P1 Mostar que $\sin 15^\circ = \frac{\sqrt{2}}{2}\sqrt{1-\frac{\sqrt{3}}{2}}$

para esto trabajemos con algo que si conocemos; e.g. $\cos 30^\circ$

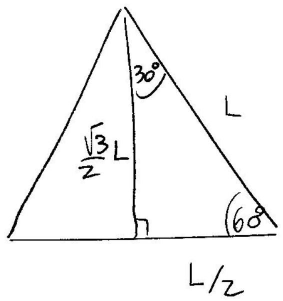
$$\cos 30^\circ = \cos(15^\circ+15^\circ) = \cos^2 15^\circ - \sin^2 15^\circ$$

$$\Rightarrow \quad \cos 30^\circ = 1 - 2\cdot \sin^2 15^\circ$$

$$\Rightarrow \quad 2\sin^2 15^\circ = 1 - \cos 30^\circ$$

$$\Rightarrow \quad \sin 15^\circ = \left[\frac{1}{2}(1-\cos 30^\circ)\right]^{1/2}$$

$\cos 30^\circ$ se consigue de un triangulo equilatero:



$$\Rightarrow \cos 30^\circ = \frac{\frac{\sqrt{3}}{2}L}{L} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \cos 30^\circ = \frac{\sqrt{3}}{2}$$

Luego:

$$\sin 15^\circ = \frac{\sqrt{2}}{2}\cdot\sqrt{1-\cos 30^\circ}$$

$$\Rightarrow \sin 15^\circ = \frac{\sqrt{2}}{2}\cdot\sqrt{1-\frac{\sqrt{3}}{2}}$$

Mostrar que $\cos 15^\circ = \frac{\sqrt{2}}{2}\sqrt{1+\frac{\sqrt{3}}{2}}$

$$\cos 30^\circ = \cos^2 15^\circ - \sin^2 15^\circ$$

$$\Rightarrow \cos 30^\circ = 2\cos^2 15^\circ - 1$$

$$\Rightarrow 2\cos^2 15^\circ = 1 + \cos 30^\circ$$

$$\Rightarrow \cos 15^\circ = \frac{1}{\sqrt{2}}\sqrt{1+\frac{\sqrt{3}}{2}}$$

$$\Rightarrow \cos 15^\circ = \frac{\sqrt{2}}{2}\sqrt{1+\frac{\sqrt{3}}{2}}$$

estimar numericamente $\sin 15° = \frac{\sqrt{2}}{2} \cdot \sqrt{1 - \frac{\sqrt{3}}{2}}$

veamos primero $\sqrt{3}$:

$$\sqrt{3} = \sqrt{4-1} = 2\sqrt{1-\frac{1}{4}}$$

$$\approx 2\left(1 - \frac{1}{2}\cdot\frac{1}{4}\right)$$

$$= 2\left(\frac{7}{8}\right) \qquad \Rightarrow \sqrt{3} \approx 2\cdot\frac{7}{8}$$

$$\Rightarrow \sin 15° = \frac{\sqrt{2}}{2}\sqrt{1-\frac{\sqrt{3}}{2}}$$

$$\approx \frac{\sqrt{2}}{2}\sqrt{1-\frac{2}{2}\cdot\frac{7}{8}}$$

$$= \frac{\sqrt{2}}{2}\sqrt{1-\frac{7}{8}}$$

$$= \frac{\sqrt{2}}{2}\cdot\sqrt{\frac{1}{8}}$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{1}{\sqrt{2}\cdot 2}$$

$$= \frac{1}{4} \qquad \Rightarrow \underline{\sin 15° \approx 0.25}$$

(valor exacto : $\sin 15° = 0.2588$)

estimar $\cos 15°$:

$$\cos 15° = \frac{\sqrt{2}}{2}\cdot\sqrt{1+\frac{\sqrt{3}}{2}}$$

$$\approx \frac{\sqrt{2}}{2}\sqrt{1+\frac{2}{2}\cdot\frac{7}{8}}$$

$$= \frac{\sqrt{2}}{2}\cdot\sqrt{1+\frac{7}{8}}$$

$$= \frac{\sqrt{2}}{2}\cdot\sqrt{\frac{15}{8}}$$

$$= \frac{\sqrt{2}}{2}\,\frac{\sqrt{15}}{2\sqrt{2}}$$

$$= \frac{\sqrt{16-1}}{4} \qquad \Rightarrow \underline{\cos 15° \approx 0.9687}$$

$$= \frac{4}{4}\sqrt{1-\frac{1}{16}} \qquad \text{valor exacto } \cos 15° = 0.9659$$

$$= 1 - \frac{1}{2}\cdot\frac{1}{32} = \frac{31}{32}$$

$$\sin 16^\circ = \sin(15^\circ + 1^\circ)$$

$$= \sin 15^\circ \cos 1^\circ + \cos 15^\circ \sin 1^\circ$$

considerando que 1° es pequeño $\Rightarrow$

$\cos 1^\circ \approx 1$

ojo! $\sin 1^\circ \not\approx 1^\circ$ ya que la aproximacion $\sin x \approx x$ es valida cuando x esta medido en radianes

$$\Rightarrow \sin 16^\circ \approx \sin 15^\circ \cdot 1 + \cos 15^\circ \cdot \frac{\pi}{180}$$

$$\approx 0.25 + 0.9687 \frac{1}{60}$$

$$\Rightarrow \sin 16^\circ \approx 0.266$$

Valor exacto $\sin 16^\circ = 0.2756$

P3

$A(\theta)$ = Area de la seccion de circunferencia
- Area del triangulo (ver fig)
$\equiv A_{\otimes} - A_{\Delta}$

sabemos que el area es proporcional al angulo

$$\Rightarrow \frac{\pi R^2}{2\pi} = \frac{A}{2\theta} \Rightarrow A_{\otimes} = \theta \cdot R^2$$

$A_{\Delta} = \frac{1}{2}$ base altura

$$= \frac{1}{2} \cdot (2R\sin\theta) R\cos\theta$$

$$= \frac{R^2}{2} \sin(2\theta)$$

$$\Rightarrow A(\theta) = \theta R^2 - \frac{R^2}{2}\sin(2\theta)$$

$$\Rightarrow A(\theta) = R^2 \left(\theta - \frac{1}{2}\sin(2\theta)\right)$$

pdq. $\cos\theta = 1 - \frac{h}{R}$:

altura del triangulo $+ h = R$

$$\Rightarrow R\cos\theta + h = R$$

$$\Rightarrow \cos\theta = \frac{R-h}{R} \qquad \Rightarrow \cos\theta = 1 - \frac{h}{R}$$

Tasa de cambio de θ:

se sabe que $Q = \frac{d}{dt} \text{Vol}$

$$\text{Vol} = A(\theta) \cdot L$$

$$\Rightarrow \frac{d\,\text{Vol}}{dt} = L \cdot \frac{dA}{dt} = L \cdot \frac{d}{dt}\left(R^2\left[\theta - \frac{1}{2}\sin 2\theta\right]\right)$$

$$= R^2 L \left[\frac{d\theta}{dt} - \frac{1}{2}\frac{d(\sin 2\theta)}{dt}\right]$$

$$= R^2 \cdot L \left[\dot{\theta} - \frac{1}{2}(\cos(2\theta) \cdot 2\frac{d\theta}{dt}\right]$$

$$\Rightarrow Q = R^2 L(\dot{\theta} - \cos 2\theta \cdot \dot{\theta})$$

$$\Rightarrow Q = \dot{\theta} \cdot R^2 L(1 - \cos 2\theta)$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{Q}{R^2 L(1 - \cos(2\theta))}$$

Tasa de cambio de h:

cuando la piscina está casi vacía, θ es pequeño

$$\Rightarrow \sin\theta \approx \theta$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

luego: $\cos\theta \approx 1 - \frac{\theta^2}{2} = 1 - \frac{h}{R}$

$$\Rightarrow \theta^2 = \frac{2h}{R} \qquad / \frac{d}{dt}$$

$$\Rightarrow 2\theta \frac{d\theta}{dt} = \frac{2}{R} \cdot \frac{dh}{dt} \qquad \Rightarrow \dot{h} = R\theta \cdot \dot{\theta}$$

$$\Rightarrow \dot{h} = R \cdot \sqrt{\frac{2h}{R}} \cdot \frac{Q}{R^2 L(1 - (\cos^2\theta - \sin^2\theta))}$$

$$= \sqrt{\frac{2h}{R}} \cdot \frac{Q}{RL} \cdot \frac{1}{2\sin^2\theta}$$

$$\Rightarrow \dot{h} \approx \sqrt{\frac{2h}{R}} \cdot \frac{Q}{RL} \cdot \frac{1}{2 \cdot \theta^2}$$

$$\approx \sqrt{\frac{2h}{R}} \cdot \frac{Q}{RL} \cdot \frac{R}{4h} \qquad \Rightarrow \underline{\frac{dh}{dt} = \frac{Q}{2L} \cdot \frac{1}{\sqrt{2h}}}$$