2°}$$

$$d_1 \approx 249.7 \qquad\qquad d_2 \approx 303.7.$$

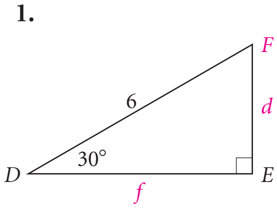
The distance to home plate is about 250 ft,* and the distance to the pitcher's mound is about 304 ft.

*In the old Comiskey Park, the distance to home plate was only 150 ft.

5.2 Exercise Set

In Exercises 1–6, solve the right triangle.

1.



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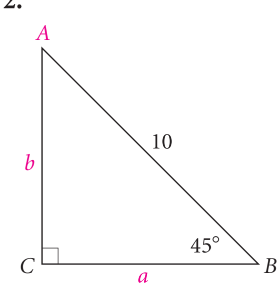

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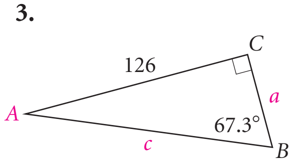

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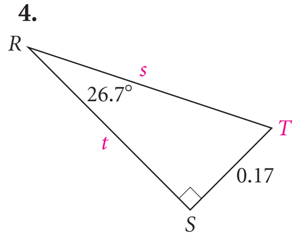

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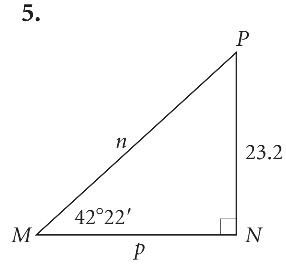

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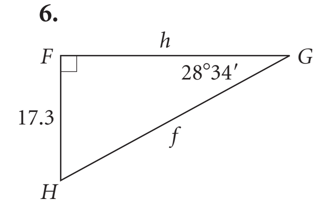
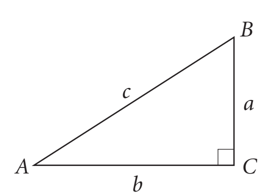


In Exercises 7–16, solve the right triangle. (Standard lettering has been used.)



7. $A = 87°43'$, $a = 9.73$

8. $a = 12.5$, $b = 18.3$

9. $b = 100$, $c = 450$

10. $B = 56.5°$, $c = 0.0447$

11. $A = 47.58°$, $c = 48.3$

12. $B = 20.6°$, $a = 7.5$

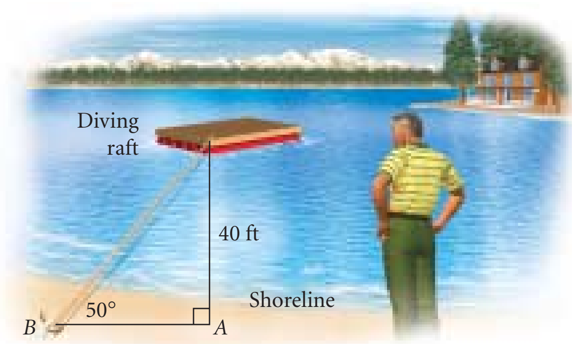
13. $A = 35°$, $b = 40$

14. $B = 69.3°$, $b = 93.4$

15. $b = 1.86$, $c = 4.02$

16. $a = 10.2$, $c = 20.4$

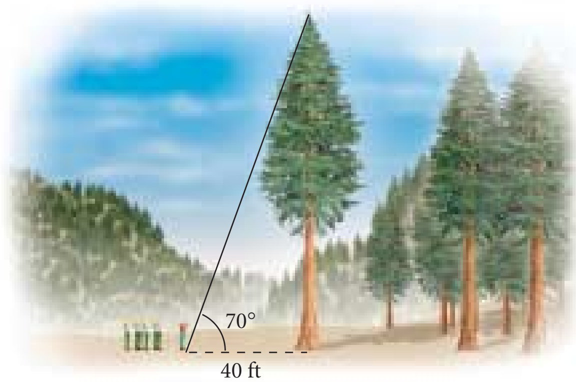
17. *Safety Line to Raft.* Each spring Bryan uses his vacation time to ready his lake property for the summer. He wants to run a new safety line from point B on the shore to the corner of the anchored diving raft. The current safety line, which runs perpendicular to the shore line to point A, is 40 ft long. He estimates the angle from B to the corner of the raft to be 50°. Approximately how much rope does he need for the new safety line if he allows 5 ft of rope at each end to fasten the rope?



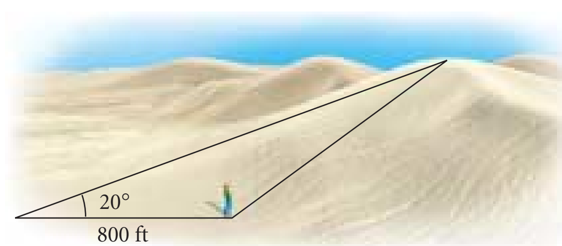
18. *Enclosing an Area.* Alicia is enclosing a triangular area in a corner of her fenced rectangular backyard for her Labrador retriever. In order for a certain tree to be included in this pen, one side needs to be 14.5 ft and make a 53° angle with the new side. How long is the new side?

19. *Easel Display.* A marketing group is designing an easel to display posters advertising their newest products. They want the easel to be 6 ft tall and the back of it to fit flush against a wall. For optimal eye contact, the best angle between the front and back legs of the easel is 23°. How far from the wall should the front legs be placed in order to obtain this angle?

20. *Height of a Tree.* A supervisor must train a new team of loggers to estimate the heights of trees. As an example, she walks off 40 ft from the base of a tree and estimates the angle of elevation to the tree's peak to be 70°. Approximately how tall is the tree?



21. *Sand Dunes National Park.* While visiting the Sand Dunes National Park in Colorado, Cole approximated the angle of elevation to the top of a sand dune to be 20°. After walking 800 ft closer, he guessed that the angle of elevation had increased by 15°. Approximately how tall is the dune he was observing?



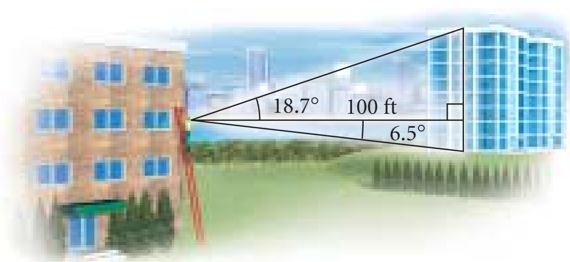
22. *Tee Shirt Design.* A new tee shirt design is to have a regular octagon inscribed in a circle, as shown in the figure. Each side of the octagon is to be 3.5 in. long. Find the radius of the circumscribed circle.

23. *Inscribed Pentagon.* A regular pentagon is inscribed in a circle of radius 15.8 cm. Find the perimeter of the pentagon.

24. *Height of a Weather Balloon.* A weather balloon is directly west of two observing stations that are 10 mi apart. The angles of elevation of the balloon from the two stations are 17.6° and 78.2°. How high is the balloon?

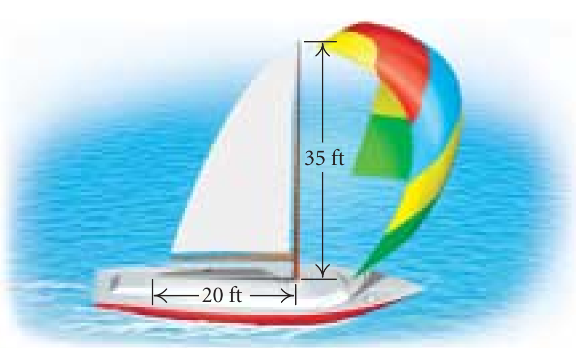
25. *Height of a Kite.* For a science fair project, a group of students tested different materials used to construct kites. Their instructor provided an instrument that accurately measures the angle of elevation. In one of the tests, the angle of elevation was 63.4° with 670 ft of string out. Assuming the string was taut, how high was the kite?

26. *Height of a Building.* A window washer on a ladder looks at a nearby building 100 ft away, noting that the angle of elevation of the top of the building is 18.7° and the angle of depression of the bottom of the building is 6.5°. How tall is the nearby building?



27. *Distance Between Towns.* From a hot-air balloon 2 km high, the angles of depression to two towns in line with the balloon are 81.2° and 13.5°. How far apart are the towns?

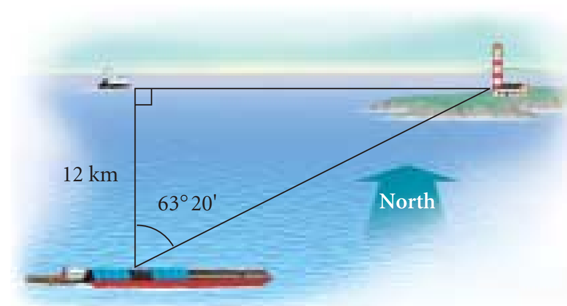
28. *Angle of Elevation.* What is the angle of elevation of the sun when a 35-ft mast casts a 20-ft shadow?



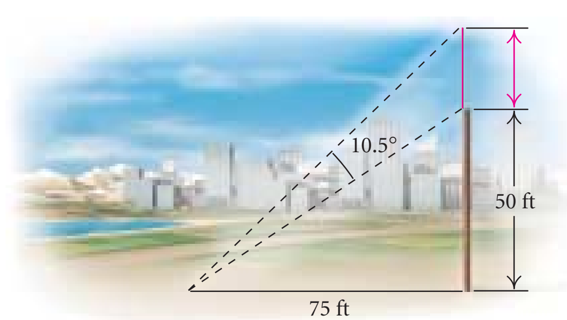
29. *Distance from a Lighthouse.* From the top of a lighthouse 55 ft above sea level, the angle of depression to a small boat is 11.3°. How far from the foot of the lighthouse is the boat?

30. *Lightning Detection.* In extremely large forests, it is not cost-effective to position forest rangers in towers or to use small aircraft to continually watch for fires. Since lightning is a frequent cause of fire, lightning detectors are now commonly used instead. These devices not only give a bearing on the location but also measure the intensity of the lightning. A detector at point Q is situated 15 mi west of a central fire station at point R. The bearing from Q to where lightning hits due south of R is S37.6°E. How far is the hit from point R?

31. *Lobster Boat.* A lobster boat is situated due west of a lighthouse. A barge is 12 km south of the lobster boat. From the barge, the bearing to the lighthouse is N63°20′E. How far is the lobster boat from the lighthouse?



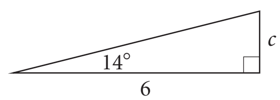
32. *Length of an Antenna.* A vertical antenna is mounted atop a 50-ft pole. From a point on level ground 75 ft from the base of the pole, the antenna subtends an angle of 10.5°. Find the length of the antenna.



Collaborative Discussion and Writing

33. In this section, the trigonometric functions have been defined as functions of acute angles. Thus the set of angles whose measures are greater than 0° and less than 90° is the domain for each function. What appear to be the ranges for the sine, the cosine, and the tangent functions given this domain?

34. Explain in your own words five ways in which length c can be determined in this triangle. Which way seems the most efficient?



Skill Maintenance

Find the distance between the points.

35. $(8, -2)$ and $(-6, -4)$

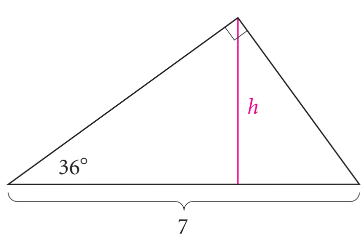
36. $(-9, 3)$ and $(0, 0)$

37. Convert to an exponential equation: $\log 0.001 = -3$.

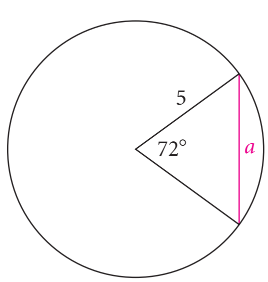
38. Convert to a logarithmic equation: $e^4 = t$.

Synthesis

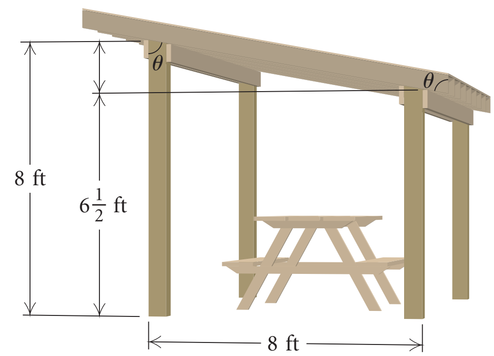
39. Find h, to the nearest tenth.



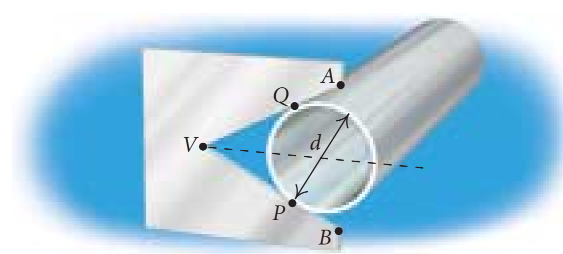
40. Find a, to the nearest tenth.



41. *Construction of Picnic Pavilions.* A construction company is mass-producing picnic pavilions for national parks, as shown in the figure. The rafter ends are to be sawed in such a way that they will be vertical when in place. The front is 8 ft high, the back is $6\frac{1}{2}$ ft high, and the distance between the front and back is 8 ft. At what angle should the rafters be cut?



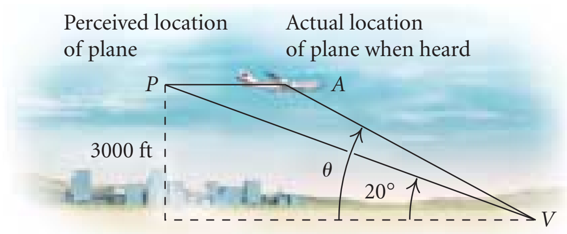
42. *Diameter of a Pipe.* A V-gauge is used to find the diameter of a pipe. The advantage of such a device is that it is rugged, it is accurate, and it has no moving parts to break down. In the figure, the measure of angle AVB is 54°. A pipe is placed in the V-shaped slot and the distance VP is used to estimate the diameter. The line VP is calibrated by listing as its units the corresponding diameters. This, in effect, establishes a function between VP and d.



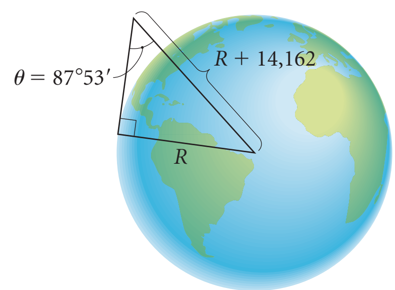
a) Suppose that the diameter of a pipe is 2 cm. What is the distance VP?
b) Suppose that the distance VP is 3.93 cm. What is the diameter of the pipe?
c) Find a formula for d in terms of VP.
d) Find a formula for VP in terms of d.

43. *Sound of an Airplane.* It is common experience to hear the sound of a low-flying airplane and look at the wrong place in the sky to see the plane. Suppose that a plane is traveling directly at you at a speed of

200 mph and an altitude of 3000 ft, and you hear the sound at what seems to be an angle of inclination of 20°. At what angle θ should you actually look in order to see the plane? Consider the speed of sound to be 1100 ft/sec.



44. *Measuring the Radius of the Earth.* One way to measure the radius of the earth is to climb to the top of a mountain whose height above sea level is known and measure the angle between a vertical line to the center of the earth from the top of the mountain and a line drawn from the top of the mountain to the horizon, as shown in the figure. The height of Mt. Shasta in California is 14,162 ft. From the top of Mt. Shasta, one can see the horizon on the Pacific Ocean. The angle formed between a line to the horizon and the vertical is found to be 87°53′. Use this information to estimate the radius of the earth, in miles.

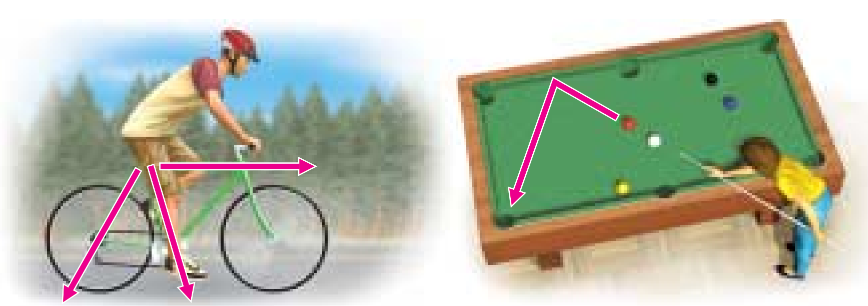


5.3 Trigonometric Functions of Any Angle

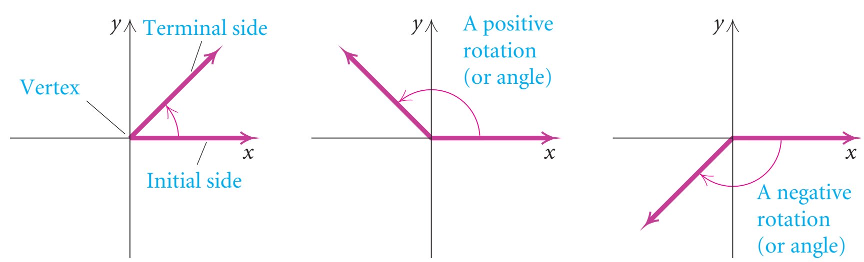
- *Find angles that are coterminal with a given angle and find the complement and the supplement of a given angle.*
- *Determine the six trigonometric function values for any angle in standard position when the coordinates of a point on the terminal side are given.*
- *Find the function values for any angle whose terminal side lies on an axis.*
- *Find the function values for an angle whose terminal side makes an angle of 30°, 45°, or 60° with the x-axis.*
- *Use a calculator to find function values and angles.*

Angles, Rotations, and Degree Measure

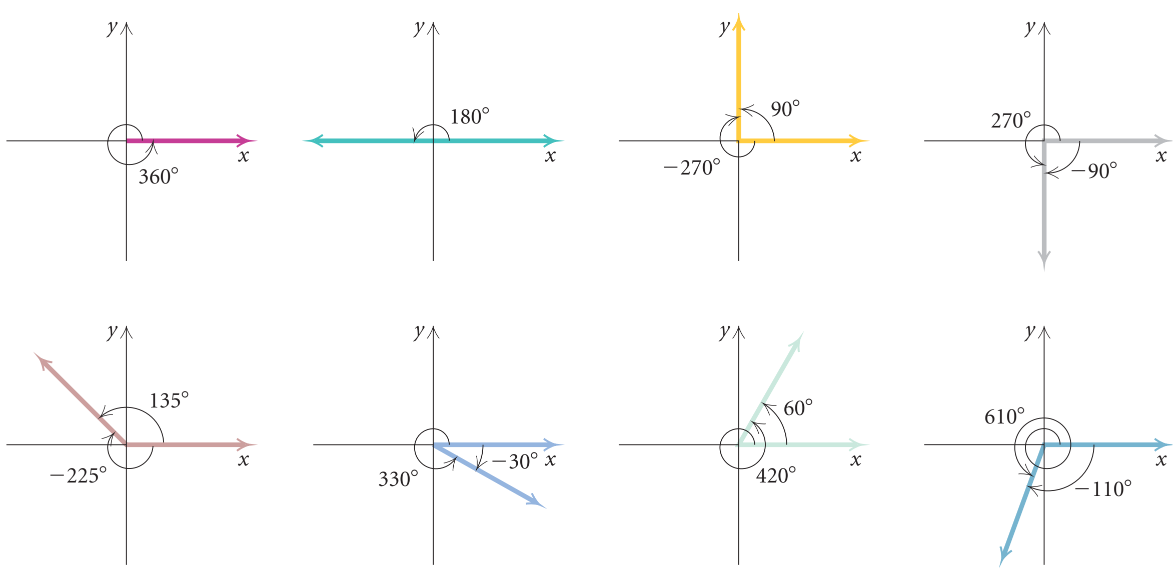
An *angle* is a familiar figure in the world around us.



An **angle** is the union of two rays with a common endpoint called the **vertex.** In trigonometry, we often think of an angle as a **rotation.** To do so, think of locating a ray along the positive x-axis with its endpoint at the origin. This ray is called the **initial side** of the angle. Though we leave that ray fixed, think of making a copy of it and rotating it. A rotation *counterclockwise* is a **positive rotation,** and a rotation *clockwise* is a **negative rotation.** The ray at the end of the rotation is called the **terminal side** of the angle. The angle formed is said to be in **standard position.**



The measure of an angle or rotation may be given in degrees. The Babylonians developed the idea of dividing the circumference of a circle into 360 equal parts, or degrees. If we let the measure of one of these parts be $1°$, then one complete positive revolution or rotation has a measure of $360°$. One half of a revolution has a measure of $180°$, one fourth of a revolution has a measure of $90°$, and so on. We can also speak of an angle of measure $60°$, $135°$, $330°$, or $420°$. The terminal sides of these angles lie in quadrants I, II, IV, and I, respectively. The negative rotations $-30°$, $-110°$, and $-225°$ represent angles with terminal sides in quadrants IV, III, and II, respectively.

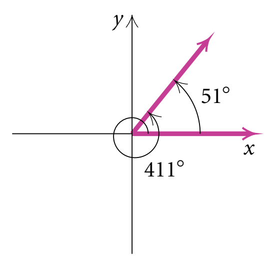


If two or more angles have the same terminal side, the angles are said to be **coterminal.** To find angles coterminal with a given angle, we add or subtract multiples of 360°. For example, 420°, shown above, has the same terminal side as 60°, since $420° = 360° + 60°$. Thus we say that angles of measure 60° and 420° are coterminal. The negative rotation that measures $-300°$ is also coterminal with 60° because $60° - 360° = -300°$. The set of all angles coterminal with 60° can be expressed as $60° + n \cdot 360°$, where n is an integer. Other examples of coterminal angles shown above are 90° and $-270°$, $-90°$ and 270°, 135° and $-225°$, $-30°$ and 330°, and $-110°$ and 610°.

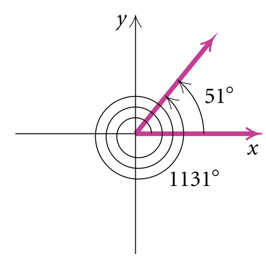
EXAMPLE 1 Find two positive and two negative angles that are coterminal with (**a**) 51° and (**b**) $-7°$.

Solution

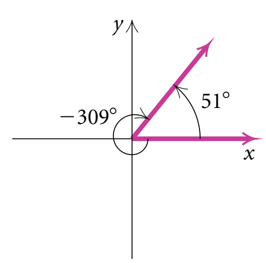
a) We add and subtract multiples of 360°. Many answers are possible.



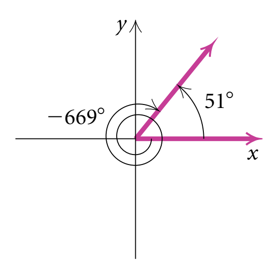
$51° + 360° = 411°$



$51° + 3(360°) = 1131°$



$51° - 360° = -309°$



$51° - 2(360°) = -669°$

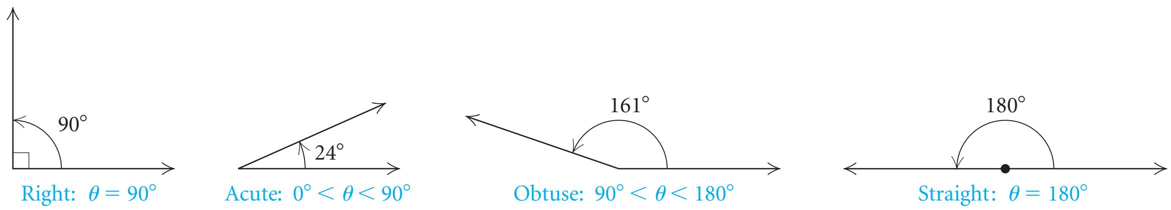
Thus angles of measure 411°, 1131°, $-309°$, and $-669°$ are coterminal with 51°.

b) We have the following:

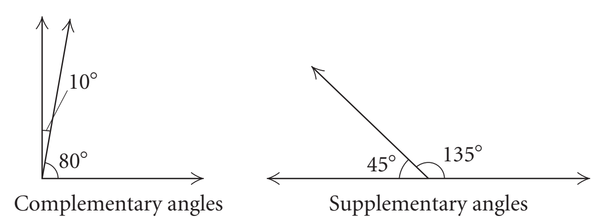
$$-7° + 360° = 353°, \qquad -7° + 2(360°) = 713°,$$
$$-7° - 360° = -367°, \qquad -7° - 10(360°) = -3607°.$$

Thus angles of measure 353°, 713°, $-367°$, and $-3607°$ are coterminal with $-7°$.

Angles can be classified by their measures, as seen in the following figure.



Recall that two acute angles are **complementary** if their sum is 90°. For example, angles that measure 10° and 80° are complementary because $10° + 80° = 90°$. Two positive angles are **supplementary** if their sum is 180°. For example, angles that measure 45° and 135° are supplementary because $45° + 135° = 180°$.



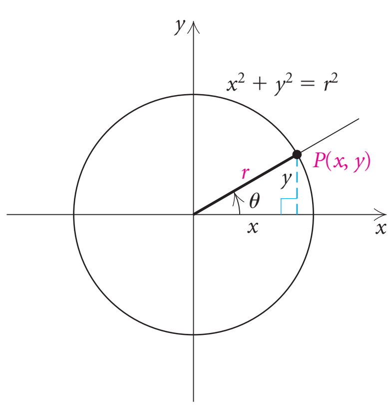
EXAMPLE 2 Find the complement and the supplement of 71.46°.

Solution We have

$$90° - 71.46° = 18.54°,$$
$$180° - 71.46° = 108.54°.$$

Thus the complement of 71.46° is 18.54° and the supplement is 108.54°.

Trigonometric Functions of Angles or Rotations



Many applied problems in trigonometry involve the use of angles that are not acute. Thus we need to extend the domains of the trigonometric functions defined in Section 5.1 to angles, or rotations, of *any* size. To do this, we first consider a right triangle with one vertex at the origin of a coordinate system and one vertex *on the positive x-axis.* (See the figure at left.) The other vertex is at P, a point on the circle whose center is at the origin and whose radius r is the length of the hypotenuse of the triangle. This triangle is a **reference triangle** for angle θ, which is in standard position. Note that y is the length of the side opposite θ and x is the length of the side adjacent to θ.

Recalling the definitions in Section 5.1, we note that three of the trigonometric functions of angle θ are defined as follows:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}, \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}, \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}.$$

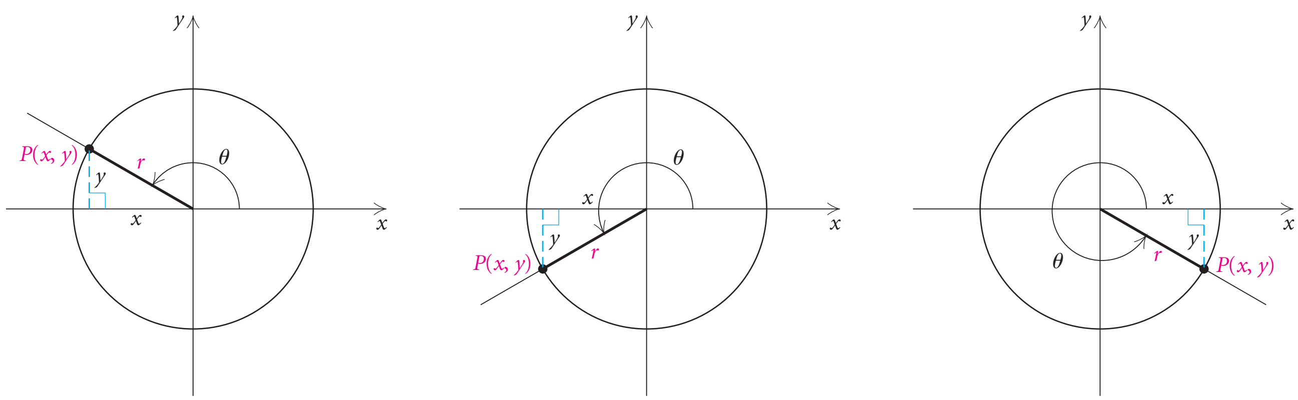
Since x and y are the coordinates of the point P and the length of the radius is the length of the hypotenuse, we can also define these functions as follows:

$$\sin\theta = \frac{y\text{-coordinate}}{\text{radius}},$$

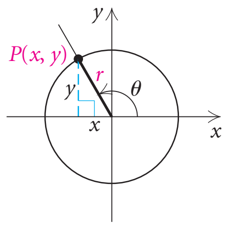
$$\cos\theta = \frac{x\text{-coordinate}}{\text{radius}},$$

$$\tan\theta = \frac{y\text{-coordinate}}{x\text{-coordinate}}.$$

We will use these definitions for functions of angles of any measure. The following figures show angles whose terminal sides lie in quadrants II, III, and IV.



A reference triangle can be drawn for angles in any quadrant, as shown. Note that the angle is in standard position; that is, it is always measured from the positive half of the x-axis. The point $P(x, y)$ is a point, other than the vertex, on the terminal side of the angle. Each of its two coordinates may be positive, negative, or zero, depending on the location of the terminal side. *The length of the radius, which is also the length of the hypotenuse of the reference triangle, is always considered positive.* (Note that $x^2 + y^2 = r^2$, or $r = \sqrt{x^2 + y^2}$.) Regardless of the location of P, we have the following definitions.



Trigonometric Functions of Any Angle θ

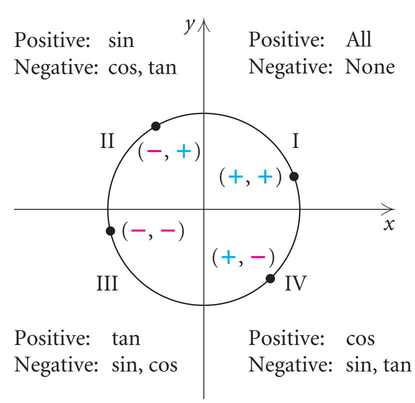
Suppose that $P(x, y)$ is any point other than the vertex on the terminal side of any angle θ in standard position, and r is the radius, or distance from the origin to $P(x, y)$. Then the trigonometric functions are defined as follows:

$$\sin\theta = \frac{y\text{-coordinate}}{\text{radius}} = \frac{y}{r}, \qquad \csc\theta = \frac{\text{radius}}{y\text{-coordinate}} = \frac{r}{y},$$

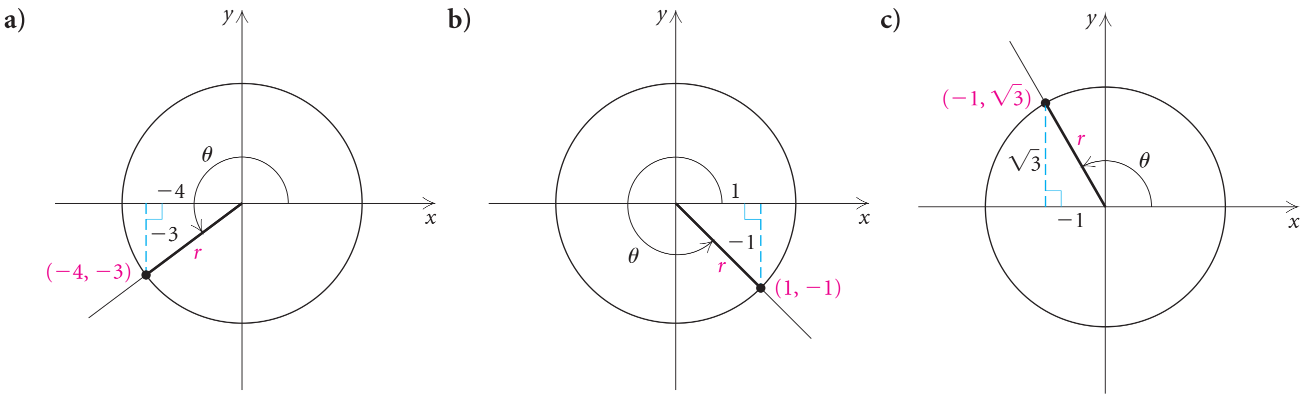
$$\cos\theta = \frac{x\text{-coordinate}}{\text{radius}} = \frac{x}{r}, \qquad \sec\theta = \frac{\text{radius}}{x\text{-coordinate}} = \frac{r}{x},$$

$$\tan\theta = \frac{y\text{-coordinate}}{x\text{-coordinate}} = \frac{y}{x}, \qquad \cot\theta = \frac{x\text{-coordinate}}{y\text{-coordinate}} = \frac{x}{y}.$$

Values of the trigonometric functions can be positive, negative, or zero, depending on where the terminal side of the angle lies. The length of the radius is always positive. Thus the signs of the function values depend only on the coordinates of the point P on the terminal side of the angle. In the first quadrant, all function values are positive because both coordinates are positive. In the second quadrant, first coordinates are negative and second coordinates are positive; thus only the sine and the cosecant values are positive. Similarly, we can determine the signs of the function values in the third and fourth quadrants. *Because of the reciprocal relationships, we need to learn only the signs for the sine, cosine, and tangent functions.*



EXAMPLE 3 Find the six trigonometric function values for each angle shown.



Solution

a) We first determine r, the distance from the origin $(0, 0)$ to the point $(-4, -3)$. The distance between $(0, 0)$ and any point (x, y) on the terminal side of the angle is

$$\begin{aligned} r &= \sqrt{(x-0)^2 + (y-0)^2} \\ &= \sqrt{x^2 + y^2}. \end{aligned}$$

Substituting -4 for x and -3 for y, we find

$$\begin{aligned} r &= \sqrt{(-4)^2 + (-3)^2} \\ &= \sqrt{16 + 9} = \sqrt{25} = 5. \end{aligned}$$

Using the definitions of the trigonometric functions, we can now find the function values of θ. We substitute -4 for x, -3 for y, and 5 for r:

$$\sin\theta = \frac{y}{r} = \frac{-3}{5} = -\frac{3}{5}, \qquad \csc\theta = \frac{r}{y} = \frac{5}{-3} = -\frac{5}{3},$$

$$\cos\theta = \frac{x}{r} = \frac{-4}{5} = -\frac{4}{5}, \qquad \sec\theta = \frac{r}{x} = \frac{5}{-4} = -\frac{5}{4},$$

$$\tan\theta = \frac{y}{x} = \frac{-3}{-4} = \frac{3}{4}, \qquad \cot\theta = \frac{x}{y} = \frac{-4}{-3} = \frac{4}{3}.$$

As expected, the tangent and the cotangent values are positive and the other four are negative. This is true for all angles in quadrant III.

b) We first determine r, the distance from the origin to the point $(1, -1)$:

$$r = \sqrt{1^2 + (-1)^2} = \sqrt{1 + 1} = \sqrt{2}.$$

Substituting 1 for x, -1 for y, and $\sqrt{2}$ for r, we find

$$\sin\theta = \frac{y}{r} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}, \qquad \csc\theta = \frac{r}{y} = \frac{\sqrt{2}}{-1} = -\sqrt{2},$$

$$\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{2}}{1} = \sqrt{2},$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1, \qquad \cot\theta = \frac{x}{y} = \frac{1}{-1} = -1.$$

c) We determine r, the distance from the origin to the point $(-1, \sqrt{3})$:

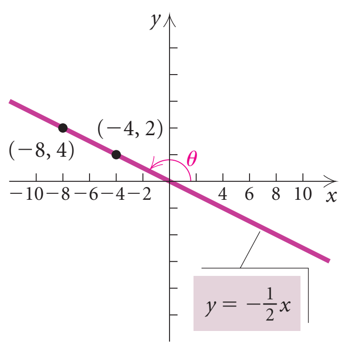
$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = \sqrt{4} = 2.$$

Substituting -1 for x, $\sqrt{3}$ for y, and 2 for r, we find the trigonometric function values of θ are

$$\sin\theta = \frac{\sqrt{3}}{2}, \qquad \csc\theta = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3},$$

$$\cos\theta = \frac{-1}{2} = -\frac{1}{2}, \qquad \sec\theta = \frac{2}{-1} = -2,$$

$$\tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}, \qquad \cot\theta = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$



Any point other than the origin on the terminal side of an angle in standard position can be used to determine the trigonometric function values of that angle. The function values are the same regardless of which point is used. To illustrate this, let's consider an angle θ in standard position whose terminal side lies on the line $y = -\frac{1}{2}x$. We can determine two second-quadrant solutions of the equation, find the length r for each point, and then compare the sine, cosine, and tangent function values using each point.

If $x = -4$, then $y = -\frac{1}{2}(-4) = 2$.
If $x = -8$, then $y = -\frac{1}{2}(-8) = 4$.
For $(-4, 2)$, $r = \sqrt{(-4)^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$.
For $(-8, 4)$, $r = \sqrt{(-8)^2 + 4^2} = \sqrt{80} = 4\sqrt{5}$.

Using $(-4, 2)$ and $r = 2\sqrt{5}$, we find that

$$\sin\theta = \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}, \qquad \cos\theta = \frac{-4}{2\sqrt{5}} = \frac{-2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5},$$

and $$\tan\theta = \frac{2}{-4} = -\frac{1}{2}.$$

Using $(-8, 4)$ and $r = 4\sqrt{5}$, we find that

$$\sin\theta = \frac{4}{4\sqrt{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}, \qquad \cos\theta = \frac{-8}{4\sqrt{5}} = \frac{-2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5},$$

and $$\tan\theta = \frac{4}{-8} = -\frac{1}{2}.$$

We see that the function values are the same using either point. Any point other than the origin on the terminal side of an angle can be used to determine the trigonometric function values.

> The trigonometric function values of θ depend only on the angle, not on the choice of the point on the terminal side that is used to compute them.

The Six Functions Related

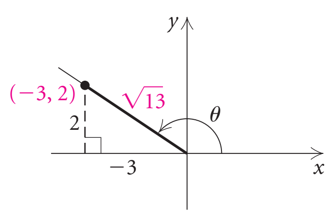
When we know one of the function values of an angle, we can find the other five if we know the quadrant in which the terminal side lies. The procedure is to sketch a reference triangle in the appropriate quadrant, use the Pythagorean theorem as needed to find the lengths of its sides, and then find the ratios of the sides.

EXAMPLE 4 Given that $\tan\theta = -\frac{2}{3}$ and θ is in the second quadrant, find the other function values.

Solution We first sketch a second-quadrant angle. Since

$$\tan\theta = \frac{y}{x} = -\frac{2}{3} = \frac{2}{-3},$$

Expressing $-\frac{2}{3}$ as $\frac{2}{-3}$ since θ is in quadrant II



we make the legs lengths 2 and 3. The hypotenuse must then have length $\sqrt{2^2 + 3^2}$, or $\sqrt{13}$. Now we read off the appropriate ratios:

$$\sin\theta = \frac{2}{\sqrt{13}}, \quad \text{or} \quad \frac{2\sqrt{13}}{13}, \qquad \csc\theta = \frac{\sqrt{13}}{2},$$

$$\cos\theta = -\frac{3}{\sqrt{13}}, \quad \text{or} \quad -\frac{3\sqrt{13}}{13}, \qquad \sec\theta = -\frac{\sqrt{13}}{3},$$

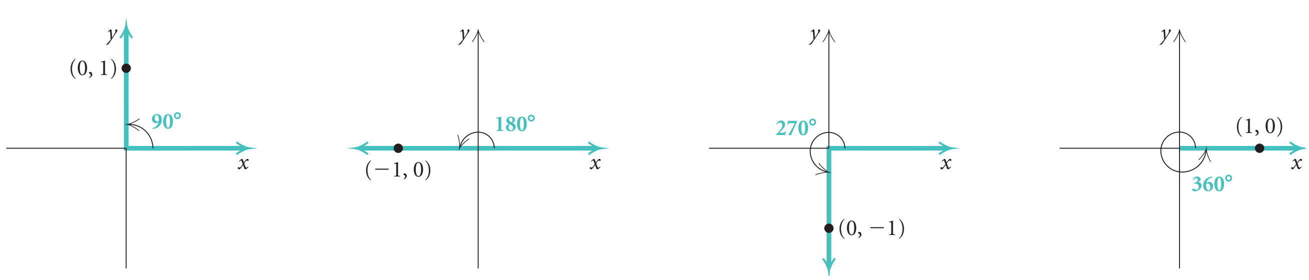
$$\tan\theta = -\frac{2}{3}, \qquad \cot\theta = -\frac{3}{2}.$$

Terminal Side on an Axis

An angle whose terminal side falls on one of the axes is a **quadrantal angle.** One of the coordinates of any point on that side is 0. The definitions of the trigonometric functions still apply, but in some cases, function values will not be defined because a denominator will be 0.

EXAMPLE 5 Find the sine, cosine, and tangent values for 90°, 180°, 270°, and 360°.

Solution We first make a drawing of each angle in standard position and label a point on the terminal side. Since the function values are the same for all points on the terminal side, we choose $(0, 1)$, $(-1, 0)$, $(0, -1)$, and $(1, 0)$ for convenience. Note that $r = 1$ for each choice.



Then by the definitions we get

$\sin 90° = \frac{1}{1} = 1,$ $\quad$ $\sin 180° = \frac{0}{1} = 0,$ $\quad$ $\sin 270° = \frac{-1}{1} = -1,$ $\quad$ $\sin 360° = \frac{0}{1} = 0,$

$\cos 90° = \frac{0}{1} = 0,$ $\quad$ $\cos 180° = \frac{-1}{1} = -1,$ $\quad$ $\cos 270° = \frac{0}{1} = 0,$ $\quad$ $\cos 360° = \frac{1}{1} = 1,$

$\tan 90° = \frac{1}{0},$ **Not defined** $\quad$ $\tan 180° = \frac{0}{-1} = 0,$ $\quad$ $\tan 270° = \frac{-1}{0},$ **Not defined** $\quad$ $\tan 360° = \frac{0}{1} = 0.$

In Example 5, all the values can be found using a calculator, but you will find that it is convenient to be able to compute them mentally. It is also helpful to note that coterminal angles have the same function values. For example, $0°$ and $360°$ are coterminal; thus, $\sin 0° = 0$, $\cos 0° = 1$, and $\tan 0° = 0$.

EXAMPLE 6 Find each of the following.

a) $\sin(-90°)$ **b)** $\csc 540°$

Solution

a) We note that $-90°$ is coterminal with $270°$. Thus,

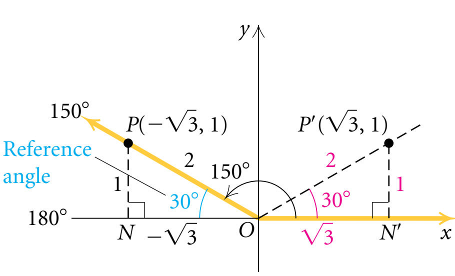
$$\sin(-90°) = \sin 270° = \frac{-1}{1} = -1.$$

b) Since $540° = 180° + 360°$, $540°$ and $180°$ are coterminal. Thus,

$$\csc 540° = \csc 180° = \frac{1}{\sin 180°} = \frac{1}{0}, \quad \text{which is not defined.}$$

Reference Angles: 30°, 45°, and 60°

We can also mentally determine trigonometric function values whenever the terminal side makes a $30°$, $45°$, or $60°$ angle with the x-axis. Consider, for example, an angle of $150°$. The terminal side makes a $30°$ angle with the x-axis, since $180° - 150° = 30°$.



As the figure shows, $\triangle ONP$ is congruent to $\triangle ON'P'$; therefore, the ratios of the sides of the two triangles are the same. Thus the trigonometric function values are the same except perhaps for the sign. We could determine the function values directly from $\triangle ONP$, but this is not necessary. If we remember that in quadrant II, the sine is positive and the cosine and the tangent are negative, we can simply use the function values of 30° that we already know and prefix the appropriate sign. Thus,

$$\sin 150° = \sin 30° = \frac{1}{2},$$

$$\cos 150° = -\cos 30° = -\frac{\sqrt{3}}{2},$$

and $$\tan 150° = -\tan 30° = -\frac{1}{\sqrt{3}}, \quad \text{or} \quad -\frac{\sqrt{3}}{3}.$$

Triangle ONP is the reference triangle and the acute angle $\angle NOP$ is called a *reference angle.*

Reference Angle

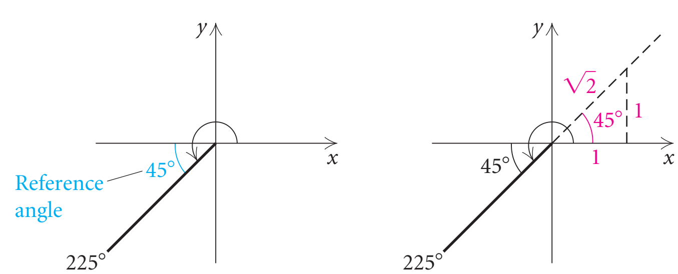
The **reference angle** for an angle is the acute angle formed by the terminal side of the angle and the x-axis.

EXAMPLE 7 Find the sine, cosine, and tangent function values for each of the following.

a) 225° **b)** −780°

Solution

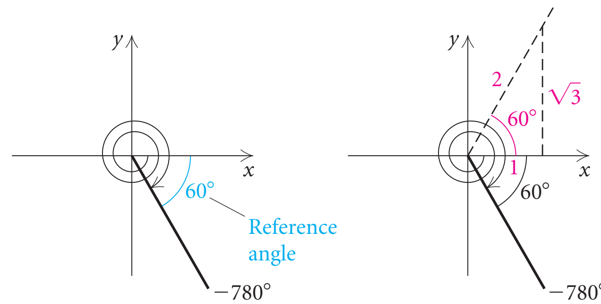
a) We draw a figure showing the terminal side of a 225° angle. The reference angle is 225° − 180°, or 45°.



Recall from Section 5.1 that $\sin 45° = \sqrt{2}/2$, $\cos 45° = \sqrt{2}/2$, and $\tan 45° = 1$. Also note that in the third quadrant, the sine and the cosine are negative and the tangent is positive. Thus we have

$$\sin 225° = -\frac{\sqrt{2}}{2}, \qquad \cos 225° = -\frac{\sqrt{2}}{2}, \quad \text{and} \quad \tan 225° = 1.$$

b) We draw a figure showing the terminal side of a $-780°$ angle. Since $-780° + 2(360°) = -60°$, we know that $-780°$ and $-60°$ are coterminal.



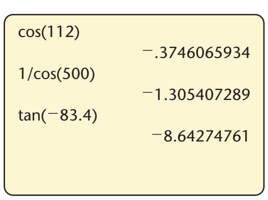
The reference angle for $-60°$ is the acute angle formed by the terminal side of the angle and the x-axis. Thus the reference angle for $-60°$ is $60°$. We know that since $-780°$ is a fourth-quadrant angle, the cosine is positive and the sine and the tangent are negative. Recalling that $\sin 60° = \sqrt{3}/2$, $\cos 60° = 1/2$, and $\tan 60° = \sqrt{3}$, we have

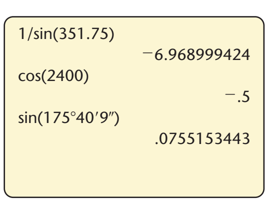
$$\sin(-780°) = -\frac{\sqrt{3}}{2}, \qquad \cos(-780°) = \frac{1}{2},$$

and $$\tan(-780°) = -\sqrt{3}.$$

Technology Connection

To find trigonometric function values of angles measured in degrees, we set the calculator in DEGREE mode. In the windows below, parts (a)–(f) of Example 8 are shown.





Function Values for Any Angle

When the terminal side of an angle falls on one of the axes or makes a $30°$, $45°$, or $60°$ angle with the x-axis, we can find exact function values without the use of a calculator. But this group is only a small subset of *all* angles. Using a calculator, we can approximate the trigonometric function values of *any* angle. In fact, we can approximate or find exact function values of all angles without using a reference angle.

EXAMPLE 8 Find each of the following function values using a calculator and round the answer to four decimal places, where appropriate.

a) $\cos 112°$ **b)** $\sec 500°$

c) $\tan(-83.4°)$ **d)** $\csc 351.75°$

e) $\cos 2400°$ **f)** $\sin 175°40'9''$

g) $\cot(-135°)$

Solution Using a calculator set in DEGREE mode, we find the values.

a) $\cos 112° \approx -0.3746$

b) $\sec 500° = \dfrac{1}{\cos 500°} \approx -1.3054$

c) $\tan(-83.4°) \approx -8.6427$

d) $\csc 351.75° = \dfrac{1}{\sin 351.75°} \approx -6.9690$

e) $\cos 2400° = -0.5$

f) $\sin 175°40'9'' \approx 0.0755$

g) $\cot(-135°) = \dfrac{1}{\tan(-135°)} = 1$

In many applications, we have a trigonometric function value and want to find the measure of a corresponding angle. When only acute angles are considered, there is only one angle for each trigonometric function value. This is not the case when we extend the domain of the trigonometric functions to the set of *all* angles. For a given function value, there is an infinite number of angles that have that function value. There can be two such angles for each value in the range from 0° to 360°. To determine a unique answer in the interval $(0°, 360°)$, the quadrant in which the terminal side lies must be specified.

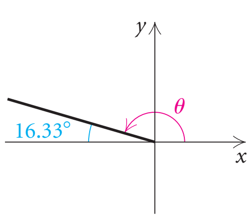
The calculator gives the reference angle as an output for each function value that is entered as an input. Knowing the reference angle and the quadrant in which the terminal side lies, we can find the specified angle.

EXAMPLE 9 Given the function value and the quadrant restriction, find θ.

a) $\sin \theta = 0.2812,\ 90° < \theta < 180°$

b) $\cot \theta = -0.1611,\ 270° < \theta < 360°$

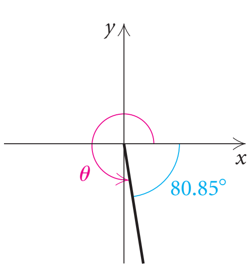
Solution



a) We first sketch the angle in the second quadrant. We use the calculator to find the acute angle (reference angle) whose sine is 0.2812. The reference angle is approximately 16.33°. We find the angle θ by subtracting 16.33° from 180°:

$$180° - 16.33° = 163.67°.$$

Thus, $\theta \approx 163.67°$.



b) We begin by sketching the angle in the fourth quadrant. Because the tangent and cotangent values are reciprocals, we know that

$$\tan \theta \approx \frac{1}{-0.1611} \approx -6.2073.$$

We use the calculator to find the acute angle (reference angle) whose tangent is 6.2073, ignoring the fact that $\tan \theta$ is negative. The reference angle is approximately 80.85°. We find angle θ by subtracting 80.85° from 360°:

$$360° - 80.85° = 279.15°.$$

Thus, $\theta \approx 279.15°$.

5.3 Exercise Set

For angles of the following measures, state in which quadrant the terminal side lies. It helps to sketch the angle in standard position.

1. $187°$ **2.** $-14.3°$

3. $245°15'$ **4.** $-120°$

5. $800°$ **6.** $1075°$

7. $-460.5°$ **8.** $315°$

9. $-912°$ **10.** $13°15'60''$

11. $537°$ **12.** $-345.14°$

Find two positive angles and two negative angles that are coterminal with the given angle. Answers may vary.

13. $74°$ **14.** $-81°$

15. $115.3°$ **16.** $275°10'$

17. $-180°$ **18.** $-310°$

Find the complement and the supplement.

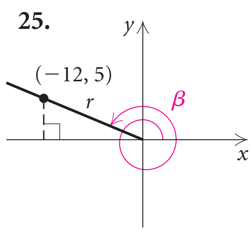
19. $17.11°$ **20.** $47°38'$

21. $12°3'14''$ **22.** $9.038°$

23. $45.2°$ **24.** $67.31°$

Find the six trigonometric function values for the angle shown.

25.



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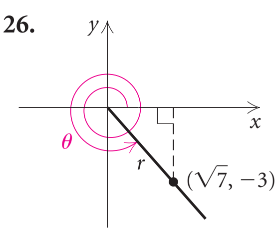

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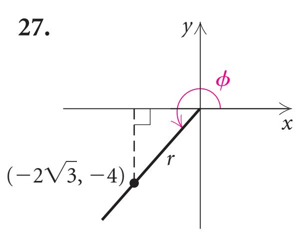
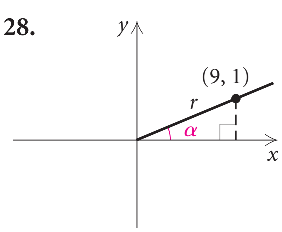


28.



The terminal side of angle θ in standard position lies on the given line in the given quadrant. Find $\sin\theta$, $\cos\theta$, and $\tan\theta$.

29. $2x + 3y = 0$; quadrant IV

30. $4x + y = 0$; quadrant II

31. $5x - 4y = 0$; quadrant I

32. $y = 0.8x$; quadrant III

A function value and a quadrant are given. Find the other five function values. Give exact answers.

33. $\sin\theta = -\frac{1}{3}$, quadrant III

34. $\tan\beta = 5$, quadrant I

35. $\cot\theta = -2$, quadrant IV

36. $\cos\alpha = -\frac{4}{5}$, quadrant II

37. $\cos\phi = \frac{3}{5}$, quadrant IV

38. $\sin\theta = -\frac{5}{13}$, quadrant III

Find the reference angle and the exact function value if it exists.

39. $\cos 150°$ **40.** $\sec(-225°)$

41. $\tan(-135°)$ **42.** $\sin(-45°)$

43. $\sin 7560°$ **44.** $\tan 270°$

45. $\cos 495°$ **46.** $\tan 675°$

47. $\csc(-210°)$ **48.** $\sin 300°$

49. $\cot 570°$ **50.** $\cos(-120°)$

51. $\tan 330°$ **52.** $\cot 855°$

53. $\sec(-90°)$ **54.** $\sin 90°$

55. $\cos(-180°)$ **56.** $\csc 90°$

57. $\tan 240°$ **58.** $\cot(-180°)$

59. sin 495°

60. sin 1050°

61. csc 225°

62. sin (−450°)

63. cos 0°

64. tan 480°

65. cot (−90°)

66. sec 315°

67. cos 90°

68. sin (−135°)

69. cos 270°

70. tan 0°

Find the signs of the six trigonometric function values for the given angles.

71. 319°

72. −57°

73. 194°

74. −620°

75. −215°

76. 290°

77. −272°

78. 91°

Use a calculator in Exercises 79–82, but do not use the trigonometric function keys.

79. Given that

$$\sin 41^\circ = 0.6561,$$
$$\cos 41^\circ = 0.7547,$$
$$\tan 41^\circ = 0.8693,$$

find the trigonometric function values for 319°.

80. Given that

$$\sin 27^\circ = 0.4540,$$
$$\cos 27^\circ = 0.8910,$$
$$\tan 27^\circ = 0.5095,$$

find the trigonometric function values for 333°.

81. Given that

$$\sin 65^\circ = 0.9063,$$
$$\cos 65^\circ = 0.4226,$$
$$\tan 65^\circ = 2.1445,$$

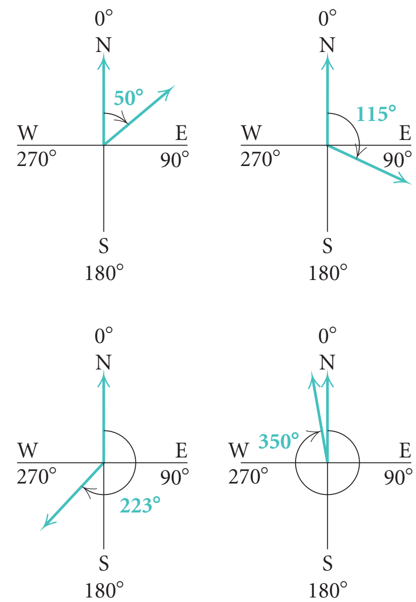
find the trigonometric function values for 115°.

82. Given that

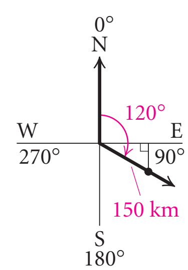
$$\sin 35^\circ = 0.5736,$$
$$\cos 35^\circ = 0.8192,$$
$$\tan 35^\circ = 0.7002,$$

find the trigonometric function values for 215°.

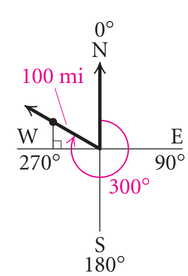
Aerial Navigation. *In aerial navigation, directions are given in degrees clockwise from north. Thus, east is 90°, south is 180°, and west is 270°. Several aerial directions or* ***bearings*** *are given below.*



83. An airplane flies 150 km from an airport in a direction of 120°. How far east of the airport is the plane then? How far south?



84. An airplane leaves an airport and travels for 100 mi in a direction of 300°. How far north of the airport is the plane then? How far west?



85. An airplane travels at 150 km/h for 2 hr in a direction of 138° from Omaha. At the end of this time, how far south of Omaha is the plane?

86. An airplane travels at 120 km/h for 2 hr in a direction of 319° from Chicago. At the end of this time, how far north of Chicago is the plane?

Find the function value. Round to four decimal places.

87. tan 310.8° **88.** cos 205.5°

89. cot 146.15° **90.** sin (−16.4°)

91. sin 118°42′ **92.** cos 273°45′

93. cos (−295.8°) **94.** tan 1086.2°

95. cos 5417° **96.** sec 240°55′

97. csc 520° **98.** sin 3824°

Given the function value and the quadrant restriction, find θ.

Function Value	Interval	θ
99. $\sin\theta = -0.9956$	$(270°, 360°)$	
100. $\tan\theta = 0.2460$	$(180°, 270°)$	
101. $\cos\theta = -0.9388$	$(180°, 270°)$	
102. $\sec\theta = -1.0485$	$(90°, 180°)$	
103. $\tan\theta = -3.0545$	$(270°, 360°)$	
104. $\sin\theta = -0.4313$	$(180°, 270°)$	
105. $\csc\theta = 1.0480$	$(0°, 90°)$	
106. $\cos\theta = -0.0990$	$(90°, 180°)$	

Collaborative Discussion and Writing

107. Why do the function values of θ depend only on the angle and not on the choice of a point on the terminal side?

108. Why is the domain of the tangent function different from the domains of the sine and the cosine functions?

Skill Maintenance

Graph the function. Sketch and label any vertical asymptotes.

109. $f(x) = \dfrac{1}{x^2 - 25}$ **110.** $g(x) = x^3 - 2x + 1$

Determine the domain and the range of the function.

111. $f(x) = \dfrac{x - 4}{x + 2}$

112. $g(x) = \dfrac{x^2 - 9}{2x^2 - 7x - 15}$

Find the zeros of the function.

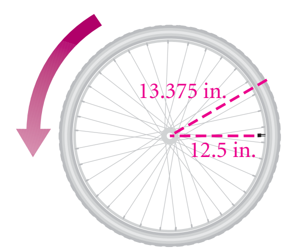
113. $f(x) = 12 - x$ **114.** $g(x) = x^2 - x - 6$

Find the x-intercepts of the graph of the function.

115. $f(x) = 12 - x$ **116.** $g(x) = x^2 - x - 6$

Synthesis

117. *Valve Cap on a Bicycle.* The valve cap on a bicycle wheel is 12.5 in. from the center of the wheel. From the position shown, the wheel starts to roll. After the wheel has turned 390°, how far above the ground is the valve cap? Assume that the outer radius of the tire is 13.375 in.



118. *Seats of a Ferris Wheel.* The seats of a ferris wheel are 35 ft from the center of the wheel. When you board the wheel, you are 5 ft above the ground. After you have rotated through an angle of 765°, how far above the ground are you?

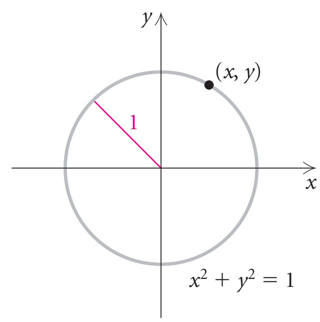
5.4 Radians, Arc Length, and Angular Speed

- *Find points on the unit circle determined by real numbers.*
- *Convert between radian and degree measure; find coterminal, complementary, and supplementary angles.*
- *Find the length of an arc of a circle; find the measure of a central angle of a circle.*
- *Convert between linear speed and angular speed.*

CIRCLES

REVIEW SECTION 1.1.

Another useful unit of angle measure is called a *radian.* To introduce radian measure, we use a circle centered at the origin with a radius of length 1. Such a circle is called a **unit circle.** Its equation is $x^2 + y^2 = 1$.



Distances on the Unit Circle

The circumference of a circle of radius r is $2\pi r$. Thus for the unit circle, where $r = 1$, the circumference is 2π. If a point starts at A and travels around the circle (Fig. 1), it will travel a distance of 2π. If it travels halfway around the circle (Fig. 2), it will travel a distance of $\frac{1}{2} \cdot 2\pi$, or π.

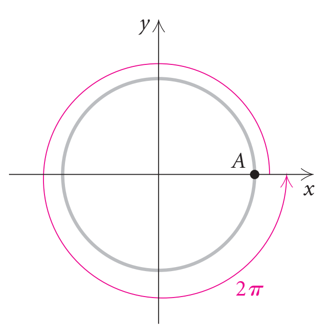


FIGURE 1

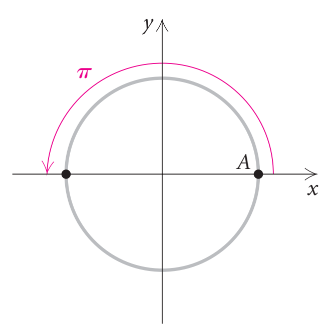


FIGURE 2

If a point C travels $\frac{1}{8}$ of the way around the circle (Fig. 3), it will travel a distance of $\frac{1}{8} \cdot 2\pi$, or $\pi/4$. Note that C is $\frac{1}{4}$ of the way from A to B. If a point D travels $\frac{1}{6}$ of the way around the circle (Fig. 4), it will travel a distance of $\frac{1}{6} \cdot 2\pi$, or $\pi/3$. Note that D is $\frac{1}{3}$ of the way from A to B.

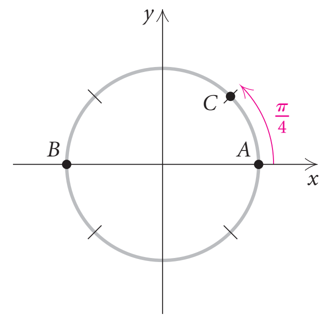


FIGURE 3

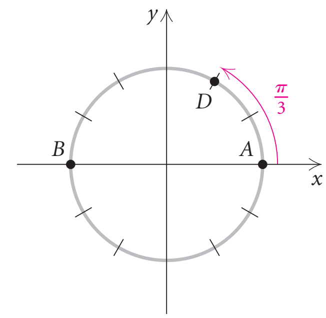


FIGURE 4

EXAMPLE 1 How far will a point travel if it goes **(a)** $\frac{1}{4}$, **(b)** $\frac{1}{12}$, **(c)** $\frac{3}{8}$, and **(d)** $\frac{5}{6}$ of the way around the unit circle?

Solution

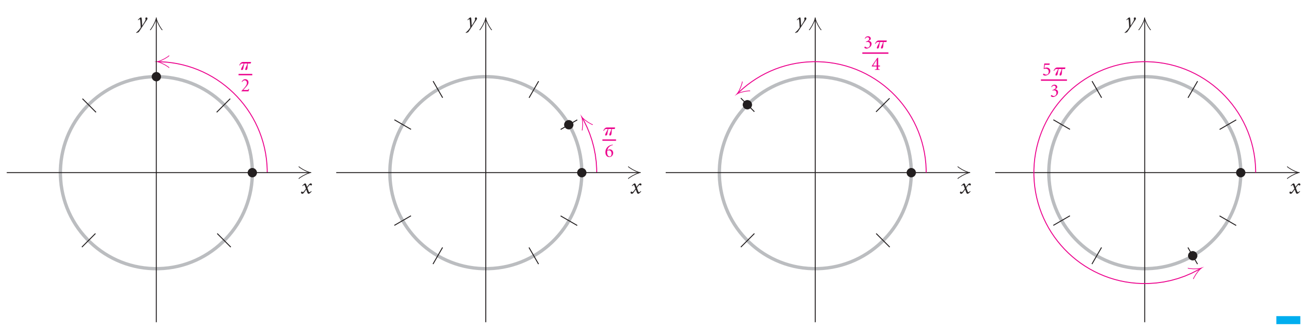
a) $\frac{1}{4}$ of the total distance around the circle is $\frac{1}{4} \cdot 2\pi$, which is $\frac{1}{2} \cdot \pi$, or $\pi/2$.

b) The distance will be $\frac{1}{12} \cdot 2\pi$, which is $\frac{1}{6}\pi$, or $\pi/6$.

c) The distance will be $\frac{3}{8} \cdot 2\pi$, which is $\frac{3}{4}\pi$, or $3\pi/4$.

d) The distance will be $\frac{5}{6} \cdot 2\pi$, which is $\frac{5}{3}\pi$, or $5\pi/3$. Think of $5\pi/3$ as $\pi + \frac{2}{3}\pi$.

These distances are illustrated in the following figures.



A point may travel completely around the circle and then continue. For example, if it goes around once and then continues $\frac{1}{4}$ of the way around, it will have traveled a distance of $2\pi + \frac{1}{4} \cdot 2\pi$, or $5\pi/2$ (Fig. 5). *Every* real number determines a point on the unit circle. For the positive number 10, for example, we start at A and travel counterclockwise a distance of 10. The point at which we stop is the point "determined" by the number 10. Note that $2\pi \approx 6.28$ and that $10 \approx 1.6(2\pi)$. Thus the point for 10 travels around the unit circle about $1\frac{3}{5}$ times.

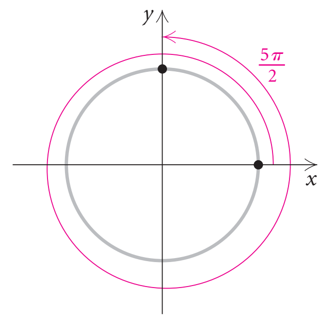


FIGURE 5

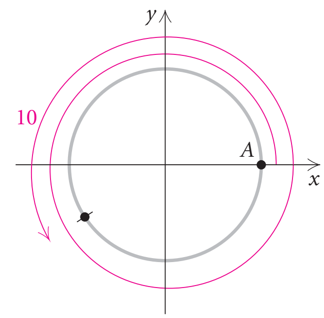
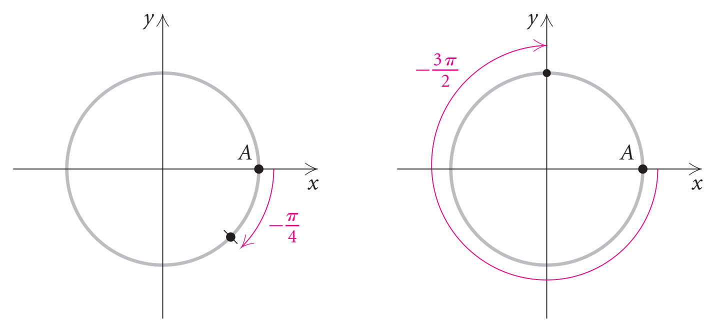


FIGURE 6

For a negative number, we move clockwise around the circle. Points for $-\pi/4$ and $-3\pi/2$ are shown in the figure below. The number 0 determines the point A.

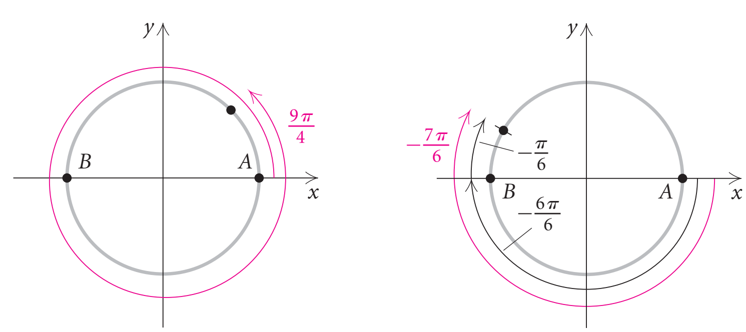


EXAMPLE 2 On the unit circle, mark the point determined by each of the following real numbers.

a) $\dfrac{9\pi}{4}$ **b)** $-\dfrac{7\pi}{6}$

Solution

a) Think of $9\pi/4$ as $2\pi + \frac{1}{4}\pi$. (See the figure on the left below.) Since $9\pi/4 > 0$, the point moves counterclockwise. The point goes completely around once and then continues $\frac{1}{4}$ of the way from A to B.

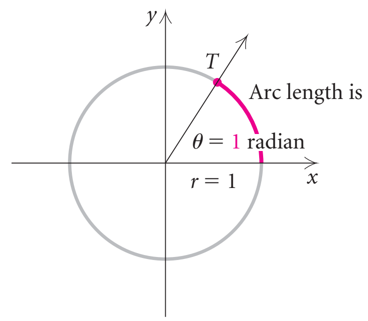


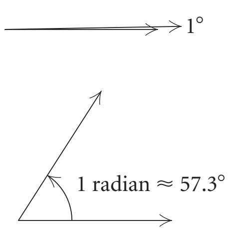
b) The number $-7\pi/6$ is negative, so the point moves clockwise. From A to B, the distance is π, or $\frac{6}{6}\pi$, so we need to go beyond B another distance of $\pi/6$, clockwise. (See the figure on the right above.)

Radian Measure

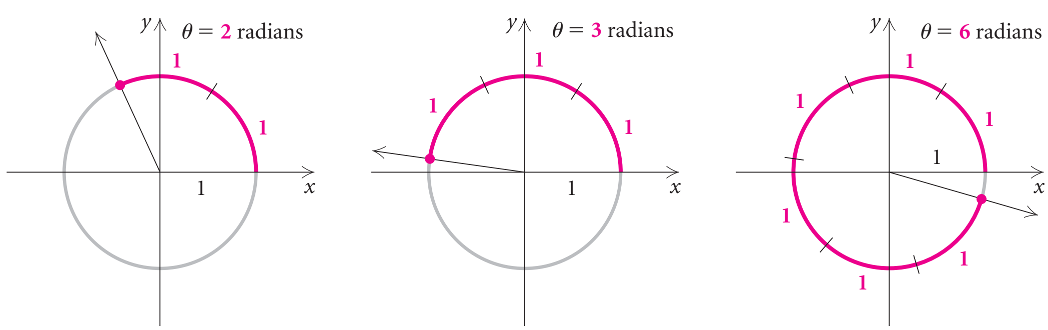
Degree measure is a common unit of angle measure in many everyday applications. But in many scientific fields and in mathematics (calculus, in particular), there is another commonly used unit of measure called the *radian.*

Consider the unit circle. Recall that this circle has radius 1. Suppose we measure, moving counterclockwise, an arc of length 1, and mark a point T on the circle.





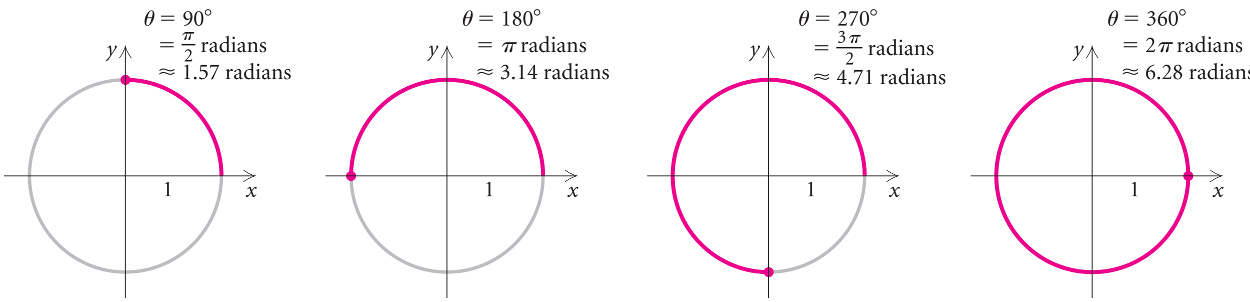
If we draw a ray from the origin through T, we have formed an angle. The measure of that angle is 1 **radian.** The word radian comes from the word *radius.* Thus measuring 1 "radius" along the circumference of the circle determines an angle whose measure is 1 *radian.* One radian is about 57.3°. Angles that measure 2 radians, 3 radians, and 6 radians are shown below.



When we make a complete (counterclockwise) revolution, the terminal side coincides with the initial side on the positive x-axis. We then have an angle whose measure is 2π radians, or about 6.28 radians, which is the circumference of the circle:

$$2\pi r = 2\pi(1) = 2\pi.$$

Thus a rotation of 360° (1 revolution) has a measure of 2π radians. A half revolution is a rotation of 180°, or π radians. A quarter revolution is a rotation of 90°, or $\pi/2$ radians, and so on.



To convert between degrees and radians, we first note that

$$360° = 2\pi \text{ radians}.$$

It follows that

$$180° = \pi \text{ radians}.$$

To make conversions, we multiply by 1, noting that:

Converting between Degree and Radian Measure

$$\frac{\pi \text{ radians}}{180°} = \frac{180°}{\pi \text{ radians}} = 1.$$

To convert from degree to radian measure, multiply by $\dfrac{\pi \text{ radians}}{180°}$.

To convert from radian to degree measure, multiply by $\dfrac{180°}{\pi \text{ radians}}$.

EXAMPLE 3 Convert each of the following to radians.

a) 120°

b) $-297.25°$

Solution

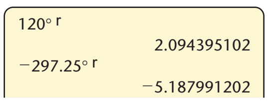
a) $$120° = 120° \cdot \frac{\pi \text{ radians}}{180°} \quad \textbf{Multiplying by 1}$$

$$= \frac{120°}{180°}\pi \text{ radians}$$

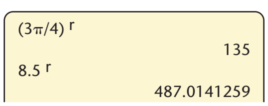
$$= \frac{2\pi}{3} \text{ radians, or about 2.09 radians}$$

Technology Connection

To convert degrees to radians, we set the calculator in RADIAN mode. Then we enter the angle measure followed by ^{r} (radians) from the ANGLE menu. Example 3 is shown here.



To convert radians to degrees, we set the calculator in DEGREE mode. Then we enter the angle measure followed by ^{r} (radians) from the ANGLE menu. Example 4 is shown here.



b) $-297.25° = -297.25° \cdot \dfrac{\pi \text{ radians}}{180°}$

$$= -\frac{297.25°}{180°}\pi \text{ radians}$$

$$= -\frac{297.25\pi}{180} \text{ radians}$$

$$\approx -5.19 \text{ radians}$$

EXAMPLE 4 Convert each of the following to degrees.

a) $\dfrac{3\pi}{4}$ radians **b)** 8.5 radians

Solution

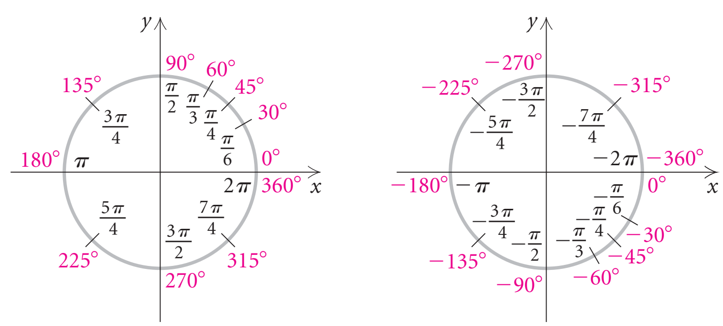
a) $\dfrac{3\pi}{4} \text{ radians} = \dfrac{3\pi}{4} \text{ radians} \cdot \dfrac{180°}{\pi \text{ radians}}$ **Multiplying by 1**

$$= \frac{3\pi}{4\pi} \cdot 180° = \frac{3}{4} \cdot 180° = 135°$$

b) $8.5 \text{ radians} = 8.5 \text{ radians} \cdot \dfrac{180°}{\pi \text{ radians}}$

$$= \frac{8.5(180°)}{\pi} \approx 487.01°$$

The radian–degree equivalents of the most commonly used angle measures are illustrated in the following figures.



When a rotation is given in radians, the word "radians" is optional and is most often omitted. **Thus if no unit is given for a rotation, the rotation is understood to be in radians.**

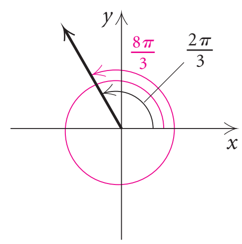
We can also find coterminal, complementary, and supplementary angles in radian measure just as we did for degree measure in Section 5.3.

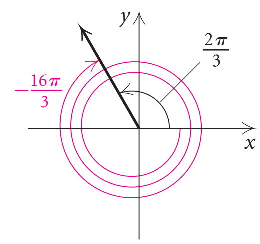
EXAMPLE 5 Find a positive angle and a negative angle that are coterminal with $2\pi/3$. Many answers are possible.

Solution To find angles coterminal with a given angle, we add or subtract multiples of 2π:

$$\frac{2\pi}{3} + 2\pi = \frac{2\pi}{3} + \frac{6\pi}{3} = \frac{8\pi}{3},$$

$$\frac{2\pi}{3} - 3(2\pi) = \frac{2\pi}{3} - \frac{18\pi}{3} = -\frac{16\pi}{3}.$$





Thus, $8\pi/3$ and $-16\pi/3$ are two of the many angles coterminal with $2\pi/3$.

EXAMPLE 6 Find the complement and the supplement of $\pi/6$.

Solution Since 90° equals $\pi/2$ radians, the complement of $\pi/6$ is

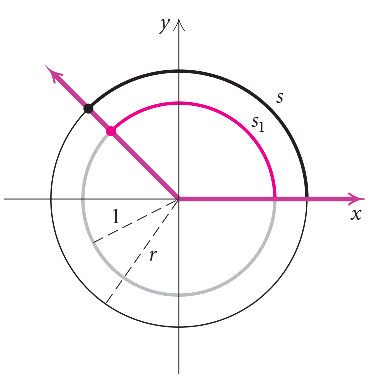
$$\frac{\pi}{2} - \frac{\pi}{6} = \frac{3\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{6}, \quad \text{or} \quad \frac{\pi}{3}.$$

Since 180° equals π radians, the supplement of $\pi/6$ is

$$\pi - \frac{\pi}{6} = \frac{6\pi}{6} - \frac{\pi}{6} = \frac{5\pi}{6}.$$

Thus the complement of $\pi/6$ is $\pi/3$ and the supplement is $5\pi/6$.

Arc Length and Central Angles



Radian measure can be determined using a circle other than a unit circle. In the figure at left, a unit circle (with radius 1) is shown along with another circle (with radius r, $r \neq 1$). The angle shown is a **central angle** of both circles.

From geometry, we know that the arcs that the angle subtends have their lengths in the same ratio as the radii of the circles. The radii of the circles are r and 1. The corresponding arc lengths are s and s_1. Thus we have the proportion

$$\frac{s}{s_1} = \frac{r}{1},$$

which also can be written as

$$\frac{s_1}{1} = \frac{s}{r}.$$

Now s_1 is the *radian measure* of the rotation in question. It is common to use a Greek letter, such as θ, for the measure of an angle or rotation and the letter s for arc length. Adopting this convention, we rewrite the proportion above as

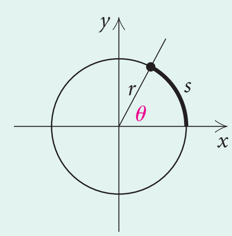
$$\theta = \frac{s}{r}.$$

In any circle, the measure (in radians) of a central angle, the arc length the angle subtends, and the length of the radius are related in this fashion Or, in general, the following is true.

Radian Measure

The **radian measure** θ of a rotation is the ratio of the distance s traveled by a point at a radius r from the center of rotation, to the length of the radius r:

$$\theta = \frac{s}{r}.$$



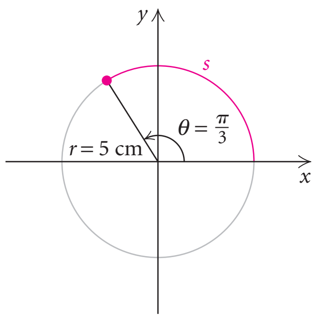
When using the formula $\theta = s/r$, θ must be in radians and s and r must be expressed in the same unit.

EXAMPLE 7 Find the measure of a rotation in radians when a point 2 m from the center of rotation travels 4 m.

Solution We have

$$\theta = \frac{s}{r}$$

$$= \frac{4\text{ m}}{2\text{ m}} = 2. \qquad \textbf{The unit is understood to be radians.}$$



EXAMPLE 8 Find the length of an arc of a circle of radius 5 cm associated with an angle of $\pi/3$ radians.

Solution We have

$$\theta = \frac{s}{r}, \quad \text{or} \quad s = r\theta.$$

Thus $s = 5\text{ cm} \cdot \pi/3$, or about 5.24 cm.

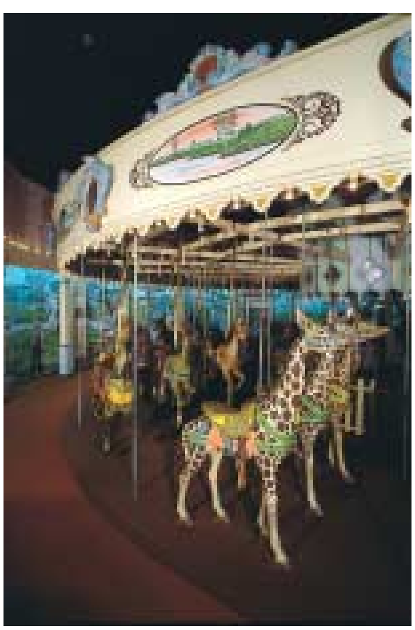
Linear Speed and Angular Speed

Linear speed is defined to be distance traveled per unit of time. If we use v for linear speed, s for distance, and t for time, then

$$v = \frac{s}{t}.$$

Similarly, **angular speed** is defined to be amount of rotation per unit of time. For example, we might speak of the angular speed of a bicycle wheel as 150 revolutions per minute or the angular speed of the earth as 2π radians per day. The Greek letter ω (omega) is generally used for angular speed. Thus for a rotation θ and time t, angular speed is defined as

$$\omega = \frac{\theta}{t}.$$



As an example of how these definitions can be applied, let's consider the refurbished carousel at the Children's Museum in Indianapolis, Indiana. It consists of three circular rows of animals. All animals, regardless of the row, travel at the same angular speed. But the animals in the outer row travel at a greater linear speed than those in the inner rows. What is the relationship between the linear speed v and the angular speed ω?

To develop the relationship we seek, recall that, for rotations measured in radians, $\theta = s/r$. This is equivalent to

$$s = r\theta.$$

We divide by time, t, to obtain

$$\frac{s}{t} = \frac{r\theta}{t} \qquad \textbf{Dividing by } t$$

$$\underbrace{\frac{s}{t}}_{v} = r \cdot \underbrace{\frac{\theta}{t}}_{\omega}$$

Now s/t is linear speed v and θ/t is angular speed ω. Thus we have the relationship we seek,

$$v = r\omega.$$

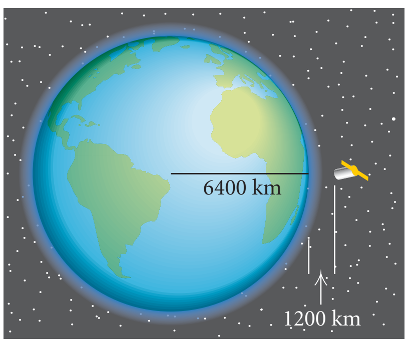
Linear Speed in Terms of Angular Speed

The **linear speed** v of a point a distance r from the center of rotation is given by

$$v = r\omega,$$

where ω is the **angular speed** in radians per unit of time.

For the formula $v = r\omega$, the units of distance for v and r must be the same, ω must be in radians per unit of time, and the units of time for v and ω must be the same.



EXAMPLE 9 *Linear Speed of an Earth Satellite.* An earth satellite in circular orbit 1200 km high makes one complete revolution every 90 min. What is its linear speed? Use 6400 km for the length of a radius of the earth.

Solution To use the formula $v = r\omega$, we need to know r and ω:

$$r = 6400 \text{ km} + 1200 \text{ km}$$ **Radius of earth plus height of satellite**

$$= 7600 \text{ km},$$

$$\omega = \frac{\theta}{t} = \frac{2\pi}{90 \text{ min}} = \frac{\pi}{45 \text{ min}}.$$ **We have, as usual, omitted the word radians.**

Now, using $v = r\omega$, we have

$$v = 7600 \text{ km} \cdot \frac{\pi}{45 \text{ min}} = \frac{7600\pi}{45} \cdot \frac{\text{km}}{\text{min}} \approx 531 \frac{\text{km}}{\text{min}}.$$

Thus the linear speed of the satellite is approximately 531 km/min.

EXAMPLE 10 *Angular Speed of a Capstan.* An anchor is hoisted at a rate of 2 ft/sec as the chain is wound around a capstan with a 1.8-yd diameter. What is the angular speed of the capstan?

Solution We will use the formula $v = r\omega$ in the form $\omega = v/r$, taking care to use the proper units. Since v is given in feet per second, we need r in feet:

$$r = \frac{d}{2} = \frac{1.8}{2}\text{ yd} \cdot \frac{3\text{ ft}}{1\text{ yd}} = 2.7\text{ ft}.$$

Then ω will be in radians per second:

$$\omega = \frac{v}{r} = \frac{2\text{ ft/sec}}{2.7\text{ ft}} = \frac{2\text{ ft}}{\text{sec}} \cdot \frac{1}{2.7\text{ ft}} \approx 0.741/\text{sec}.$$

Thus the angular speed is approximately 0.741 radian/sec.

The formulas $\theta = \omega t$ and $v = r\omega$ can be used in combination to find distances and angles in various situations involving rotational motion.

EXAMPLE 11 *Angle of Revolution.* A 2004 Tundra V8 is traveling at a speed of 65 mph. Its tires have an outside diameter of 30.56 in. Find the angle through which a tire turns in 10 sec.

Solution Recall that $\omega = \theta/t$, or $\theta = \omega t$. Thus we can find θ if we know ω and t. To find ω, we use the formula $v = r\omega$. The linear speed v of a point on the outside of the tire is the speed of the Tundra, 65 mph. For convenience, we first convert 65 mph to feet per second:

$$\begin{aligned} v &= 65\,\frac{\text{mi}}{\text{hr}} \cdot \frac{1\text{ hr}}{60\text{ min}} \cdot \frac{1\text{ min}}{60\text{ sec}} \cdot \frac{5280\text{ ft}}{1\text{ mi}} \\ &\approx 95.333\,\frac{\text{ft}}{\text{sec}}. \end{aligned}$$

The radius of the tire is half the diameter. Now $r = d/2 = 30.56\text{ in.}/2 = 15.28$ in. We will convert to feet, since v is in feet per second:

$$\begin{aligned} r &= 15.28\text{ in.} \cdot \frac{1\text{ ft}}{12\text{ in.}} \\ &= \frac{15.28}{12}\text{ ft} \approx 1.27\text{ ft}. \end{aligned}$$

Using $v = r\omega$, we have

$$95.333\,\frac{\text{ft}}{\text{sec}} = 1.27\text{ ft}\cdot\omega,$$

so

$$\omega = \frac{95.333\text{ ft/sec}}{1.27\text{ ft}} \approx \frac{75.07}{\text{sec}}.$$

Then in 10 sec,

$$\theta = \omega t = \frac{75.07}{\text{sec}}\cdot 10\text{ sec} \approx 751.$$

Thus the angle, in radians, through which a tire turns in 10 sec is 751.

Study Tip

The *Student's Solutions Manual* is an excellent resource if you need additional help with an exercise in the exercise sets. It contains worked-out solutions to the odd-numbered exercises in the exercise sets.

5.4 Exercise Set

For each of Exercises 1–4, sketch a unit circle and mark the points determined by the given real numbers.

1. a) $\frac{\pi}{4}$ b) $\frac{3\pi}{2}$ c) $\frac{3\pi}{4}$
d) π e) $\frac{11\pi}{4}$ f) $\frac{17\pi}{4}$

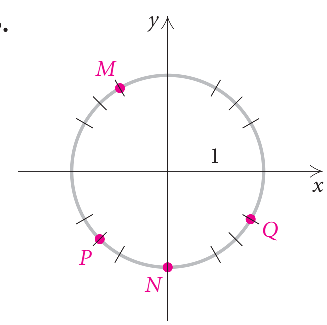
2. a) $\frac{\pi}{2}$ b) $\frac{5\pi}{4}$ c) 2π
d) $\frac{9\pi}{4}$ e) $\frac{13\pi}{4}$ f) $\frac{23\pi}{4}$

3. a) $\frac{\pi}{6}$ b) $\frac{2\pi}{3}$ c) $\frac{7\pi}{6}$
d) $\frac{10\pi}{6}$ e) $\frac{14\pi}{6}$ f) $\frac{23\pi}{4}$

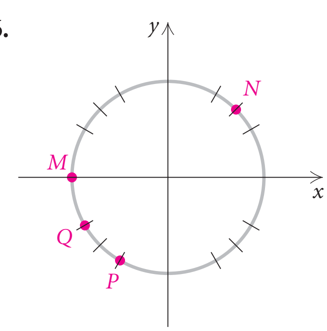
4. a) $-\frac{\pi}{2}$ b) $-\frac{3\pi}{4}$ c) $-\frac{5\pi}{6}$
d) $-\frac{5\pi}{2}$ e) $-\frac{17\pi}{6}$ f) $-\frac{9\pi}{4}$

Find two real numbers between -2π and 2π that determine each of the points on the unit circle.

5.



6.



For Exercises 7 and 8, sketch a unit circle and mark the approximate location of the point determined by the given real number.

7. a) 2.4 **b)** 7.5
c) 32 **d)** 320

8. a) 0.25 **b)** 1.8
c) 47 **d)** 500

Find a positive angle and a negative angle that are coterminal with the given angle. Answers may vary.

9. $\frac{\pi}{4}$ **10.** $\frac{5\pi}{3}$

11. $\frac{7\pi}{6}$ **12.** π

13. $-\frac{2\pi}{3}$ **14.** $-\frac{3\pi}{4}$

Find the complement and the supplement.

15. $\frac{\pi}{3}$ **16.** $\frac{5\pi}{12}$

17. $\frac{3\pi}{8}$ **18.** $\frac{\pi}{4}$

19. $\frac{\pi}{12}$ **20.** $\frac{\pi}{6}$

Convert to radian measure. Leave the answer in terms of π.

21. 75° **22.** 30°

23. 200° **24.** −135°

25. −214.6° **26.** 37.71°

27. −180° **28.** 90°

29. 12.5° **30.** 6.3°

31. −340° **32.** −60°

Convert to radian measure. Round the answer to two decimal places.

33. 240° **34.** 15°

35. −60° **36.** 145°

37. 117.8° **38.** −231.2°

39. 1.354° **40.** 584°

41. 345° **42.** −75°

43. 95° **44.** 24.8°

Convert to degree measure. Round the answer to two decimal places.

45. $-\frac{3\pi}{4}$ **46.** $\frac{7\pi}{6}$

47. 8π **48.** $-\frac{\pi}{3}$

49. 1 **50.** −17.6

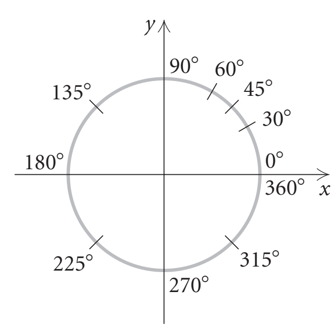
51. 2.347 **52.** 25

53. $\frac{5\pi}{4}$ **54.** -6π

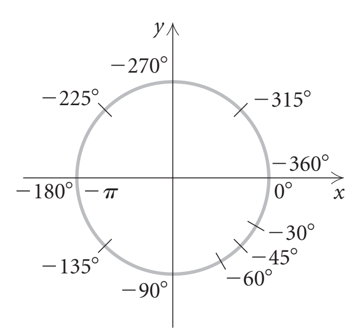
55. −90 **56.** 37.12

57. $\frac{2\pi}{7}$ **58.** $\frac{\pi}{9}$

59. Certain positive angles are marked here in degrees. Find the corresponding radian measures.



60. Certain negative angles are marked here in degrees. Find the corresponding radian measures.



Arc Length and Central Angles. *Complete the table. Round the answers to two decimal places.*

	Distance, s (arc length)	Radius, r	Angle, θ
61.	8 ft	$3\frac{1}{2}$ ft	
62.	200 cm		45°
63.	16 yd		5
64.		4.2 in.	$\frac{5\pi}{12}$

65. In a circle with a 120-cm radius, an arc 132 cm long subtends an angle of how many radians? how many degrees, to the nearest degree?

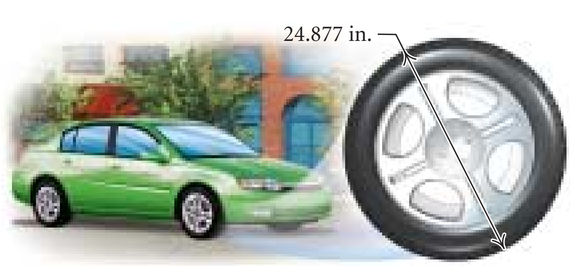
66. In a circle with a 10-ft diameter, an arc 20 ft long subtends an angle of how many radians? how many degrees, to the nearest degree?

67. In a circle with a 2-yd radius, how long is an arc associated with an angle of 1.6 radians?

68. In a circle with a 5-m radius, how long is an arc associated with an angle of 2.1 radians?

69. *Angle of Revolution.* Through how many radians does the minute hand of a clock rotate from 12:40 P.M. to 1:30 P.M.?

70. *Angle of Revolution.* A tire on a 2004 Saturn Ion has an outside diameter of 24.877 in. Through what angle (in radians) does the tire turn while traveling 1 mi?



71. *Linear Speed.* A flywheel with a 15-cm diameter is rotating at a rate of 7 radians/sec. What is the linear speed of a point on its rim, in centimeters per minute?

72. *Linear Speed.* A wheel with a 30-cm radius is rotating at a rate of 3 radians/sec. What is the linear speed of a point on its rim, in meters per minute?

73. *Angular Speed of a Printing Press.* This text was printed on a four-color web heatset offset press. A cylinder on this press has a 21-in. diameter. The linear speed of a point on the cylinder's surface is 18.33 feet per second. What is the angular speed of the cylinder, in revolutions per hour? Printers often refer to the angular speed as impressions per hour (IPH). (*Source*: Bill Delano, Von Hoffman, St Louis, Missouri)

74. *Linear Speeds on a Carousel.* When Alicia and Zoe ride the carousel described earlier in this section, Alicia always selects a horse on the outside row, whereas Zoe prefers the row closest to the center. These rows are 19 ft 3 in. and 13 ft 11 in. from the center, respectively. The angular speed of the carousel is 2.4 revolutions per minute. What is the difference, in miles per hour, in the linear speeds of Alicia and Zoe? (*Source*: The Children's Museum, Indianapolis, IN)

75. *Linear Speed at the Equator.* The earth has a 4000-mi radius and rotates one revolution every 24 hr. What is the linear speed of a point on the equator, in miles per hour?

76. *Linear Speed of the Earth.* The earth is about 93,000,000 mi from the sun and traverses its orbit, which is nearly circular, every 365.25 days. What is the linear velocity of the earth in its orbit, in miles per hour?

77. *Determining the Speed of a River.* A water wheel has a 10-ft radius. To get a good approximation of the speed of the river, you count the revolutions of the wheel and find that it makes 14 revolutions per minute (rpm). What is the speed of the river, in miles per hour?

78. *The Tour de France.* Lance Armstrong won the 2003 Tour de France bicycle race. The wheel of his bicycle had a 67-cm diameter. His overall average linear speed during the race was 40.940 km/h. What was the angular speed of the wheel, in revolutions per hour? (*Source*: Velo.news.com)

79. *John Deere Tractor.* A rear wheel on a John Deere 8300 farm tractor has a 23-in. radius. Find the angle (in radians) through which a wheel rotates in 12 sec if the tractor is traveling at a speed of 22 mph.

Technology Connection

80. In each of Exercises 33–44, convert to radian measure using a graphing calculator.

81. In each of Exercises 45–58, convert to degree measure using a graphing calculator.

Collaborative Discussion and Writing

82. Explain in your own words why it is preferable to omit the word, or unit, *radians* in radian measures.

83. In circular motion with a fixed angular speed, the length of the radius is directly proportional to the linear speed. Explain why with an example.

84. Two new cars are each driven at an average speed of 60 mph for an extended highway test drive of 2000 mi. The diameter of the wheels of the two cars are 15 in. and 16 in., respectively. If the cars use tires of equal durability and profile, differing only by the diameter, which car will probably need new tires first? Explain your answer.

Skill Maintenance

In each of Exercises 85–92, fill in the blanks with the correct terms. Some of the given choices will not be used.

inverse
a horizontal line
a vertical line
exponential function
logarithmic function
natural
common
logarithm
one-to-one
a relation
vertical asymptote
horizontal asymptote
even function

odd function
sine of θ
cosine of θ
tangent of θ

85. The domain of a ____________ function f is the range of the inverse f^{-1}.

86. The ____________ is the length of the side adjacent to θ divided by the length of the hypotenuse.

87. The function $f(x) = a^x$, where x is a real number, $a > 0$ and $a \neq 1$, is called the ____________, base a.

88. The graph of a rational function may or may not cross a ____________.

89. If the graph of a function f is symmetric with respect to the origin, we say that it is an ____________.

90. Logarithms, base e, are called ____________ logarithms.

91. If it is possible for a ____________ to intersect the graph of a function more than once, then the function is not one-to-one and its ____________ is not a function.

92. A ____________ is an exponent.

Synthesis

93. On the earth, one degree of latitude is how many kilometers? how many miles? (Assume that the radius of the earth is 6400 km, or 4000 mi, approximately.)

94. A point on the unit circle has y-coordinate $-\sqrt{21}/5$. What is its x-coordinate? Check using a calculator.

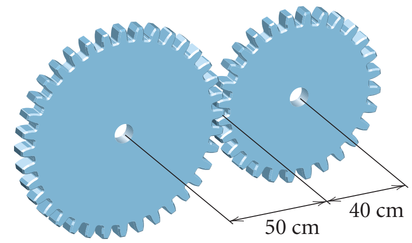
95. A **mil** is a unit of angle measure. A right angle has a measure of 1600 mils. Convert each of the following to degrees, minutes, and seconds.

a) 100 mils **b)** 350 mils

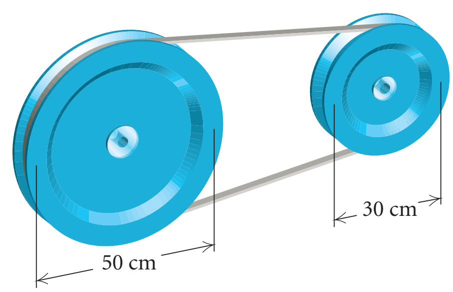
96. A **grad** is a unit of angle measure similar to a degree. A right angle has a measure of 100 grads. Convert each of the following to grads.

a) $48°$ **b)** $\dfrac{5\pi}{7}$

97. *Angular Speed of a Gear Wheel.* One gear wheel turns another, the teeth being on the rims. The wheels have 40-cm and 50-cm radii, and the smaller wheel rotates at 20 rpm. Find the angular speed of the larger wheel, in radians per second.



98. *Angular Speed of a Pulley.* Two pulleys, 50 cm and 30 cm in diameter, respectively, are connected by a belt. The larger pulley makes 12 revolutions per minute. Find the angular speed of the smaller pulley, in radians per second.



99. *Distance between Points on the Earth.* To find the distance between two points on the earth when their latitude and longitude are known, we can use a right triangle for an excellent approximation if the points are not too far apart. Point A is at latitude $38°27'30''$ N, longitude $82°57'15''$ W; and point B is at latitude $38°28'45''$ N, longitude $82°56'30''$ W. Find the distance from A to B in nautical miles. (One minute of latitude is one nautical mile.)

100. *Hands of a Clock.* At what times between noon and 1:00 P.M. are the hands of a clock perpendicular?

5.5 Circular Functions: Graphs and Properties

- *Given the coordinates of a point on the unit circle, find its reflections across the x-axis, the y-axis, and the origin.*
- *Determine the six trigonometric function values for a real number when the coordinates of the point on the unit circle determined by that real number are given.*
- *Find function values for any real number using a calculator.*
- *Graph the six circular functions and state their properties.*

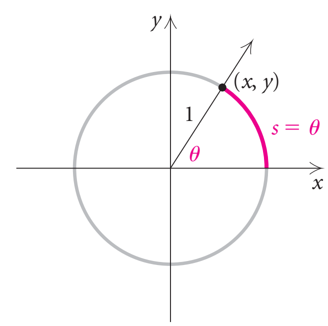
The domains of the trigonometric functions, defined in Sections 5.1 and 5.3, have been sets of angles or rotations measured in a real number of degree units. We can also consider the domains to be sets of real numbers, or radians, introduced in Section 5.4. Many applications in calculus that use the trigonometric functions refer only to radians.

Let's again consider radian measure and the unit circle. We defined radian measure for θ as

$$\theta = \frac{s}{r}.$$

When $r = 1$,

$$\theta = \frac{s}{1}, \quad \text{or} \quad \theta = s.$$



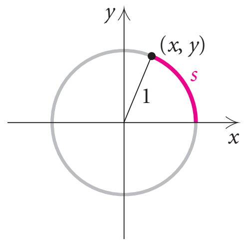
The arc length s on the unit circle is the same as the radian measure of the angle θ.

In the figure above, the point (x, y) is the point where the terminal side of the angle with radian measure s intersects the unit circle. We can now extend our definitions of the trigonometric functions using domains composed of real numbers, or radians.

In the definitions, *s can be considered the radian measure of an angle or the measure of an arc length on the unit circle. Either way, s is a real number.* To each real number s, there corresponds an arc length s on the unit circle. Trigonometric functions with domains composed of real numbers are called **circular functions.**

Study Tip

Take advantage of the numerous detailed art pieces in this text. They provide a visual image of the concept being discussed. Taking the time to study each figure is an efficient way to learn and retain the concepts.

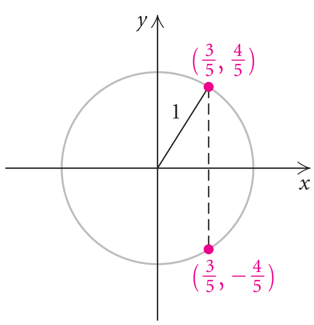


Basic Circular Functions

For a real number s that determines a point (x, y) on the unit circle:

$$\sin s = \text{second coordinate} = y,$$
$$\cos s = \text{first coordinate} = x,$$
$$\tan s = \frac{\text{second coordinate}}{\text{first coordinate}} = \frac{y}{x} \quad (x \neq 0),$$
$$\csc s = \frac{1}{\text{second coordinate}} = \frac{1}{y} \quad (y \neq 0),$$
$$\sec s = \frac{1}{\text{first coordinate}} = \frac{1}{x} \quad (x \neq 0),$$
$$\cot s = \frac{\text{first coordinate}}{\text{second coordinate}} = \frac{x}{y} \quad (y \neq 0).$$

We can consider the domains of trigonometric functions to be real numbers rather than angles. We can determine these values for a specific real number if we know the coordinates of the point on the unit circle determined by that number. As with degree measure, we can also find these function values directly using a calculator.



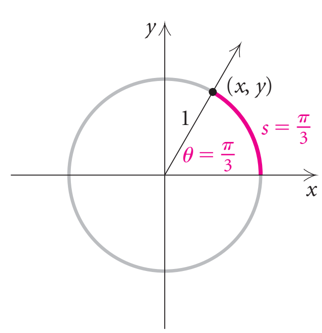
Reflections on the Unit Circle

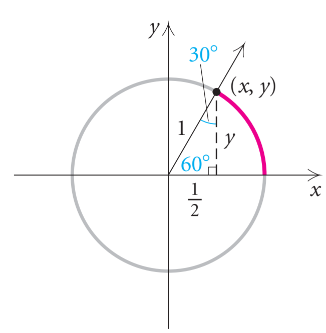
Let's consider the unit circle and a few of its points. For any point (x, y) on the unit circle, $x^2 + y^2 = 1$, we know that $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. If we know the x- or y-coordinate of a point on the unit circle, we can find the other coordinate. If $x = \frac{3}{5}$, then

$$\left(\tfrac{3}{5}\right)^2 + y^2 = 1$$
$$y^2 = 1 - \tfrac{9}{25} = \tfrac{16}{25}$$
$$y = \pm\tfrac{4}{5}.$$

Thus, $\left(\frac{3}{5}, \frac{4}{5}\right)$ and $\left(\frac{3}{5}, -\frac{4}{5}\right)$ are points on the unit circle. There are two points with an x-coordinate of $\frac{3}{5}$.

Now let's consider the radian measure $\pi/3$ and determine the coordinates of the point on the unit circle determined by $\pi/3$. We construct a right triangle by dropping a perpendicular segment from the point to the x-axis.



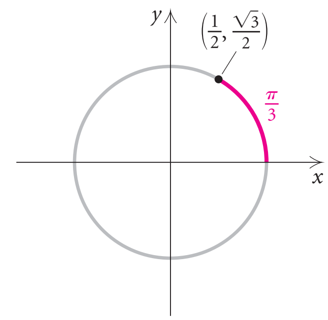


Since $\pi/3 = 60°$, we have a 30°–60° right triangle in which the side opposite the 30° angle is one half of the hypotenuse. The hypotenuse, or radius, is 1, so the side opposite the 30° angle is $\frac{1}{2} \cdot 1$, or $\frac{1}{2}$. Using the Pythagorean theorem, we can find the other side:

$$\left(\frac{1}{2}\right)^2 + y^2 = 1$$

$$y^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$y = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$



We know that y is positive since the point is in the first quadrant. Thus the coordinates of the point determined by $\pi/3$ are $x = 1/2$ and $y = \sqrt{3}/2$, or $(1/2, \sqrt{3}/2)$. We can always check to see if a point is on the unit circle by substituting into the equation $x^2 + y^2 = 1$:

$$\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1.$$

Because a unit circle is symmetric with respect to the x-axis, the y-axis, and the origin, we can use the coordinates of one point on the unit circle to find coordinates of its reflections.

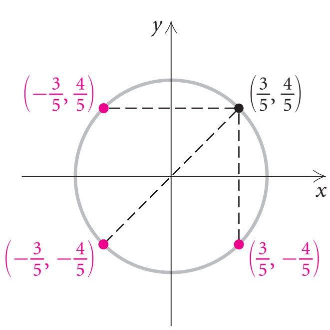
EXAMPLE 1 Each of the following points lies on the unit circle. Find their reflections across the x-axis, the y-axis, and the origin.

a) $\left(\frac{3}{5}, \frac{4}{5}\right)$ **b)** $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

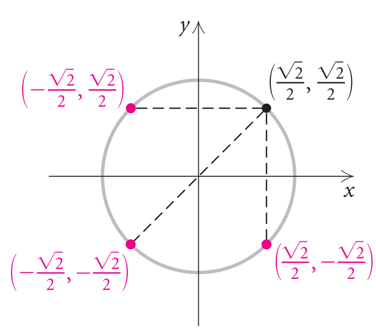
c) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Solution

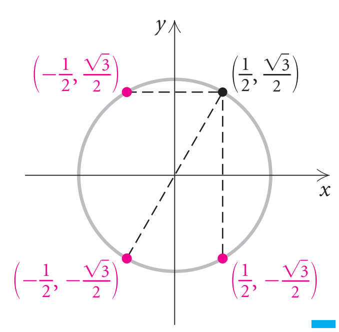
a)



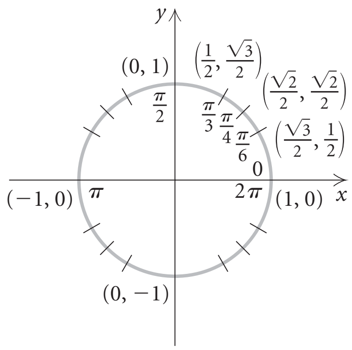
b)



c)



Finding Function Values



Knowing the coordinates of only a few points on the unit circle along with their reflections allows us to find trigonometric function values of the most frequently used real numbers, or radians.

EXAMPLE 2 Find each of the following function values.

a) $\tan \dfrac{\pi}{3}$

b) $\cos \dfrac{3\pi}{4}$

c) $\sin \left(-\dfrac{\pi}{6}\right)$

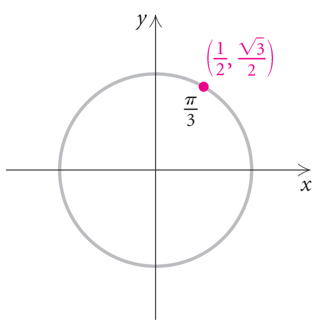
d) $\cos \dfrac{4\pi}{3}$

e) $\cot \pi$

f) $\csc \left(-\dfrac{7\pi}{2}\right)$

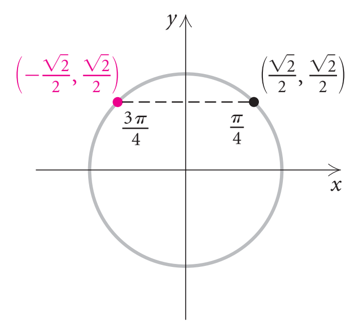
Solution We locate the point on the unit circle determined by the rotation, and then find its coordinates using reflection if necessary.

a) The coordinates of the point determined by $\pi/3$ are $\left(1/2, \sqrt{3}/2\right)$.



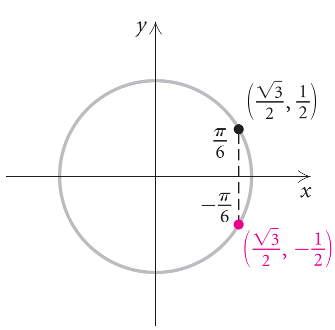
Thus, $\tan \dfrac{\pi}{3} = \dfrac{y}{x} = \dfrac{\sqrt{3}/2}{1/2} = \sqrt{3}$.

b) The reflection of $\left(\sqrt{2}/2, \sqrt{2}/2\right)$ across the y-axis is $\left(-\sqrt{2}/2, \sqrt{2}/2\right)$.



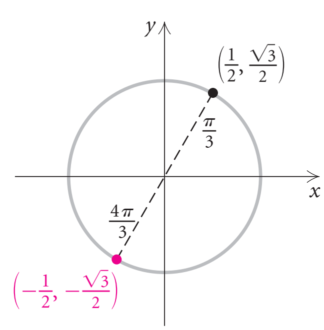
Thus, $\cos \dfrac{3\pi}{4} = x = -\dfrac{\sqrt{2}}{2}$.

c) The reflection of $\left(\sqrt{3}/2, 1/2\right)$ across the x-axis is $\left(\sqrt{3}/2, -1/2\right)$.



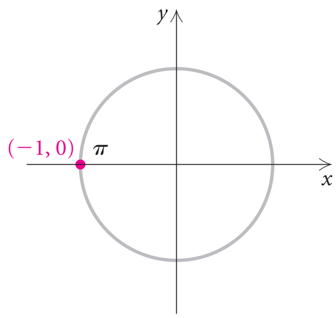
Thus, $\sin \left(-\dfrac{\pi}{6}\right) = y = -\dfrac{1}{2}$.

d) The reflection of $\left(1/2, \sqrt{3}/2\right)$ across the origin is $\left(-1/2, -\sqrt{3}/2\right)$.



Thus, $\cos \dfrac{4\pi}{3} = x = -\dfrac{1}{2}$.

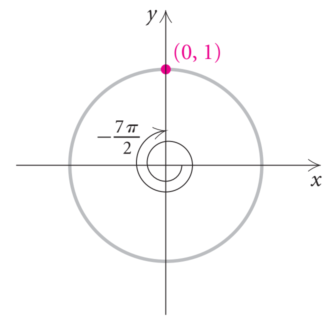
e) The coordinates of the point determined by π are $(-1, 0)$.



Thus, $\cot \pi = \dfrac{x}{y} = \dfrac{-1}{0}$, which is not defined.

We can also think of $\cot \pi$ as the reciprocal of $\tan \pi$. Since $\tan \pi = y/x = 0/-1 = 0$ and the reciprocal of 0 is not defined, we know that $\cot \pi$ is not defined.

f) The coordinates of the point determined by $-7\pi/2$ are $(0, 1)$.

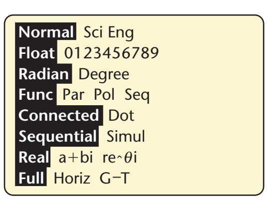


Thus, $\csc\left(-\dfrac{7\pi}{2}\right) = \dfrac{1}{y} = \dfrac{1}{1} = 1.$

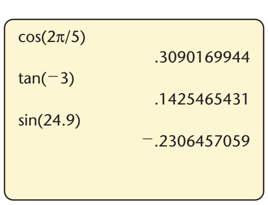
Using a calculator, we can find trigonometric function values of any real number without knowing the coordinates of the point that it determines on the unit circle. Most calculators have both degree and radian modes. When finding function values of radian measures, or real numbers, we *must* set the calculator in RADIAN mode.

Technology Connection

To find trigonometric function values of angles measured in radians, we set the calculator in RADIAN mode.



Parts (a)–(c) of Example 3 are shown in the window below.



EXAMPLE 3 Find each of the following function values of radian measures using a calculator. Round the answers to four decimal places.

a) $\cos \dfrac{2\pi}{5}$

b) $\tan(-3)$

c) $\sin 24.9$

d) $\sec \dfrac{\pi}{7}$

Solution Using a calculator set in RADIAN mode, we find the values.

a) $\cos \dfrac{2\pi}{5} \approx 0.3090$

b) $\tan(-3) \approx 0.1425$

c) $\sin 24.9 \approx -0.2306$

d) $\sec \dfrac{\pi}{7} = \dfrac{1}{\cos \dfrac{\pi}{7}} \approx 1.1099$

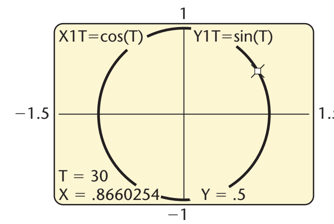
Note in part (d) that the secant function value can be found by taking the reciprocal of the cosine value. Thus we can enter $\cos \pi/7$ and use the reciprocal key.

Technology Connection

Exploration

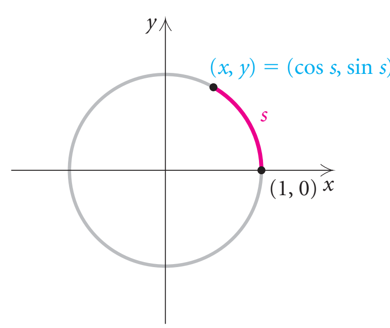
We can graph the unit circle using a graphing calculator. We use PARAMETRIC mode with the following window and let $X_{1T} = \cos T$ and $Y_{1T} = \sin T$. Here we use DEGREE mode.

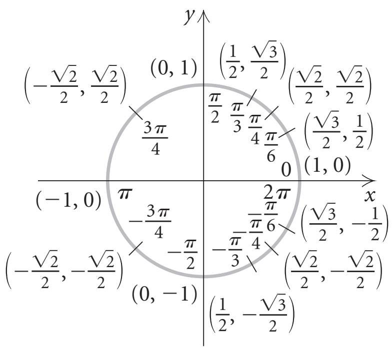
WINDOW
- Tmin = 0
- Tmax = 360
- Tstep = 15
- Xmin = −1.5
- Xmax = 1.5
- Xscl = 1
- Ymin = −1
- Ymax = 1
- Yscl = 1



Using the trace key and an arrow key to move the cursor around the unit circle, we see the T, X, and Y values appear on the screen. What do they represent? Repeat this exercise in RADIAN mode. What do the T, X, and Y values represent? (For more on parametric equations, see Appendix B.)

From the definitions on p. 467, we can relabel any point (x, y) on the unit circle as $(\cos s, \sin s)$, where s is any real number.





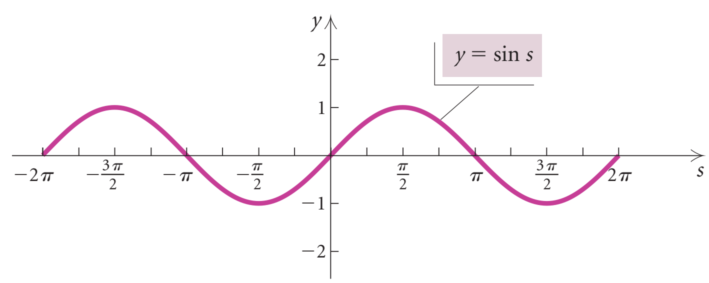
Graphs of the Sine and Cosine Functions

Properties of functions can be observed from their graphs. We begin by graphing the sine and cosine functions. We make a table of values, plot the points, and then connect those points with a smooth curve. It is helpful to first draw a unit circle and label a few points with coordinates. We can either use the coordinates as the function values or find approximate sine and cosine values directly with a calculator.

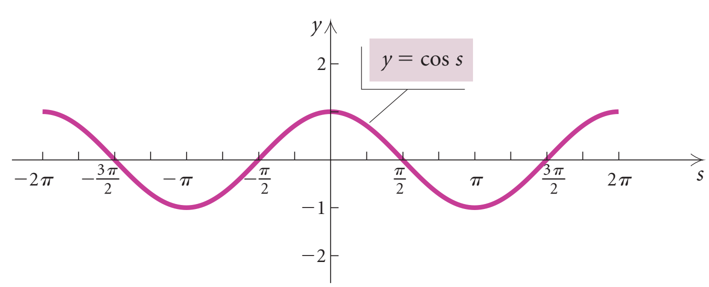
s	$\sin s$	$\cos s$
0	0	1
$\pi/6$	0.5	0.8660
$\pi/4$	0.7071	0.7071
$\pi/3$	0.8660	0.5
$\pi/2$	1	0
$3\pi/4$	0.7071	−0.7071
π	0	−1
$5\pi/4$	−0.7071	−0.7071
$3\pi/2$	−1	0
$7\pi/4$	−0.7071	0.7071
2π	0	1

s	$\sin s$	$\cos s$
0	0	1
$-\pi/6$	−0.5	0.8660
$-\pi/4$	−0.7071	0.7071
$-\pi/3$	−0.8660	0.5
$-\pi/2$	−1	0
$-3\pi/4$	−0.7071	−0.7071
$-\pi$	0	−1
$-5\pi/4$	0.7071	−0.7071
$-3\pi/2$	1	0
$-7\pi/4$	0.7071	0.7071
-2π	0	1

The graphs are as follows.



The sine function

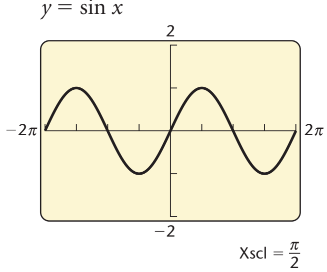


The cosine function

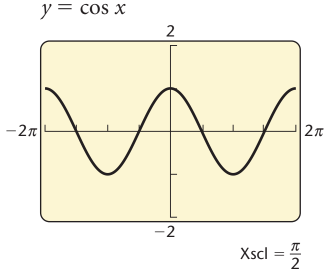
Technology Connection

The graphing calculator provides an efficient way to graph trigonometric functions. Here we use RADIAN mode to graph $y = \sin x$ and $y = \cos x$.

$y = \sin x$



$y = \cos x$



The sine and cosine functions are continuous functions. Note in the graph of the sine function that function values increase from 0 at $s = 0$ to 1 at $s = \pi/2$, then decrease to 0 at $s = \pi$, decrease further to -1 at $s = 3\pi/2$, and increase to 0 at 2π. The reverse pattern follows when s decreases from 0 to -2π. Note in the graph of the cosine function that function values start at 1 when $s = 0$, and decrease to 0 at $s = \pi/2$. They decrease further to -1 at $s = \pi$, then increase to 0 at $s = 3\pi/2$, and increase further to 1 at $s = 2\pi$. An identical pattern follows when s decreases from 0 to -2π.

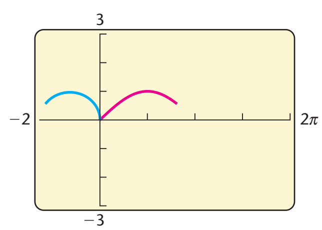
From the unit circle and the graphs of the functions, we know that the domain of both the sine and cosine functions is the entire set of real numbers, $(-\infty, \infty)$. The range of each function is the set of all real numbers from -1 to 1, $[-1, 1]$.

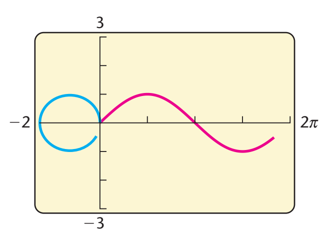
Domain and Range of Sine and Cosine Functions

The *domain* of the sine and cosine functions is $(-\infty, \infty)$.

The *range* of the sine and cosine functions is $[-1, 1]$.

Technology Connection





Another way to construct the sine and cosine graphs is by considering the unit circle and transferring vertical distances for the sine function and horizontal distances for the cosine function. Using a graphing calculator, we can visualize the transfer of these distances. We use the calculator set in PARAMETRIC and RADIAN modes and let $X_{1T} = \cos T - 1$ and $Y_{1T} = \sin T$ for the unit circle centered at $(-1, 0)$ and $X_{2T} = T$ and $Y_{2T} = \sin T$ for the sine curve. Use the following window settings.

Tmin = 0	Xmin = −2	Ymin = −3
Tmax = 2π	Xmax = 2π	Ymax = 3
Tstep = .1	Xscl = $\pi/2$	Yscl = 1

With the calculator set in SIMULTANEOUS mode, we can actually watch the sine function (in red) "unwind" from the unit circle (in blue). In the two screens at left, we partially illustrate this animated procedure.

Consult your calculator's instruction manual for specific keystrokes and graph both the sine curve and the cosine curve in this manner. (For more on parametric equations, see Appendix B.)

A function with a repeating pattern is called **periodic.** The sine and cosine functions are examples of periodic functions. The values of these functions repeat themselves every 2π units. In other words, for any s, we have

$$\sin(s + 2\pi) = \sin s \quad \text{and} \quad \cos(s + 2\pi) = \cos s.$$

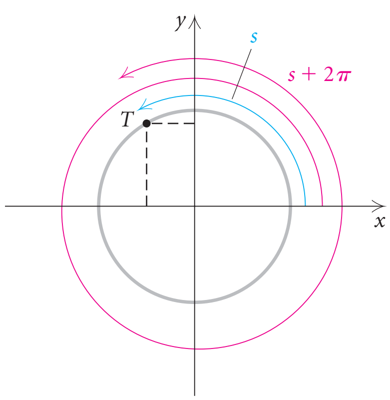
To see this another way, think of the part of the graph between 0 and 2π and note that the rest of the graph consists of copies of it. If we translate the graph of $y = \sin x$ or $y = \cos x$ to the left or right 2π units, we will obtain the original graph. We say that each of these functions has a period of 2π.

Periodic Function

A function f is said to be **periodic** if there exists a positive constant p such that

$$f(s + p) = f(s)$$

for all s in the domain of f. The smallest such positive number p is called the period of the function.



The period p can be thought of as the length of the shortest recurring interval.

We can also use the unit circle to verify that the period of the sine and cosine functions is 2π. Consider any real number s and the point T that it determines on a unit circle, as shown at left. If we increase s by 2π, the point determined by $s + 2\pi$ is again the point T. Hence for any real number s,

$$\sin(s + 2\pi) = \sin s \quad \text{and} \quad \cos(s + 2\pi) = \cos s.$$

It is also true that $\sin(s + 4\pi) = \sin s$, $\sin(s + 6\pi) = \sin s$, and so on. In fact, for *any* integer k, the following equations are identities:

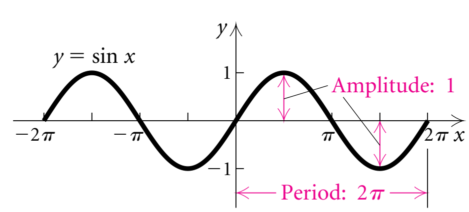
$$\sin[s + k(2\pi)] = \sin s \quad \text{and} \quad \cos[s + k(2\pi)] = \cos s,$$

or

$$\sin s = \sin(s + 2k\pi) \quad \text{and} \quad \cos s = \cos(s + 2k\pi).$$

The **amplitude** of a periodic function is defined as one half of the distance between its maximum and minimum function values. It is always positive. Both the graphs and the unit circle verify that the maximum value of the sine and cosine functions is 1, whereas the minimum value of each is -1. Thus,

$$\text{the amplitude of the sine function} = \tfrac{1}{2}\left|1 - (-1)\right| = 1$$



Technology Connection

Exploration

Using the TABLE feature on a graphing calculator, compare the y-values for $y_1 = \sin x$ and $y_2 = \sin(-x)$ and for $y_3 = \cos x$ and $y_4 = \cos(-x)$. We set TblMin $= 0$ and $\triangle$Tbl $= \pi/12$.

X	Y1	Y2
0	0	0
.2618	.25882	−.2588
.5236	.5	−.5
.7854	.70711	−.7071
1.0472	.86603	−.866
1.309	.96593	−.9659
1.5708	1	−1

X = 0

X	Y3	Y4
0	1	1
.2618	.96593	.96593
.5236	.86603	.86603
.7854	.70711	.70711
1.0472	.5	.5
1.309	.25882	.25882
1.5708	0	0

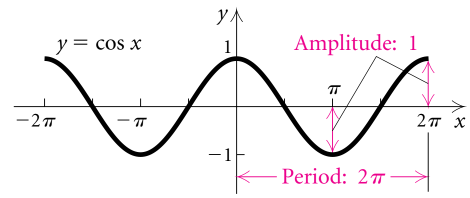
X = 0

What appears to be the relationship between $\sin x$ and $\sin(-x)$ and between $\cos x$ and $\cos(-x)$?

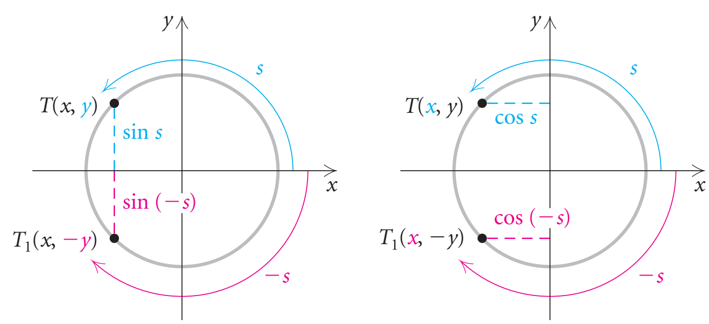
EVEN AND ODD FUNCTIONS
REVIEW SECTION 1.7.

and

the amplitude of the cosine function is $\frac{1}{2}\left|1 - (-1)\right| = 1.$



Consider any real number s and its opposite, $-s$. These numbers determine points T and T_1 on a unit circle that are symmetric with respect to the x-axis.



Because their second coordinates are opposites of each other, we know that for any number s,

$$\sin(-s) = -\sin s.$$

Because their first coordinates are the same, we know that for any number s,

$$\cos(-s) = \cos s.$$

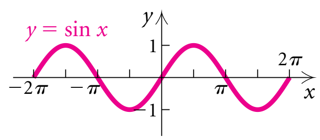
Thus we have shown that the sine function is *odd* and the cosine function is *even*.

The following is a summary of the properties of the sine and cosine functions.

CONNECTING THE CONCEPTS

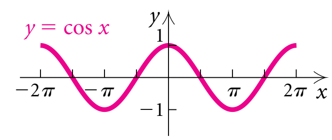
COMPARING THE SINE AND COSINE FUNCTIONS

SINE FUNCTION



1. Continuous
2. Period: 2π
3. Domain: All real numbers
4. Range: $[-1, 1]$
5. Amplitude: 1
6. Odd: $\sin(-s) = -\sin s$

COSINE FUNCTION

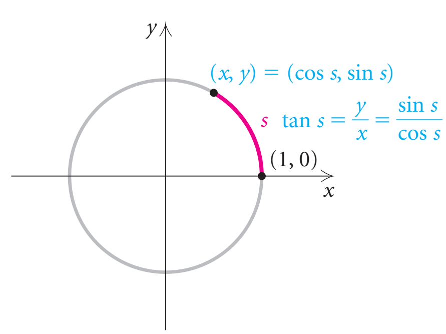


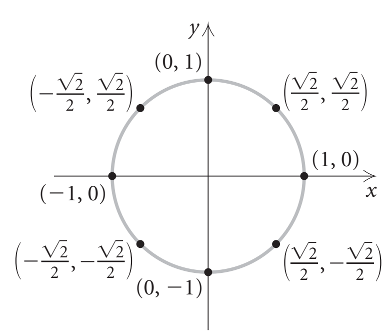
1. Continuous
2. Period: 2π
3. Domain: All real numbers
4. Range: $[-1, 1]$
5. Amplitude: 1
6. Even: $\cos(-s) = \cos s$

Graphs of the Tangent, Cotangent, Cosecant, and Secant Functions

To graph the tangent function, we could make a table of values using a calculator, but in this case it is easier to begin with the definition of tangent and the coordinates of a few points on the unit circle. We recall that

$$\tan s = \frac{y}{x} = \frac{\sin s}{\cos s}.$$





The tangent function is not defined when x, the first coordinate, is 0. That is, it is not defined for any number s whose cosine is 0:

$$s = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots.$$

We draw vertical asymptotes at these locations (see Fig. 1 below).

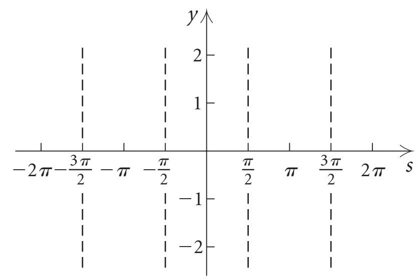


FIGURE 1

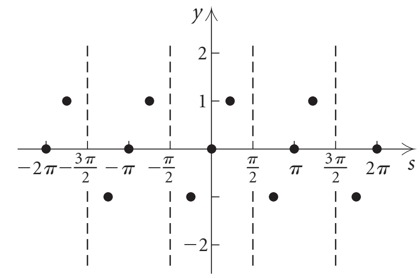


FIGURE 2

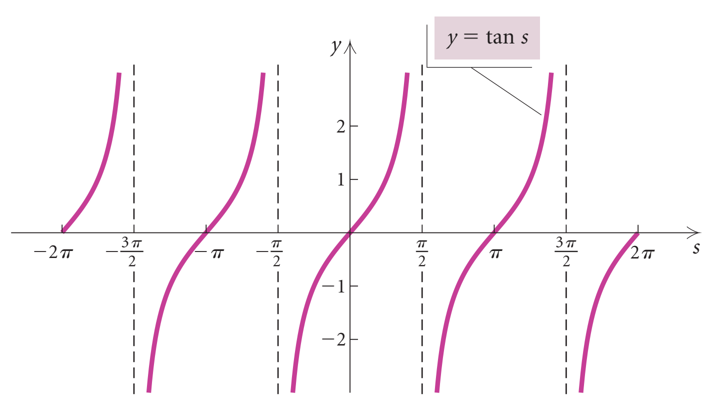
We also note that

$$\tan s = 0 \text{ at } s = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots,$$

$$\tan s = 1 \text{ at } s = \ldots -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots,$$

$$\tan s = -1 \text{ at } s = \ldots -\frac{9\pi}{4}, -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots.$$

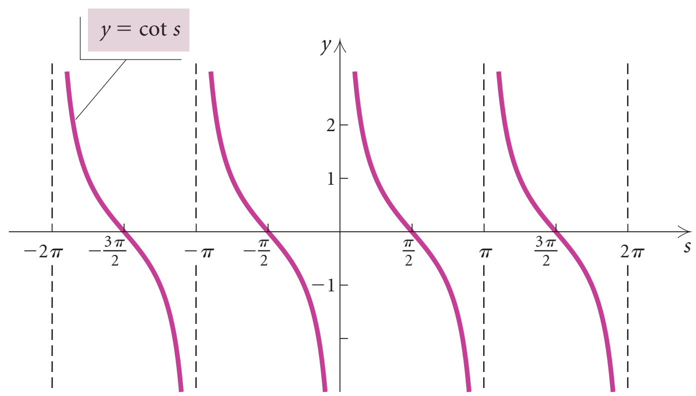
We can add these ordered pairs to the graph (see Fig. 2 above) and investigate the values in $(-\pi/2, \pi/2)$ using a calculator. Note that the function value is 0 when $s = 0$, and the values increase without bound as s increases toward $\pi/2$. The graph gets closer and closer to an asymptote as s gets closer to $\pi/2$, but it never touches the line. As s decreases from 0 to $-\pi/2$, the values decrease without bound. Again the graph gets closer and closer to an asymptote, but it never touches it. We now complete the graph.



The tangent function

From the graph, we see that the tangent function is continuous except where it is not defined. The period of the tangent function is π. Note that although there is a period, there is no amplitude because there are no maximum and minimum values. When $\cos s = 0$, $\tan s$ is not defined ($\tan s = \sin s/\cos s$). Thus the domain of the tangent function is the set of all real numbers except $(\pi/2) + k\pi$, where k is an integer. The range of the function is the set of all real numbers.

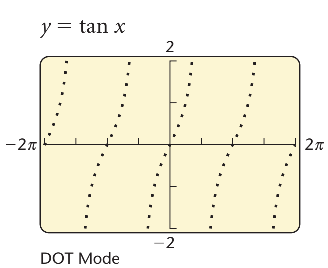
The cotangent function ($\cot s = \cos s/\sin s$) is not defined when y, the second coordinate, is 0—that is, it is not defined for any number s whose sine is 0. Thus the cotangent is not defined for $s = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. The graph of the function is shown below.

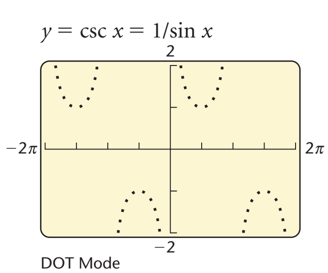


The cotangent function

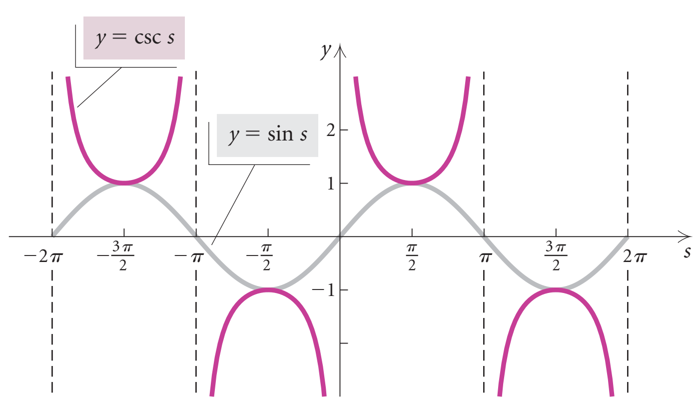
Technology Connection

When graphing trigonometric functions that are not defined for all real numbers, it is best to use DOT mode for the graph. Here we illustrate $y = \tan x$ and $y = \csc x$.

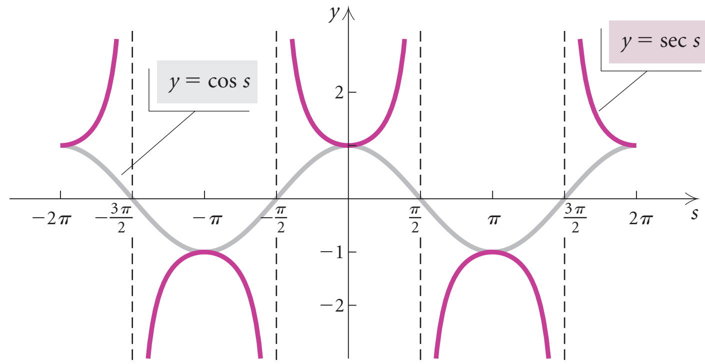




The cosecant and sine functions are reciprocal functions, as are the secant and cosine functions. The graphs of the cosecant and secant functions can be constructed by finding the reciprocals of the values of the sine and cosine functions, respectively. Thus the functions will be positive together and negative together. The cosecant function is not defined for those numbers s whose sine is 0. The secant function is not defined for those numbers s whose cosine is 0. In the graphs below, the sine and cosine functions are shown by the gray curves for reference.



The cosecant function



The secant function

The following is a summary of the basic properties of the tangent, cotangent, cosecant, and secant functions. These functions are continuous except where they are not defined.

CONNECTING THE CONCEPTS

COMPARING THE TANGENT, COTANGENT, COSECANT, AND SECANT FUNCTIONS

TANGENT FUNCTION

1. Period: π
2. Domain: All real numbers except $(\pi/2) + k\pi$, where k is an integer
3. Range: All real numbers

COTANGENT FUNCTION

1. Period: π
2. Domain: All real numbers except $k\pi$, where k is an integer
3. Range: All real numbers

COSECANT FUNCTION

1. Period: 2π
2. Domain: All real numbers except $k\pi$, where k is an integer
3. Range: $(-\infty, -1] \cup [1, \infty)$

SECANT FUNCTION

1. Period: 2π
2. Domain: All real numbers except $(\pi/2) + k\pi$, where k is an integer
3. Range: $(-\infty, -1] \cup [1, \infty)$

In this chapter, we have used the letter s for arc length and have avoided the letters x and y, which generally represent first and second coordinates. Nevertheless, we can represent the arc length on a unit circle by any variable, such as s, t, x, or θ. Each arc length determines a point that can be labeled with an ordered pair. The first coordinate of that ordered pair is the cosine of the arc length, and the second coordinate is the sine of the arc length. The identities we have developed hold no matter what symbols are used for variables—for example, $\cos(-s) = \cos s$, $\cos(-x) = \cos x$, $\cos(-\theta) = \cos\theta$, and $\cos(-t) = \cos t$.

5.5 Exercise Set

The following points are on the unit circle. Find the coordinates of their reflections across **(a)** *the x-axis,* **(b)** *the y-axis, and* **(c)** *the origin.*

1. $\left(-\frac{3}{4}, \frac{\sqrt{7}}{4}\right)$

2. $\left(\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$

3. $\left(\frac{2}{5}, -\frac{\sqrt{21}}{5}\right)$

4. $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

5. The number $\pi/4$ determines a point on the unit circle with coordinates $\left(\sqrt{2}/2, \sqrt{2}/2\right)$. What are the coordinates of the point determined by $-\pi/4$?

6. A number β determines a point on the unit circle with coordinates $\left(-2/3, \sqrt{5}/3\right)$. What are the coordinates of the point determined by $-\beta$?

Find the function value using coordinates of points on the unit circle. Give exact answers.

7. $\sin \pi$

8. $\cos\left(-\frac{\pi}{3}\right)$

9. $\cot \frac{7\pi}{6}$

10. $\tan \frac{11\pi}{4}$

11. $\sin(-3\pi)$

12. $\csc \frac{3\pi}{4}$

13. $\cos \frac{5\pi}{6}$

14. $\tan\left(-\frac{\pi}{4}\right)$

15. $\sec \frac{\pi}{2}$

16. $\cos 10\pi$

17. $\cos \frac{\pi}{6}$

18. $\sin \frac{2\pi}{3}$

19. $\sin \frac{5\pi}{4}$

20. $\cos \frac{11\pi}{6}$

21. $\sin(-5\pi)$

22. $\tan \frac{3\pi}{2}$

23. $\cot \frac{5\pi}{2}$

24. $\tan \frac{5\pi}{3}$

Find the function value using a calculator set in RADIAN *mode. Round the answer to four decimal places, where appropriate.*

25. $\tan \frac{\pi}{7}$

26. $\cos\left(-\frac{2\pi}{5}\right)$

27. $\sec 37$

28. $\sin 11.7$

29. $\cot 342$

30. $\tan 1.3$

31. $\cos 6\pi$

32. $\sin \frac{\pi}{10}$

33. $\csc 4.16$

34. $\sec \frac{10\pi}{7}$

35. $\tan \frac{7\pi}{4}$

36. $\cos 2000$

37. $\sin\left(-\frac{\pi}{4}\right)$

38. $\cot 7\pi$

39. $\sin 0$

40. $\cos(-29)$

41. $\tan \frac{2\pi}{9}$

42. $\sin \frac{8\pi}{3}$

43. a) Sketch a graph of $y = \sin x$.
b) By reflecting the graph in part (a), sketch a graph of $y = \sin(-x)$.
c) By reflecting the graph in part (a), sketch a graph of $y = -\sin x$.
d) How do the graphs in parts (b) and (c) compare?

44. a) Sketch a graph of $y = \cos x$.
b) By reflecting the graph in part (a), sketch a graph of $y = \cos(-x)$.
c) By reflecting the graph in part (a), sketch a graph of $y = -\cos x$.
d) How do the graphs in parts (a) and (b) compare?

45. a) Sketch a graph of $y = \sin x$.
b) By translating, sketch a graph of $y = \sin(x + \pi)$.
c) By reflecting the graph of part (a), sketch a graph of $y = -\sin x$.
d) How do the graphs of parts (b) and (c) compare?

46. **a)** Sketch a graph of $y = \sin x$.
b) By translating, sketch a graph of $y = \sin(x - \pi)$.
c) By reflecting the graph of part (a), sketch a graph of $y = -\sin x$.
d) How do the graphs of parts (b) and (c) compare?

47. **a)** Sketch a graph of $y = \cos x$.
b) By translating, sketch a graph of $y = \cos(x + \pi)$.
c) By reflecting the graph of part (a), sketch a graph of $y = -\cos x$.
d) How do the graphs of parts (b) and (c) compare?

48. **a)** Sketch a graph of $y = \cos x$.
b) By translating, sketch a graph of $y = \cos(x - \pi)$.
c) By reflecting the graph of part (a), sketch a graph of $y = -\cos x$.
d) How do the graphs of parts (b) and (c) compare?

49. Of the six circular functions, which are even? Which are odd?

50. Of the six circular functions, which have period π? Which have period 2π?

Consider the coordinates on the unit circle for Exercises 51–54.

51. In which quadrants is the tangent function positive? negative?

52. In which quadrants is the sine function positive? negative?

53. In which quadrants is the cosine function positive? negative?

54. In which quadrants is the cosecant function positive? negative?

Technology Connection

Use a graphing calculator to determine the domain, the range, the period, and the amplitude of the function.

55. $y = (\sin x)^2$

56. $y = |\cos x| + 1$

57. Using a calculator, consider $(\sin x)/x$, where x is between 0 and $\pi/2$. As x approaches 0, this function approaches a limiting value. What is it?

58. Using graphs, determine all numbers x that satisfy

$$\sin x < \cos x.$$

Collaborative Discussion and Writing

59. Describe how the graphs of the sine and cosine functions are related.

60. Explain why both the sine and cosine functions are continuous, but the tangent function, defined as sine/cosine, is not continuous.

Skill Maintenance

Graph both functions on the same set of axes, and describe how g is a transformation of f.

61. $f(x) = x^2,\ g(x) = 2x^2 - 3$

62. $f(x) = x^2,\ g(x) = (x - 2)^2$

63. $f(x) = |x|,\ g(x) = \frac{1}{2}|x - 4| + 1$

64. $f(x) = x^3,\ g(x) = -x^3$

Write an equation for a function that has a graph with the given characteristics.

65. The shape of $y = x^3$, but reflected across the x-axis, shifted right 2 units, and shifted down 1 unit

66. The shape of $y = 1/x$, but shrunk vertically by a factor of $\frac{1}{4}$ and shifted up 3 units

Synthesis

Complete. (For example, $\sin(x + 2\pi) = \sin x$.*)*

67. $\cos(-x) =$ ________

68. $\sin(-x) =$ ________

69. $\sin(x + 2k\pi),\ k \in \mathbb{Z} =$ ________

70. $\cos(x + 2k\pi),\ k \in \mathbb{Z} =$ ________

71. $\sin(\pi - x) =$ ________

72. $\cos(\pi - x) =$ ________

73. $\cos(x - \pi) =$ ________

74. $\cos(x + \pi) =$ ________

75. $\sin(x + \pi) =$ ________

76. $\sin(x - \pi) =$ ________

77. Find all numbers x that satisfy the following.
a) $\sin x = 1$
b) $\cos x = -1$
c) $\sin x = 0$

78. Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 2x$ and $g(x) = \cos x$.

Determine the domain of the function.

79. $f(x) = \sqrt{\cos x}$

80. $g(x) = \dfrac{1}{\sin x}$

81. $f(x) = \dfrac{\sin x}{\cos x}$

82. $g(x) = \log(\sin x)$

Graph.

83. $y = 3 \sin x$

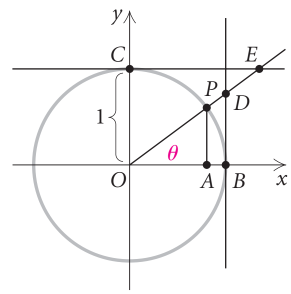
84. $y = \sin |x|$

85. $y = \sin x + \cos x$

86. $y = |\cos x|$

87. One of the motivations for developing trigonometry with a unit circle is that you can actually "see" $\sin \theta$ and $\cos \theta$ on the circle. Note in the figure at right that $AP = \sin \theta$ and $OA = \cos \theta$. It turns out that you can also "see" the other four trigonometric functions. Prove each of the following.

a) $BD = \tan \theta$ **b)** $OD = \sec \theta$
c) $OE = \csc \theta$ **d)** $CE = \cot \theta$



5.6 Graphs of Transformed Sine and Cosine Functions

- *Graph transformations of $y = \sin x$ and $y = \cos x$ in the form*

$$y = A \sin(Bx - C) + D$$

and

$$y = A \cos(Bx - C) + D$$

and determine the amplitude, the period, and the phase shift.
- *Graph sums of functions.*

Variations of Basic Graphs

In Section 5.5, we graphed all six trigonometric functions. In this section, we will consider variations of the graphs of the sine and cosine functions. For example, we will graph equations like the following:

$$y = 5 \sin \tfrac{1}{2}x, \qquad y = \cos(2x - \pi), \quad \text{and} \quad y = \tfrac{1}{2} \sin x - 3.$$

In particular, we are interested in graphs of functions in the form

$$y = A \sin(Bx - C) + D$$

and

$$y = A \cos(Bx - C) + D,$$

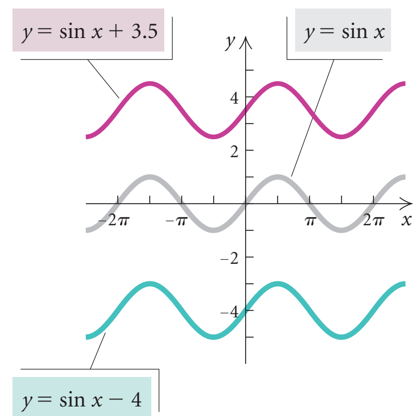
where A, B, C, and D are constants. These constants have the effect of translating, reflecting, stretching, and shrinking the basic graphs. Let's first examine the effect of each constant individually. Then we will consider the combined effects of more than one constant.

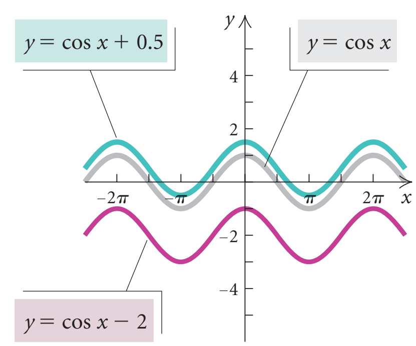
TRANSFORMATIONS OF FUNCTIONS

REVIEW SECTION 1.5.

The Constant *D*

Let's observe the effect of the constant D in the graphs below.





The constant D in

$$y = A \sin (Bx - C) + D \quad \text{and} \quad y = A \cos (Bx - C) + D$$

translates the graphs up D units if $D > 0$ or down $|D|$ units if $D < 0$.

EXAMPLE 1 Sketch a graph of $y = \sin x + 3$.

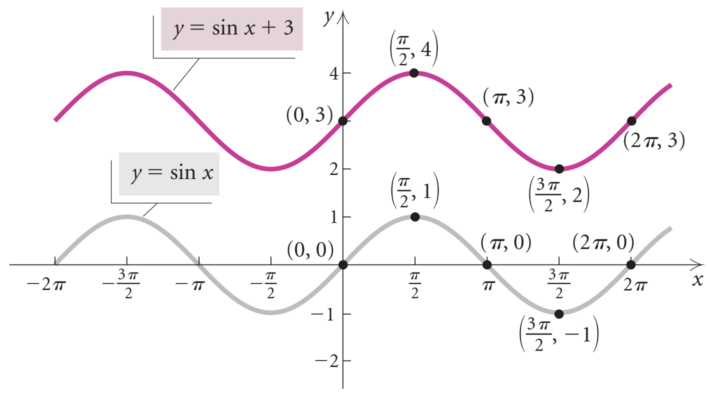
Solution The graph of $y = \sin x + 3$ is a *vertical* translation of the graph of $y = \sin x$ up 3 units. One way to sketch the graph is to first consider $y = \sin x$ on an interval of length 2π, say, $[0, 2\pi]$. The zeros of the function and the maximum and minimum values can be considered key points. These are

$$(0, 0), \quad \left(\frac{\pi}{2}, 1\right), \quad (\pi, 0), \quad \left(\frac{3\pi}{2}, -1\right), \quad (2\pi, 0).$$

These key points are transformed up 3 units to obtain the key points of the graph of $y = \sin x + 3$. These are

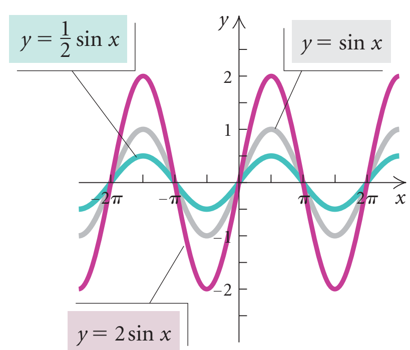
$$(0, 3), \quad \left(\frac{\pi}{2}, 4\right), \quad (\pi, 3), \quad \left(\frac{3\pi}{2}, 2\right), \quad (2\pi, 3).$$

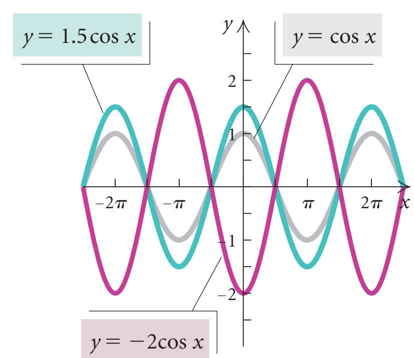
The graph of $y = \sin x + 3$ can be sketched on the interval $[0, 2\pi]$ and extended to obtain the rest of the graph by repeating the graph on intervals of length 2π.



The Constant A

Next, we consider the effect of the constant A. What can we observe in the following graphs? What is the effect of the constant A on the graph of the basic function when **(a)** $0 < A < 1$? **(b)** $A > 1$? **(c)** $-1 < A < 0$? **(d)** $A < -1$?





If $|A| > 1$, then there will be a vertical stretching. If $|A| < 1$, then there will be a vertical shrinking. If $A < 0$, the graph is also reflected across the x-axis.

Amplitude

The **amplitude** of the graphs of $y = A \sin(Bx - C) + D$ and $y = A \cos(Bx - C) + D$ is $|A|$.

EXAMPLE 2 Sketch a graph of $y = 2\cos x$. What is the amplitude?

Solution The constant 2 in $y = 2\cos x$ has the effect of stretching the graph of $y = \cos x$ vertically by a factor of 2 units. Since the function values of $y = \cos x$ are such that $-1 \le \cos x \le 1$, the function values of $y = 2\cos x$ are such that $-2 \le 2\cos x \le 2$. The maximum value of $y = 2\cos x$ is 2, and the minimum value is -2. Thus the *amplitude*, A, is $\frac{1}{2}|2 - (-2)|$, or 2.

We draw the graph of $y = \cos x$ and consider its key points,

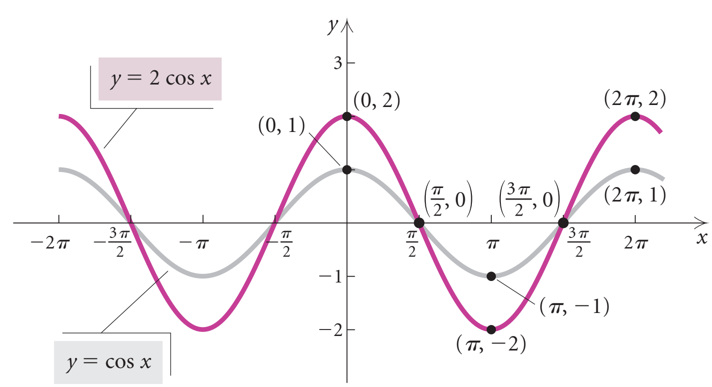
$$(0, 1),\quad \left(\frac{\pi}{2}, 0\right),\quad (\pi, -1),\quad \left(\frac{3\pi}{2}, 0\right),\quad (2\pi, 1),$$

on the interval $[0, 2\pi]$.

We then multiply the second coordinates by 2 to obtain the key points of $y = 2\cos x$. These are

$$(0, 2),\quad \left(\frac{\pi}{2}, 0\right),\quad (\pi, -2),\quad \left(\frac{3\pi}{2}, 0\right),\quad (2\pi, 2).$$

We plot these points and sketch the graph on the interval $[0, 2\pi]$. Then we repeat this part of the graph on adjacent intervals of length 2π.



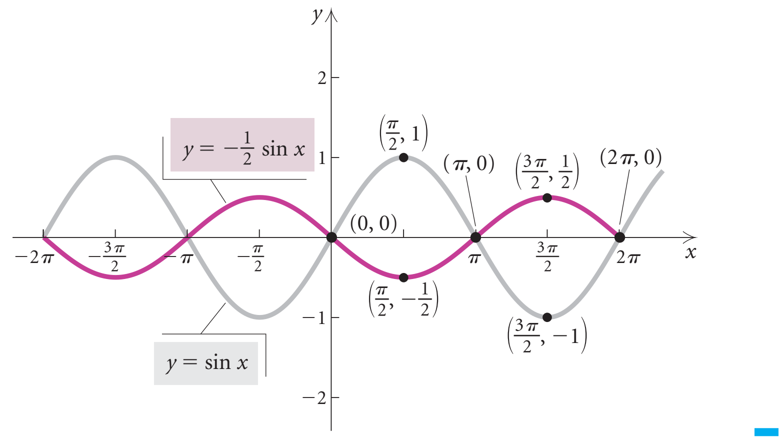
EXAMPLE 3 Sketch a graph of $y = -\frac{1}{2}\sin x$. What is the amplitude?

Solution The amplitude of the graph is $\left|-\frac{1}{2}\right|$, or $\frac{1}{2}$. The graph of $y = -\frac{1}{2}\sin x$ is a vertical shrinking and a reflection of the graph of $y = \sin x$ across the x-axis. In graphing, the key points of $y = \sin x$,

$$(0, 0),\quad \left(\frac{\pi}{2}, 1\right),\quad (\pi, 0),\quad \left(\frac{3\pi}{2}, -1\right),\quad (2\pi, 0),$$

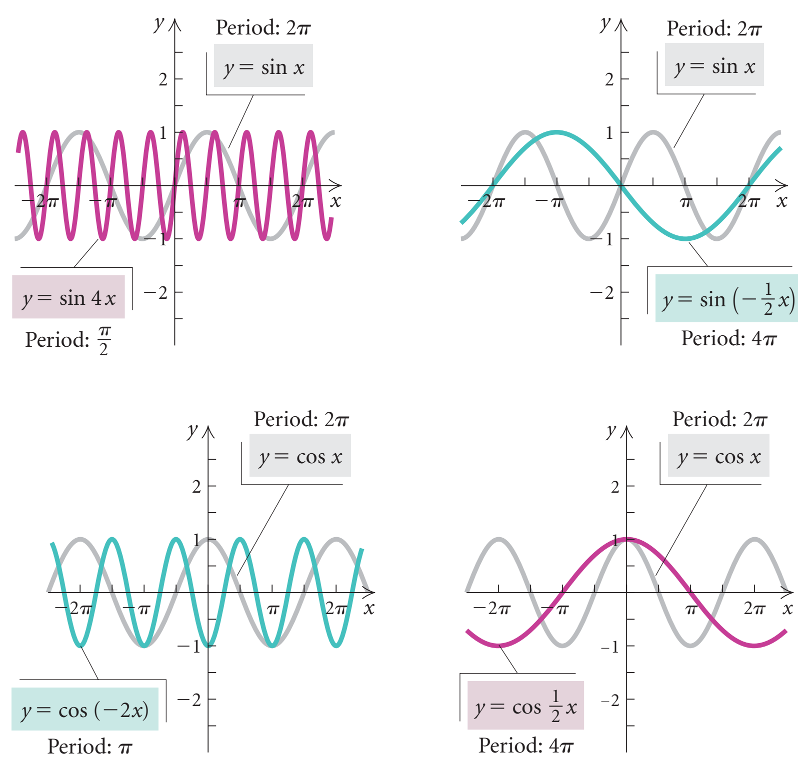
are transformed to

$$(0, 0),\quad \left(\frac{\pi}{2}, -\frac{1}{2}\right),\quad (\pi, 0),\quad \left(\frac{3\pi}{2}, \frac{1}{2}\right),\quad (2\pi, 0).$$



The Constant *B*

Now, we consider the effect of the constant B. Changes in the constants A and D *do not* change the period. But what effect, if any, does a change in B have on the period of the function? Let's observe the period of each of the following graphs.



If $|B| < 1$, then there will be a horizontal stretching. If $|B| > 1$, then there will be a horizontal shrinking. If $B < 0$, the graph is also reflected across the y-axis.

Period

The **period** of the graphs of $y = A \sin(Bx - C) + D$ and $y = A \cos(Bx - C) + D$ is $\left|\dfrac{2\pi}{B}\right|$.*

EXAMPLE 4 Sketch a graph of $y = \sin 4x$. What is the period?

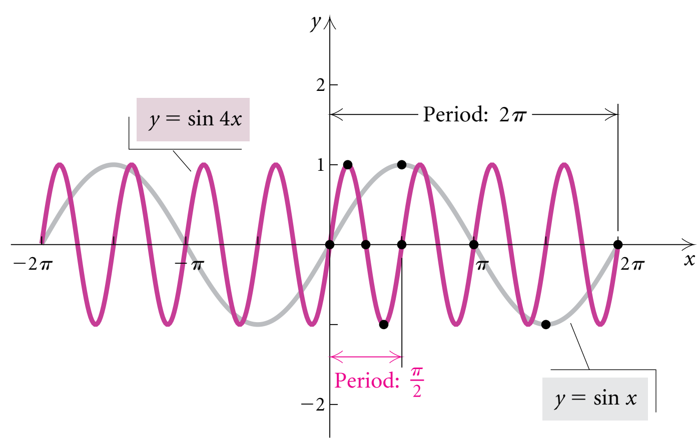
Solution The constant B has the effect of changing the period. The graph of $y = f(4x)$ is obtained from the graph of $y = f(x)$ by shrinking the graph horizontally. The period of $y = \sin 4x$ is $|2\pi/4|$, or $\pi/2$. The new graph is obtained by dividing the first coordinate of each ordered-pair solution of $y = f(x)$ by 4. The key points of $y = \sin x$ are

$$(0, 0),\quad \left(\frac{\pi}{2}, 1\right),\quad (\pi, 0),\quad \left(\frac{3\pi}{2}, -1\right),\quad (2\pi, 0).$$

These are transformed to the key points of $y = \sin 4x$, which are

$$(0, 0),\quad \left(\frac{\pi}{8}, 1\right),\quad \left(\frac{\pi}{4}, 0\right),\quad \left(\frac{3\pi}{8}, -1\right),\quad \left(\frac{\pi}{2}, 0\right).$$

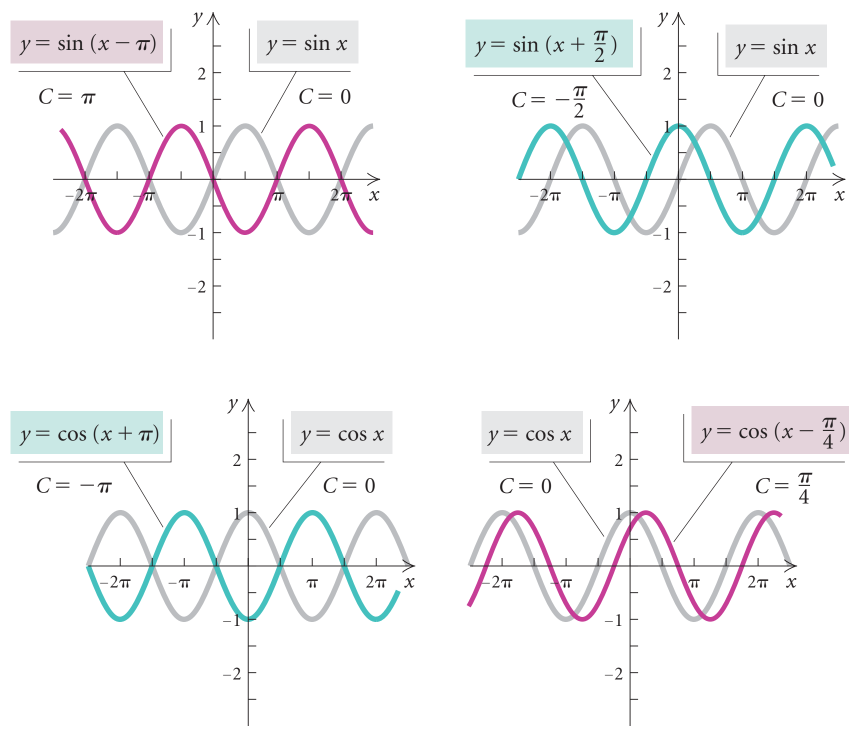
We plot these key points and sketch in the graph on the shortened interval $[0, \pi/2]$. Then we repeat the graph on other intervals of length $\pi/2$.



*The period of the graphs of $y = A \tan(Bx - C) + D$, $y = A \cot(Bx - C) + D$, $y = A \sec(Bx - C) + D$, and $y = A \csc(Bx - C) + D$ is $|\pi/B|$.

The Constant C

Next, we examine the effect of the constant C. The curve in each of the following graphs has an amplitude of 1 and a period of 2π, but there are six distinct graphs. What is the effect of the constant C?



For each of the functions of the form

$$y = A\sin(Bx - C) + D \quad \text{and} \quad y = A\cos(Bx - C) + D$$

that are graphed above, the coefficient of x, which is B, is 1. In this case, the effect of the constant C on the graph of the basic function is a horizontal translation of $|C|$ units. In Example 5, which follows, $B = 1$. We will consider functions where $B \neq 1$ in Examples 6 and 7. When $B \neq 1$, the horizontal translation will be $|C/B|$.

EXAMPLE 5 Sketch a graph of $y = \sin\left(x - \dfrac{\pi}{2}\right)$.

Solution The amplitude is 1, and the period is 2π. The graph of $y = f(x - c)$ is obtained from the graph of $y = f(x)$ by translating the graph horizontally—to the right c units if $c > 0$ and to the left $|c|$ units if $c < 0$. The graph of $y = \sin(x - \pi/2)$ is a translation of the graph of

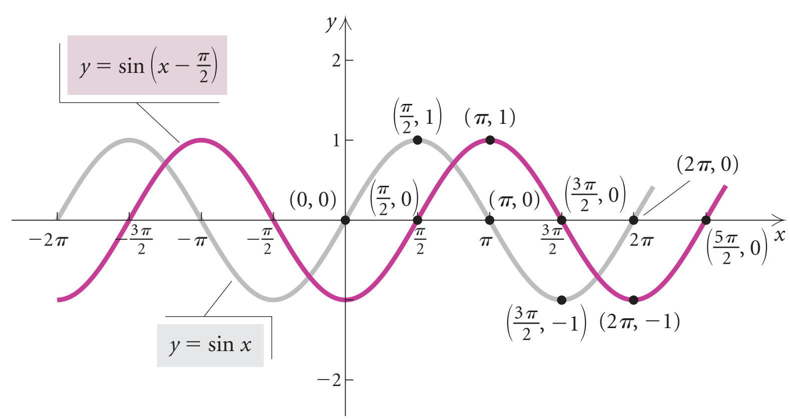
$y = \sin x$ to the right $\pi/2$ units. The value $\pi/2$ is called the **phase shift.** The key points of $y = \sin x$,

$$(0, 0), \quad \left(\frac{\pi}{2}, 1\right), \quad (\pi, 0), \quad \left(\frac{3\pi}{2}, -1\right), \quad (2\pi, 0),$$

are transformed by adding $\pi/2$ to each of the first coordinates to obtain the following key points of $y = \sin(x - \pi/2)$:

$$\left(\frac{\pi}{2}, 0\right), \quad (\pi, 1), \quad \left(\frac{3\pi}{2}, 0\right), \quad (2\pi, -1), \quad \left(\frac{5\pi}{2}, 0\right).$$

We plot these key points and sketch the curve on the interval $[\pi/2, 5\pi/2]$. Then we repeat the graph on other intervals of length 2π.



Combined Transformations

Now we consider combined transformations of graphs. It is helpful to rewrite

$$y = A\sin(Bx - C) + D \qquad \text{and} \qquad y = A\cos(Bx - C) + D$$

as

$$y = A\sin\left[B\left(x - \frac{C}{B}\right)\right] + D \quad \text{and} \quad y = A\cos\left[B\left(x - \frac{C}{B}\right)\right] + D.$$

EXAMPLE 6 Sketch a graph of $y = \cos(2x - \pi)$.

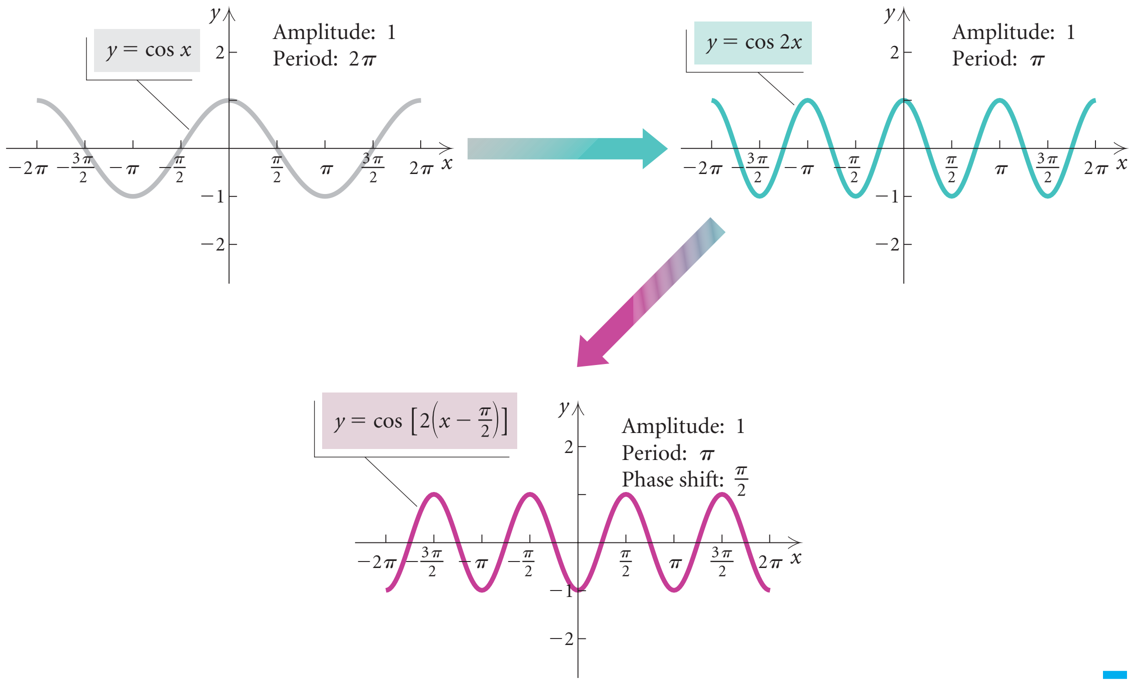
Solution The graph of

$$y = \cos(2x - \pi)$$

is the same as the graph of

$$y = 1 \cdot \cos\left[2\left(x - \frac{\pi}{2}\right)\right] + 0.$$

The amplitude is 1. The factor 2 shrinks the period by half, making the period $|2\pi/2|$, or π. The phase shift $\pi/2$ translates the graph of $y = \cos 2x$ to the right $\pi/2$ units. Thus, to form the graph, we first graph $y = \cos x$, followed by $y = \cos 2x$ and then $y = \cos[2(x - \pi/2)]$.



Phase Shift

The **phase shift** of the graphs

$$y = A\sin(Bx - C) + D = A\sin\left[B\left(x - \frac{C}{B}\right)\right] + D$$

and

$$y = A\cos(Bx - C) + D = A\cos\left[B\left(x - \frac{C}{B}\right)\right] + D$$

is the quantity $\dfrac{C}{B}$.

If $C/B > 0$, the graph is translated to the right C/B units. If $C/B < 0$, the graph is translated to the left $|C/B|$ units. Be sure that the horizontal stretching or shrinking based on the constant B is done before the translation based on the phase shift C/B.

Let's now summarize the effect of the constants. We carry out the procedures in the order listed.

Transformations of Sine and Cosine Functions

To graph

$$y = A\sin(Bx - C) + D = A\sin\left[B\left(x - \frac{C}{B}\right)\right] + D$$

and

$$y = A\cos(Bx - C) + D = A\cos\left[B\left(x - \frac{C}{B}\right)\right] + D,$$

follow the steps listed below in the order in which they are listed.

1. Stretch or shrink the graph horizontally according to B.

$\lvert B\rvert < 1$	Stretch horizontally
$\lvert B\rvert > 1$	Shrink horizontally
$B < 0$	Reflect across the y-axis

The *period* is $\left|\frac{2\pi}{B}\right|$.*

2. Stretch or shrink the graph vertically according to A.

$\lvert A\rvert < 1$	Shrink vertically
$\lvert A\rvert > 1$	Stretch vertically
$A < 0$	Reflect across the x-axis

The *amplitude* is $|A|$.

3. Translate the graph horizontally according to C/B.

$\frac{C}{B} < 0$	$\left\lvert\frac{C}{B}\right\rvert$ units to the left
$\frac{C}{B} > 0$	$\frac{C}{B}$ units to the right

The *phase shift* is $\frac{C}{B}$.

4. Translate the graph vertically according to D.

$D < 0$	$\lvert D\rvert$ units down
$D > 0$	D units up

EXAMPLE 7 Sketch a graph of $y = 3\sin(2x + \pi/2) + 1$. Find the amplitude, the period, and the phase shift.

Solution We first note that

$$y = 3\sin\left(2x + \frac{\pi}{2}\right) + 1 = 3\sin\left[2\left(x - \left(-\frac{\pi}{4}\right)\right)\right] + 1.$$

*When graphing transformations of the tangent, cotangent, secant, and cosecant functions, note that the period is $|\pi/B|$.

Then we have the following:

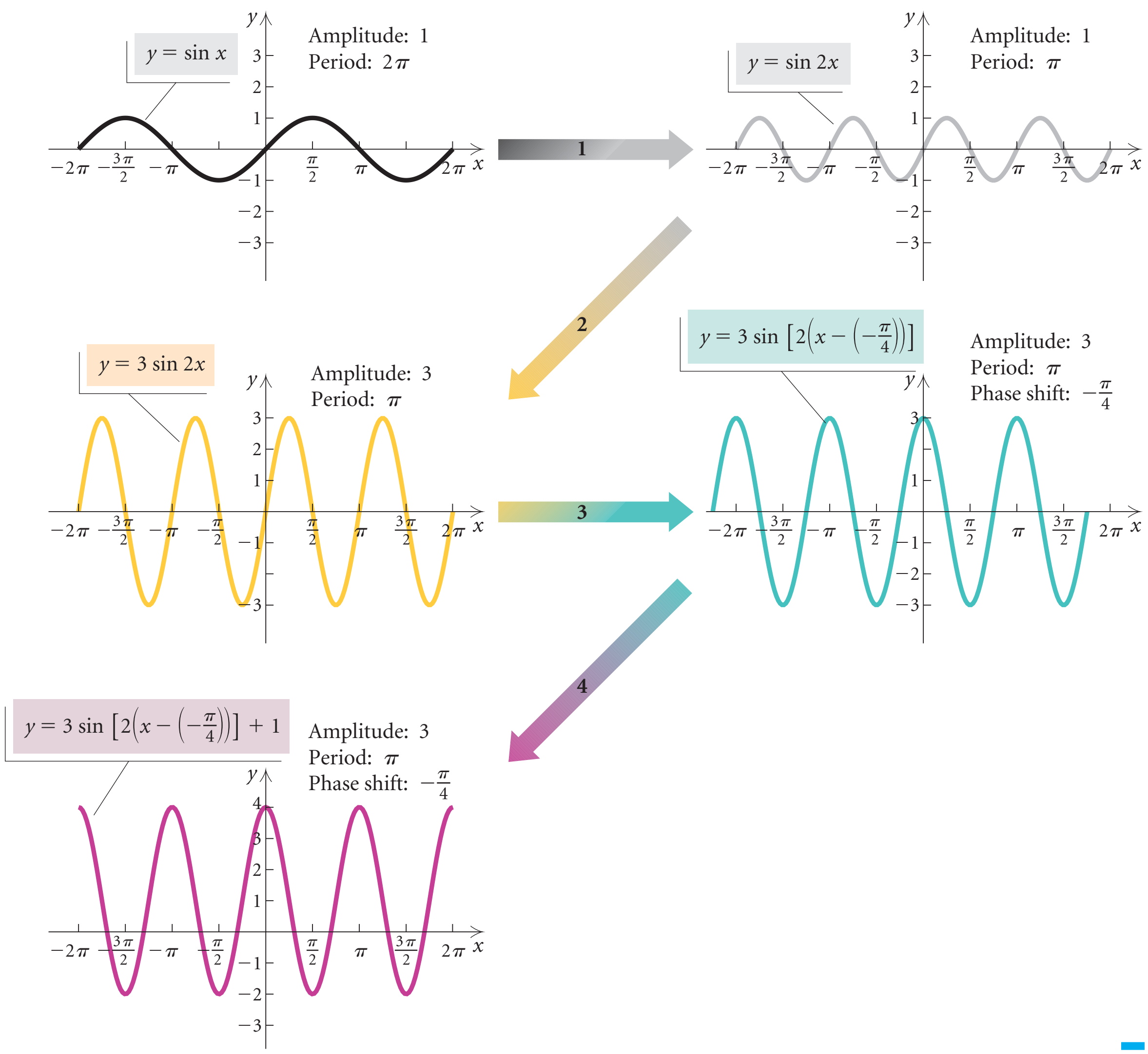
$$\text{Amplitude} = |A| = |3| = 3,$$

$$\text{Period} = \left|\frac{2\pi}{B}\right| = \left|\frac{2\pi}{2}\right| = \pi,$$

$$\text{Phase shift} = \frac{C}{B} = \frac{-\pi/2}{2} = -\frac{\pi}{4}.$$

To create the final graph, we begin with the basic sine curve, $y = \sin x$. Then we sketch graphs of each of the following equations in sequence.

1. $y = \sin 2x$
2. $y = 3 \sin 2x$
3. $y = 3 \sin\left[2\left(x - \left(-\frac{\pi}{4}\right)\right)\right]$
4. $y = 3 \sin\left[2\left(x - \left(-\frac{\pi}{4}\right)\right)\right] + 1$



All the graphs in Examples 1–7 can be checked using a graphing calculator. Even though it is faster and more accurate to graph using a calculator, graphing by hand gives us a greater understanding of the effect of changing the constants A, B, C, and D.

Graphing calculators are especially convenient when a period or a phase shift is not a multiple of $\pi/4$.

EXAMPLE 8 Graph $y = 3\cos 2\pi x - 1$. Find the amplitude, the period, and the phase shift.

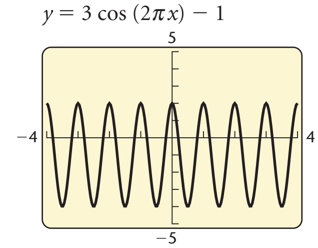
Solution First we note the following:

$$\text{Amplitude} = |A| = |3| = 3,$$

$$\text{Period} = \left|\frac{2\pi}{B}\right| = \left|\frac{2\pi}{2\pi}\right| = |1| = 1,$$

$$\text{Phase shift} = \frac{C}{B} = \frac{0}{2\pi} = 0.$$

There is no phase shift in this case because the constant $C = 0$. The graph has a vertical translation of the graph of the cosine function down 1 unit, an amplitude of 3, and a period of 1, so we can use $[-4, 4, -5, 5]$ as the viewing window.

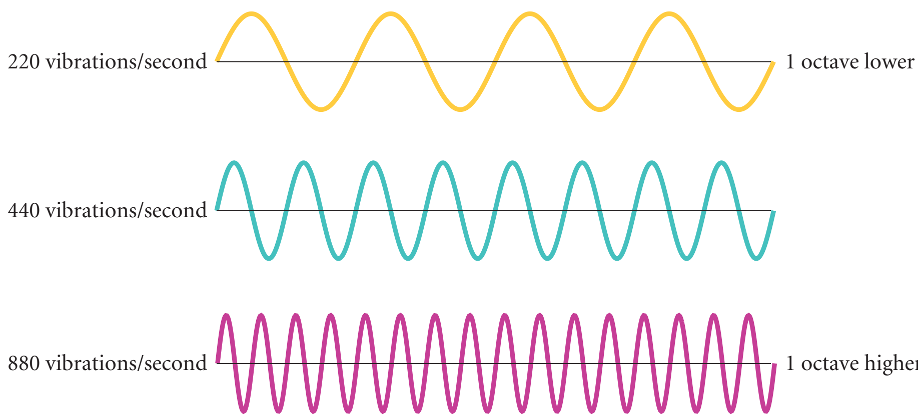


The transformation techniques that we learned in this section for graphing the sine and cosine functions can also be applied in the same manner to the other trigonometric functions. Transformations of this type appear in the synthesis exercises in Exercise Set 5.6.

An **oscilloscope** is an electronic device that converts electrical signals into graphs like those in the preceding examples. These graphs are often called sine waves. By manipulating the controls, we can change the amplitude, the period, and the phase of sine waves. The oscilloscope has many applications, and the trigonometric functions play a major role in many of them.

Graphs of Sums: Addition of Ordinates

The output of an electronic synthesizer used in the recording and playing of music can be converted into sine waves by an oscilloscope. The following graphs illustrate simple tones of different frequencies. The frequency of a simple tone is the number of vibrations in the signal of the tone per second. The loudness or intensity of the tone is reflected in the height of the graph (its amplitude). The three tones in the diagrams below all have the same intensity but different frequencies.

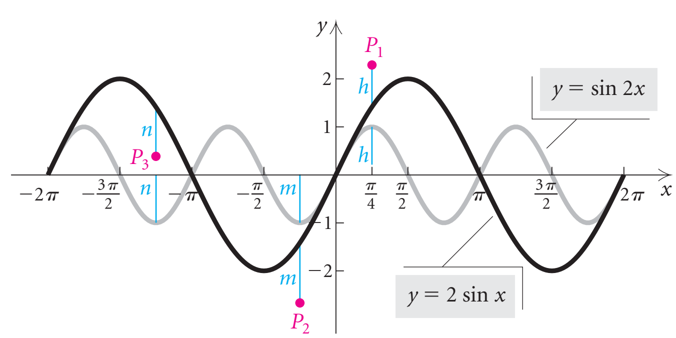


Musical instruments can generate extremely complex sine waves. On a single instrument, overtones can become superimposed on a simple tone. When multiple notes are played simultaneously, graphs become very complicated. This can happen when multiple notes are played on a single instrument or a group of instruments, or even when the same simple note is played on different instruments.

Combinations of simple tones produce interesting curves. Consider two tones whose graphs are $y_1 = 2 \sin x$ and $y_2 = \sin 2x$. The combination of the two tones produces a new sound whose graph is $y = 2 \sin x + \sin 2x$, as shown in the following example.

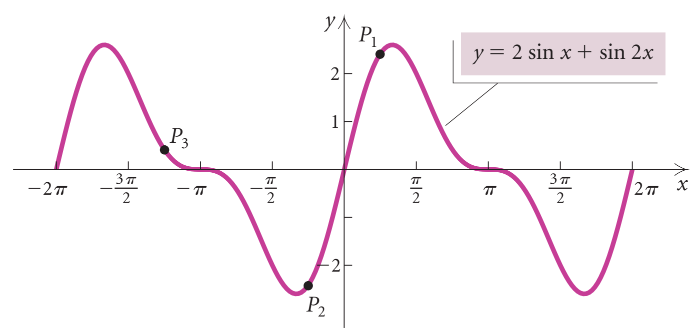
EXAMPLE 9 Graph: $y = 2 \sin x + \sin 2x$.

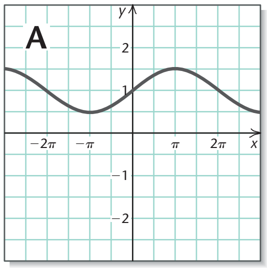
Solution We graph $y = 2 \sin x$ and $y = \sin 2x$ using the same set of axes.

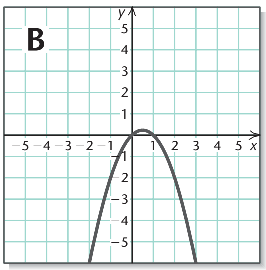


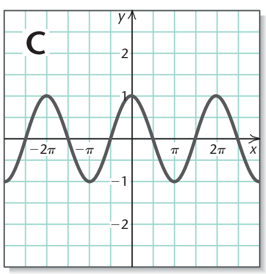
Now we graphically add some y-coordinates, or ordinates, to obtain points on the graph that we seek. At $x = \pi/4$, we transfer the distance h, which is the value of $\sin 2x$, up to add it to the value of $2 \sin x$. Point P_1 is on the graph that we seek. At $x = -\pi/4$, we use a similar procedure, but this time both ordinates are negative. Point P_2 is on the graph.

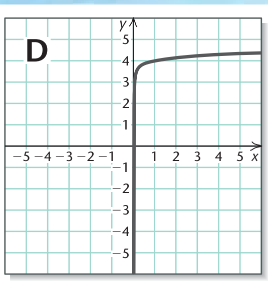
At $x = -5\pi/4$, we add the negative ordinate of $\sin 2x$ to the positive ordinate of $2 \sin x$. Point P_3 is also on the graph. We continue to plot points in this fashion and then connect them to get the desired graph, shown below. This method is called **addition of ordinates,** because we add the y-values (ordinates) of $y = \sin 2x$ to the y-values (ordinates) of $y = 2 \sin x$. Note that the period of $2 \sin x$ is 2π and the period of $\sin 2x$ is π. The period of the sum $2 \sin x + \sin 2x$ is 2π, the least common multiple of 2π and π.

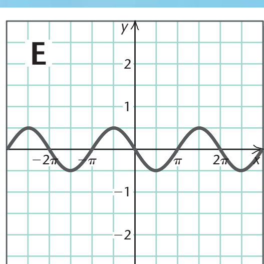












Visualizing the Graph

Match the function with its graph.

1. $f(x) = -\sin x$

2. $f(x) = 2x^3 - x + 1$

3. $y = \dfrac{1}{2}\cos\left(x + \dfrac{\pi}{2}\right)$

4. $f(x) = \cos\left(\dfrac{1}{2}x\right)$

5. $y = -x^2 + x$

6. $y = \dfrac{1}{2}\log x + 4$

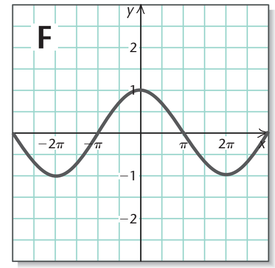
7. $f(x) = 2^{x-1}$

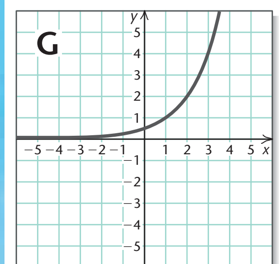
8. $f(x) = \dfrac{1}{2}\sin\left(\dfrac{1}{2}x\right) + 1$

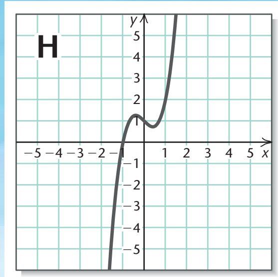
9. $f(x) = -\cos(x - \pi)$

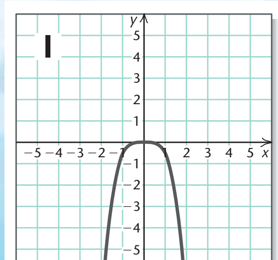
10. $f(x) = -\dfrac{1}{2}x^4$

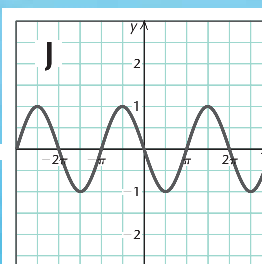
Answers on page A-39











5.6 Exercise Set

Determine the amplitude, the period, and the phase shift of the function and sketch the graph of the function.

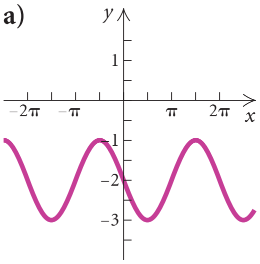
1. $y = \sin x + 1$
2. $y = \frac{1}{4} \cos x$
3. $y = -3 \cos x$
4. $y = \sin(-2x)$
5. $y = \frac{1}{2} \cos x$
6. $y = \sin\left(\frac{1}{2}x\right)$
7. $y = \sin(2x)$
8. $y = \cos x - 1$
9. $y = 2 \sin\left(\frac{1}{2}x\right)$
10. $y = \cos\left(x - \frac{\pi}{2}\right)$
11. $y = \frac{1}{2} \sin\left(x + \frac{\pi}{2}\right)$
12. $y = \cos x - \frac{1}{2}$
13. $y = 3 \cos(x - \pi)$
14. $y = -\sin\left(\frac{1}{4}x\right) + 1$
15. $y = \frac{1}{3} \sin x - 4$
16. $y = \cos\left(\frac{1}{2}x + \frac{\pi}{2}\right)$
17. $y = -\cos(-x) + 2$
18. $y = \frac{1}{2} \sin\left(2x - \frac{\pi}{4}\right)$

Determine the amplitude, the period, and the phase shift of the function.

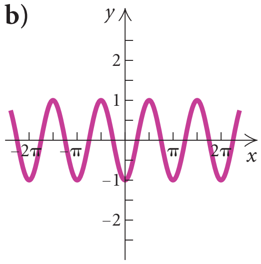
19. $y = 2 \cos\left(\frac{1}{2}x - \frac{\pi}{2}\right)$
20. $y = 4 \sin\left(\frac{1}{4}x + \frac{\pi}{8}\right)$
21. $y = -\frac{1}{2} \sin\left(2x + \frac{\pi}{2}\right)$
22. $y = -3 \cos(4x - \pi) + 2$
23. $y = 2 + 3 \cos(\pi x - 3)$
24. $y = 5 - 2 \cos\left(\frac{\pi}{2}x + \frac{\pi}{2}\right)$
25. $y = -\frac{1}{2} \cos(2\pi x) + 2$
26. $y = -2 \sin(-2x + \pi) - 2$
27. $y = -\sin\left(\frac{1}{2}x - \frac{\pi}{2}\right) + \frac{1}{2}$
28. $y = \frac{1}{3} \cos(-3x) + 1$
29. $y = \cos(-2\pi x) + 2$
30. $y = \frac{1}{2} \sin(2\pi x + \pi)$
31. $y = -\frac{1}{4} \cos(\pi x - 4)$
32. $y = 2 \sin(2\pi x + 1)$

In Exercises 33–40, match the function with one of the graphs (a)–(h), which follow.

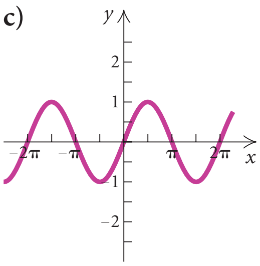
a)



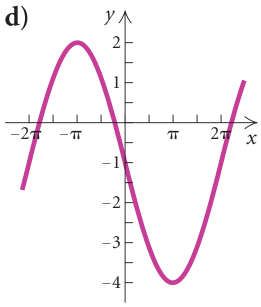
b)



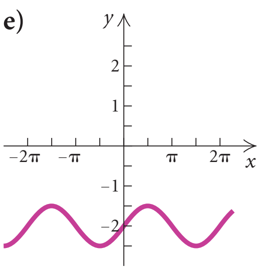
c)



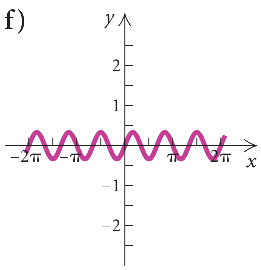
d)



e)



f)



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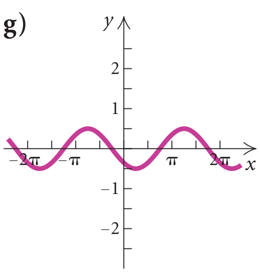

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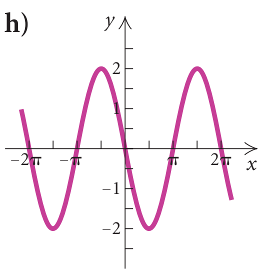


33. $y = -\cos 2x$

34. $y = \frac{1}{2} \sin x - 2$

35. $y = 2 \cos \left(x + \frac{\pi}{2}\right)$

36. $y = -3 \sin \frac{1}{2} x - 1$

37. $y = \sin (x - \pi) - 2$

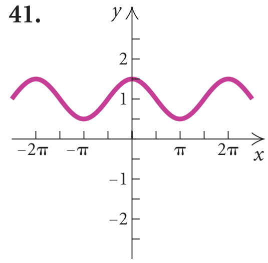
38. $y = -\frac{1}{2} \cos \left(x - \frac{\pi}{4}\right)$

39. $y = \frac{1}{3} \sin 3x$

40. $y = \cos \left(x - \frac{\pi}{2}\right)$

In Exercises 41–44, determine the equation of the function that is graphed.

41.



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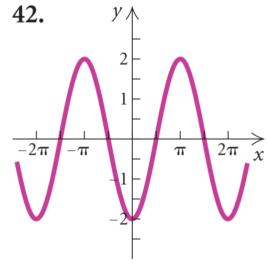

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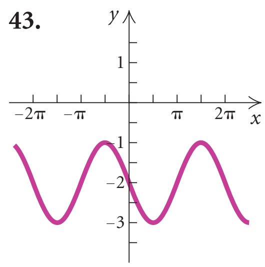

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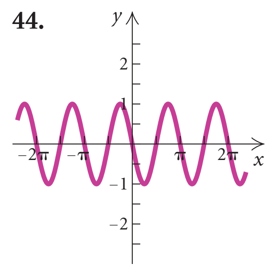


Graph using addition of ordinates.

45. $y = 2 \cos x + \cos 2x$

46. $y = 3 \cos x + \cos 3x$

47. $y = \sin x + \cos 2x$

48. $y = 2 \sin x + \cos 2x$

49. $y = \sin x - \cos x$

50. $y = 3 \cos x - \sin x$

51. $y = 3 \cos x + \sin 2x$

52. $y = 3 \sin x - \cos 2x$

Technology Connection

Use a graphing calculator to graph the function.

53. $y = x + \sin x$

54. $y = -x - \sin x$

55. $y = \cos x - x$

56. $y = -(\cos x - x)$

57. $y = \cos 2x + 2x$

58. $y = \cos 3x + \sin 3x$

59. $y = 4 \cos 2x - 2 \sin x$

60. $y = 7.5 \cos x + \sin 2x$

Use a graphing calculator to graph each of the following on the given interval and approximate the zeros.

61. $f(x) = \frac{\sin x}{x}$; $[-12, 12]$

62. $f(x) = \frac{\cos x - 1}{x}$; $[-12, 12]$

63. $f(x) = x^3 \sin x$; $[-5, 5]$

64. $f(x) = \frac{(\sin x)^2}{x}$; $[-4, 4]$

65. *Temperature During an Illness.* The temperature T of a patient during a 12-day illness is given by

$$T(t) = 101.6° + 3° \sin\left(\frac{\pi}{8}t\right).$$

a) Graph the function on the interval $[0, 12]$.
b) What are the maximum and the minimum temperatures during the illness?

66. *Periodic Sales.* A company in a northern climate has sales of skis as given by

$$S(t) = 10\left(1 - \cos\frac{\pi}{6}t\right),$$

where t is the time, in months ($t = 0$ corresponds to July 1), and $S(t)$ is in thousands of dollars.

a) Graph the function on a 12-month interval $[0, 12]$.
b) What is the period of the function?
c) What is the minimum amount of sales and when does it occur?
d) What is the maximum amount of sales and when does it occur?

Collaborative Discussion and Writing

67. In the equations $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$, which constants translate the graphs and which constants stretch and shrink the graphs? Describe in your own words the effect of each constant.

68. In the transformation steps listed in this section, why must step (1) precede step (3)? Give an example that illustrates this.

Skill Maintenance

Classify the function as linear, quadratic, cubic, quartic, rational, exponential, logarithmic, or trigonometric.

69. $f(x) = \dfrac{x + 4}{4}$

70. $y = \dfrac{1}{2}\log x - 4$

71. $y = x^4 - x - 2$

72. $\dfrac{3}{4}x + \dfrac{1}{2}y = -5$

73. $f(x) = \sin x - 3$

74. $f(x) = 0.5e^{x-2}$

75. $y = \dfrac{2}{5}$

76. $y = \sin x + \cos x$

77. $y = x^2 - x^3$

78. $f(x) = \left(\dfrac{1}{2}\right)^x$

Synthesis

Find the maximum and minimum values of the function.

79. $y = 2\cos\left[3\left(x - \dfrac{\pi}{2}\right)\right] + 6$

80. $y = \dfrac{1}{2}\sin(2x - 6\pi) - 4$

The transformation techniques that we learned in this section for graphing the sine and cosine functions can also be applied to the other trigonometric functions. Sketch a graph of each of the following.

81. $y = -\tan x$

82. $y = \tan(-x)$

83. $y = -2 + \cot x$

84. $y = -\dfrac{3}{2}\csc x$

85. $y = 2 \tan \frac{1}{2} x$

86. $y = \cot 2x$

87. $y = 2 \sec (x - \pi)$

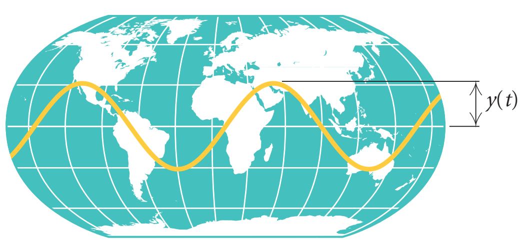
88. $y = 4 \tan \left(\frac{1}{4} x + \frac{\pi}{8}\right)$

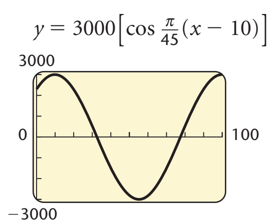
89. $y = 2 \csc \left(\frac{1}{2} x - \frac{3\pi}{4}\right)$

90. $y = 4 \sec (2x - \pi)$

91. *Satellite Location.* A satellite circles the earth in such a way that it is y miles from the equator (north or south, height not considered) t minutes after its launch, where

$$y(t) = 3000\left[\cos \frac{\pi}{45}(t - 10)\right].$$



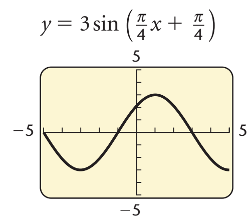


What are the amplitude, the period, and the phase shift?

92. *Water Wave.* The cross-section of a water wave is given by

$$y = 3 \sin \left(\frac{\pi}{4} x + \frac{\pi}{4}\right),$$

where y is the vertical height of the water wave and x is the distance from the origin to the wave.



What are the amplitude, the period, and the phase shift?

93. *Damped Oscillations.* Suppose that the motion of a spring is given by

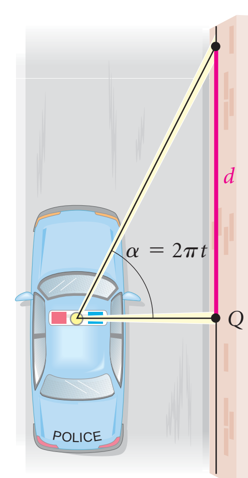
$$d(t) = 6e^{-0.8t} \cos (6\pi t) + 4,$$

where d is the distance, in inches, of a weight from the point at which the spring is attached to a ceiling, after t seconds. How far do you think the spring is from the ceiling when the spring stops bobbing?

94. *Rotating Beacon.* A police car is parked 10 ft from a wall. On top of the car is a beacon rotating in such a way that the light is at a distance $d(t)$ from point Q after t seconds, where

$$d(t) = 10 \tan (2\pi t).$$

When d is positive, as shown in the figure, the light is pointing north of Q, and when d is negative, the light is pointing south of Q.



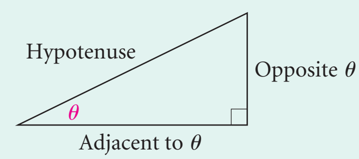
Explain the meaning of the values of t for which the function is not defined.

Chapter 5 Summary and Review

Important Properties and Formulas

Trigonometric Function Values of an Acute Angle θ

Let θ be an acute angle of a right triangle. The six trigonometric functions of θ are as follows:



$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \quad \cos\theta = \frac{\text{adj}}{\text{hyp}}, \quad \tan\theta = \frac{\text{opp}}{\text{adj}},$$
$$\csc\theta = \frac{\text{hyp}}{\text{opp}}, \quad \sec\theta = \frac{\text{hyp}}{\text{adj}}, \quad \cot\theta = \frac{\text{adj}}{\text{opp}}.$$

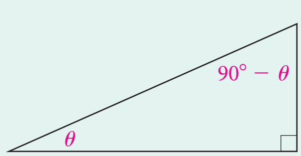
Reciprocal Functions

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

Function Values of Special Angles

	0°	**30°**	**45°**	**60°**	**90°**
sin	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
tan	0	$\sqrt{3}/3$	1	$\sqrt{3}$	Not defined

Cofunction Identities



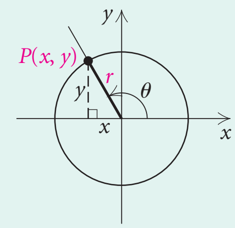
$$\sin\theta = \cos(90° - \theta), \quad \cos\theta = \sin(90° - \theta),$$
$$\tan\theta = \cot(90° - \theta), \quad \cot\theta = \tan(90° - \theta),$$
$$\sec\theta = \csc(90° - \theta), \quad \csc\theta = \sec(90° - \theta)$$

(*continued*)

Trigonometric Functions of Any Angle θ

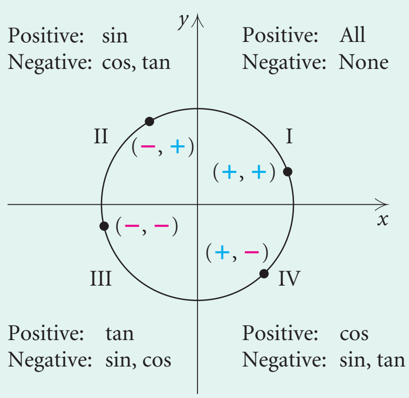
If $P(x, y)$ is any point on the terminal side of any angle θ in standard position, and r is the distance from the origin to $P(x, y)$, where $r = \sqrt{x^2 + y^2}$, then

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x},$$
$$\csc\theta = \frac{r}{y}, \quad \sec\theta = \frac{r}{x}, \quad \cot\theta = \frac{x}{y}$$

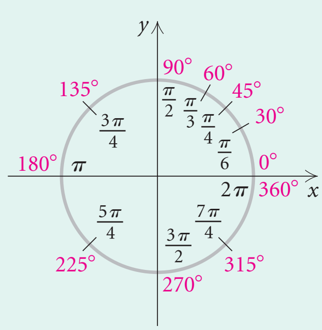


Signs of Function Values

The signs of the function values depend only on the coordinates of the point P on the terminal side of an angle.



Radian–Degree Equivalents

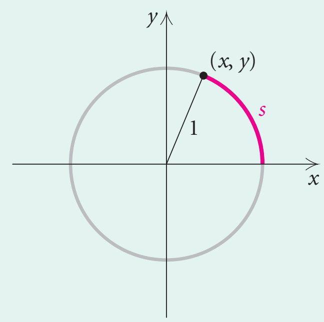


Linear Speed in Terms of Angular Speed

$$v = r\omega$$

Basic Circular Functions

For a real number s that determines a point (x, y) on the unit circle:



$$\sin s = y,$$
$$\cos s = x,$$
$$\tan s = \frac{y}{x}.$$

Sine is an odd function: $\sin(-s) = -\sin s$

Cosine is an even function: $\cos(-s) = \cos s$

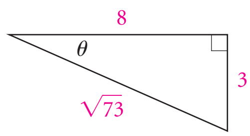
Transformations of Sine and Cosine Functions

To graph $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$:

1. Stretch or shrink the graph horizontally according to B. $\left(\text{Period} = \left|\frac{2\pi}{B}\right|\right)$
2. Stretch or shrink the graph vertically according to A. (Amplitude $= |A|$)
3. Translate the graph horizontally according to C/B. $\left(\text{Phase shift} = \frac{C}{B}\right)$
4. Translate the graph vertically according to D.

Review Exercises

1. Find the six trigonometric function values of θ.



2. Given that $\sin \beta = \frac{\sqrt{91}}{10}$, find the other five trigonometric function values.

Find the exact function value, if it exists.

3. $\cos 45°$

4. $\cot 60°$

5. $\cos 495°$

6. $\sin 150°$

7. $\sec(-270°)$

8. $\tan(-600°)$

9. $\csc 60°$

10. $\cot(-45°)$

11. Convert 22.27° to degrees, minutes, and seconds. Round to the nearest second.

12. Convert 47°33′27″ to decimal degree notation. Round to two decimal places.

Find the function value. Round to four decimal places.

13. $\tan 2184°$

14. $\sec 27.9°$

15. $\cos 18°13'42''$

16. $\sin 245°24'$

17. $\cot(-33.2°)$

18. $\sin 556.13°$

Find θ in the interval indicated. Round the answer to the nearest tenth of a degree.

19. $\cos \theta = -0.9041$, $(180°, 270°)$

20. $\tan \theta = 1.0799$, $(0°, 90°)$

Find the exact acute angle θ, in degrees, given the function value.

21. $\sin \theta = \frac{\sqrt{3}}{2}$

22. $\tan \theta = \sqrt{3}$

23. $\cos \theta = \frac{\sqrt{2}}{2}$

24. $\sec \theta = \frac{2\sqrt{3}}{3}$

25. Given that $\sin 59.1° \approx 0.8581$, $\cos 59.1° \approx 0.5135$, and $\tan 59.1° \approx 1.6709$, find the six function values for 30.9°.

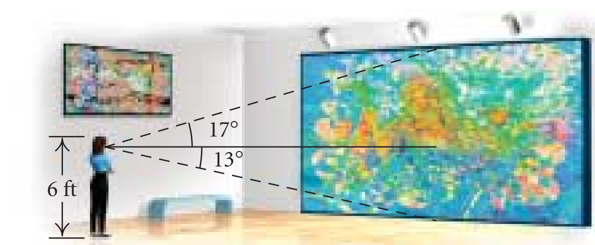
Solve each of the following right triangles. Standard lettering has been used.

26. $a = 7.3$, $c = 8.6$

27. $a = 30.5$, $B = 51.17°$

28. One leg of a right triangle bears east. The hypotenuse is 734 m long and bears N57°23′E. Find the perimeter of the triangle.

29. An observer's eye is 6 ft above the floor. A mural is being viewed. The bottom of the mural is at floor level. The observer looks down 13° to see the bottom and up 17° to see the top. How tall is the mural?



For angles of the following measures, state in which quadrant the terminal side lies.

30. 142°11′5″

31. −635.2°

32. −392°

Find a positive angle and a negative angle that are coterminal with the given angle. Answers may vary.

33. 65°

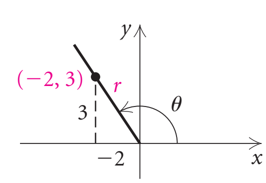
34. $\frac{7\pi}{3}$

Find the complement and the supplement.

35. 13.4°

36. $\frac{\pi}{6}$

37. Find the six trigonometric function values for the angle θ shown.



38. Given that $\tan\theta = 2/\sqrt{5}$ and that the terminal side is in quadrant III, find the other five function values.

39. An airplane travels at 530 mph for $3\frac{1}{2}$ hr in a direction of 160° from Minneapolis, Minnesota. At the end of that time, how far south of Minneapolis is the airplane?

40. On a unit circle, mark and label the points determined by $7\pi/6$, $-3\pi/4$, $-\pi/3$, and $9\pi/4$.

For angles of the following measures, convert to radian measure in terms of π, and convert to radian measure not in terms of π. Round the answer to two decimal places.

41. 145.2°

42. −30°

Convert to degree measure. Round the answer to two decimal places.

43. $\dfrac{3\pi}{2}$

44. 3

45. −4.5

46. 11π

47. Find the length of an arc of a circle, given a central angle of $\pi/4$ and a radius of 7 cm.

48. An arc 18 m long on a circle of radius 8 m subtends an angle of how many radians? how many degrees, to the nearest degree?

49. At one time, inside La Madeleine French Bakery and Cafe in Houston, Texas, there was one of the few remaining working watermills in the world. The 300-yr-old French-built waterwheel had a radius of 7 ft and made one complete revolution in 70 sec. What was the linear speed in feet per minute of a point on the rim? (*Source*: La Madeleine French Bakery and Cafe, Houston, TX)

50. An automobile wheel has a diameter of 14 in. If the car travels at a speed of 55 mph, what is the angular velocity, in radians per hour, of a point on the edge of the wheel?

51. The point $\left(\frac{3}{5}, -\frac{4}{5}\right)$ is on a unit circle. Find the coordinates of its reflections across the x-axis, the y-axis, and the origin.

Find the exact function value, if it exists.

52. $\cos\pi$

53. $\tan\dfrac{5\pi}{4}$

54. $\sin\dfrac{5\pi}{3}$

55. $\sin\left(-\dfrac{7\pi}{6}\right)$

56. $\tan\dfrac{\pi}{6}$

57. $\cos(-13\pi)$

Find the function value. Round to four decimal places.

58. $\sin 24$

59. $\cos(-75)$

60. $\cot 16\pi$

61. $\tan\dfrac{3\pi}{7}$

62. $\sec 14.3$

63. $\cos\left(-\dfrac{\pi}{5}\right)$

64. Graph each of the six trigonometric functions from -2π to 2π.

65. What is the period of each of the six trigonometric functions?

66. Complete the following table.

Function	Domain	Range
sine		
cosine		
tangent		

67. Complete the following table with the sign of the specified trigonometric function value in each of the four quadrants.

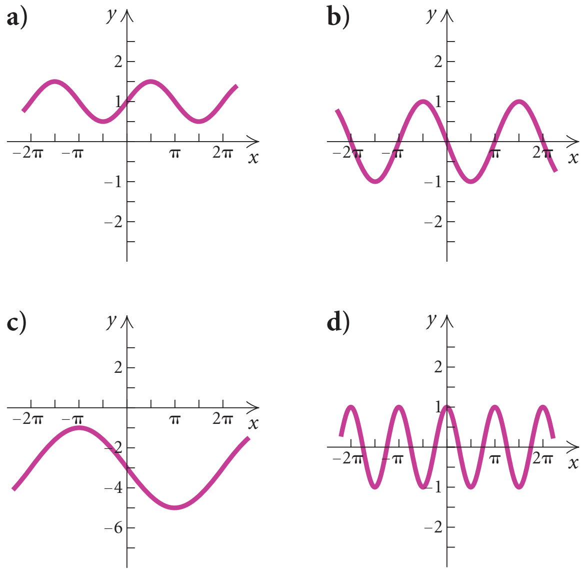
Function	I	II	III	IV
sine				
cosine				
tangent				

Determine the amplitude, the period, and the phase shift of the function, and sketch the graph of the function.

68. $y = \sin\left(x + \dfrac{\pi}{2}\right)$

69. $y = 3 + \dfrac{1}{2}\cos\left(2x - \dfrac{\pi}{2}\right)$

In Exercises 70–73, match the function with one of the graphs (a)–(d), which follow.



70. $y = \cos 2x$

71. $y = \frac{1}{2} \sin x + 1$

72. $y = -2 \sin \frac{1}{2}x - 3$

73. $y = -\cos\left(x - \frac{\pi}{2}\right)$

74. Sketch a graph of $y = 3 \cos x + \sin x$ for values of x between 0 and 2π.

Collaborative Discussion and Writing

75. Compare the terms radian and degree.

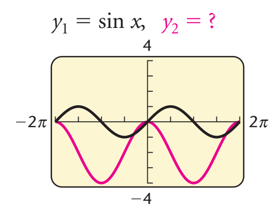
76. Describe the shape of the graph of the cosine function. How many maximum values are there of the cosine function? Where do they occur?

77. Does $5 \sin x = 7$ have a solution for x? Why or why not?

Synthesis

78. Graph $y = 3 \sin (x/2)$, and determine the domain, the range, and the period.

79. In the graph below, $y_1 = \sin x$ is shown and y_2 is shown in red. Express y_2 as a transformation of the graph of y_1.

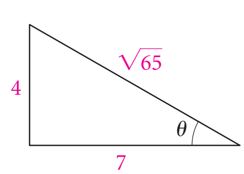


80. Find the domain of $y = \log(\cos x)$.

81. Given that $\sin x = 0.6144$ and that the terminal side is in quadrant II, find the other basic circular function values.

Chapter 5 Test

1. Find the six trigonometric function values of θ.



Find the exact function value, if it exists.

2. $\sin 120°$

3. $\tan(-45°)$

4. $\cos 3\pi$

5. $\sec \frac{5\pi}{4}$

6. Convert $38°27'56''$ to decimal degree notation. Round to two decimal places.

Find the function values. Round to four decimal places.

7. $\tan 526.4°$

8. $\sin(-12°)$

9. $\sec \frac{5\pi}{9}$

10. $\cos 76.07$

11. Find the exact acute angle θ, in degrees, for which $\sin \theta = \frac{1}{2}$.

12. Given that $\sin 28.4° \approx 0.4756$, $\cos 28.4° \approx 0.8796$, and $\tan 28.4° \approx 0.5407$, find the six trigonometric function values for $61.6°$.

13. Solve the right triangle with $b = 45.1$ and $A = 35.9°$. Standard lettering has been used.

14. Find a positive angle and a negative angle coterminal with a $112°$ angle.

15. Find the supplement of $\frac{5\pi}{6}$.

16. Given that $\sin \theta = -4/\sqrt{41}$ and that the terminal side is in quadrant IV, find the other five trigonometric function values.

17. Convert $210°$ to radian measure in terms of π.

18. Convert $\frac{3\pi}{4}$ to degree measure.

19. Find the length of an arc of a circle given a central angle of $\pi/3$ and a radius of 16 cm.

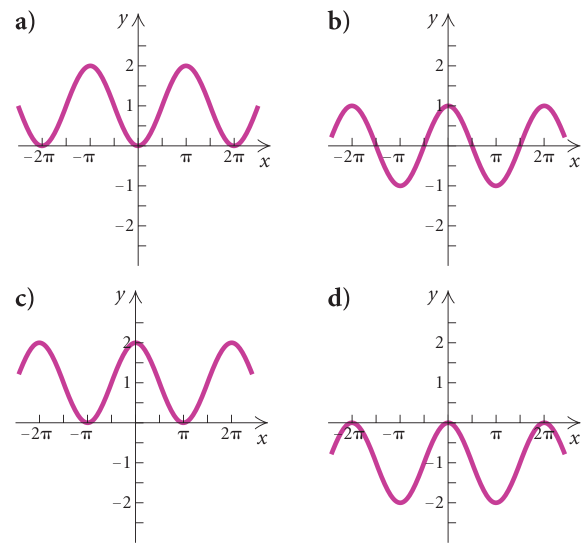
Consider the function $y = -\sin(x - \pi/2) + 1$ *for Exercises 20–23.*

20. Find the amplitude.

21. Find the period.

22. Find the phase shift.

23. Which is the graph of the function?



24. *Height of a Kite.* The angle of elevation of a kite is $65°$ with 490 ft of string out. Assuming the string is taut, how high is the kite?

25. *Location.* A pickup-truck camper travels at 50 mph for 6 hr in a direction of $115°$ from Buffalo, Wyoming. At the end of that time, how far east of Buffalo is the camper?

26. *Linear Speed.* A ferris wheel has a radius of 6 m and revolves at 1.5 rpm. What is the linear speed, in meters per minute?

Synthesis

27. Determine the domain of $f(x) = \frac{-3}{\sqrt{\cos x}}$.