P3]

$$\sum_{k=0}^{n} \frac{\binom{n}{k}}{(k+1)(k+2)} = \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} \cdot \frac{n!}{k!\,(n-k)!}$$

$$= \sum_{k=0}^{n} \frac{n!}{(k+2)!\,(n-k)!} \cdot \frac{(n+1)(n+2)}{(n+1)(n+2)}$$

$$= \sum_{k=0}^{n} \frac{(n+2)!}{(k+2)!\,(n-k)!} \cdot \frac{1}{(n+1)(n+2)}$$

$$= \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} \binom{n+2}{k+2}$$

$$= \frac{1}{(n+1)(n+2)} \sum_{k=2}^{n+2} \binom{n+2}{k}$$

$$= \frac{1}{(n+1)(n+2)} \left[\sum_{k=0}^{n+2} \binom{n+2}{k} - \binom{n+2}{0} - \binom{n+2}{1} \right]$$

$$= \frac{1}{(n+1)(n+2)} \left[2^{n+2} - 1 - (n+2) \right]$$

$$= \boxed{\frac{2^{n+2} - n - 3}{(n+1)(n+2)}}$$

P4) Utilizamos T. de Newton para

$$(1+x)^{2n}+(1-x)^{2n}$$

$$= \sum_{k=0}^{2n} \binom{2n}{k} x^k \, 1^{n-k} + \sum_{k=0}^{2n} \binom{2n}{k} (-x)^k \, 1^{n-k}$$

$$= \sum_{k=0}^{2n} \binom{2n}{k} x^k + \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k x^k$$

$$= \sum_{k=0}^{2n} \binom{2n}{k} x^k \left[1+(-1)^k\right]$$

para los terminos k impares vale cero.

$$\left[1+(-1)^{(2i-1)}\right] = 1+(-1) = 0$$

para los terminos k pares vale 2.

$$\left[1+(-1)^{2i}\right] = 1+1 = 2$$

$$= \sum_{i=0}^{n} \binom{2n}{2i} x^2 \left[1+(-1)^{2i}\right] \quad \text{par } k = 2i \quad k \in \{0,1,\dots,n\}$$

$$= \sum_{i=0}^{n} \binom{2n}{2i} x^2 \cdot 2$$

$$\Rightarrow (1+x)^{2n}+(1-x)^{2n} = 2\sum_{i=0}^{n} \binom{2n}{2i} x^2$$

Esto vale $\forall x \in \mathbb{R}$ en particular para $x=1$

$$\Rightarrow \quad (1+1)^{2n} + \underbrace{(1-1)^{2n}}_{0} = 2\sum_{i=0}^{n} \binom{2n}{2i} 1^2$$

$$2^{2n} = 2\sum_{i=0}^{n} \binom{2n}{2i}$$

$$\boxed{2^{2n-1} = \sum_{i=0}^{n} \binom{2n}{2i}}$$

P5 a) Pdq! $\binom{n}{i}\binom{i}{k} = \binom{n}{k}\binom{n-k}{i-k}$

$$\binom{n}{i}\binom{i}{k} = \frac{n!}{\cancel{i!}(n-i)!} \cdot \frac{\cancel{i!}}{k!\,(i-k)!}$$

$$= \frac{n!}{k!\,(n-i)!\,(i-k)!} \qquad i \geq k$$

$$= \frac{n!}{k!\,(n-k)!\,(n-k+1)\cdots(n-i)\,(i-k)!}$$

$$= \binom{n}{k} \cdot \frac{1}{(n-k+1)\cdots(n-i)\,(i-k)!} \cdot \underbrace{\frac{(n-k)!}{(n-k)!}}_{=1}$$

$$= \binom{n}{k} \frac{(n-k)!}{(n-k)!\,(n-k+1)\cdots(n-i)\,(i-k)!}$$

$$= \binom{n}{k} \frac{(n-k)!}{(n-i)!\,(i-k)!} = \binom{n}{k}\binom{n-k}{i-k}$$

k, n fijos

(b) $$\sum_{i=k}^{n} \binom{n}{i}\binom{i}{k} = \sum_{i=k}^{n} \binom{n}{k}\binom{n-k}{i-k} \quad \text{por (a)}$$

$$= \binom{n}{k} \sum_{i=k}^{n} \binom{n-k}{i-k} = \binom{n}{k} \sum_{i=0}^{n-k} \binom{n-k}{i}$$

$$= \binom{n}{k} 2^{n-k}$$