A Fruitful Interaction Between Algebra, Geometry, and Topology: Varieties Through the Lens of Group Actions

Rubí E. Rodríguez and Anita M. Rojas

Beautiful theories have emerged from mixing different areas of mathematics, when building *bridges* between areas provides the right paths to achieve the comprehension of a certain subject. Mathematicians know that if an object has several underlying structures, then the answer to some question considering one aspect of the object may come from another aspect of itself. A remarkable example of this is Descartes's use of coordinates; this milestone was the key to translate questions from geometry to algebra. But there are many other examples: Galois theory, algebraic geometry, mathematical physics, and the list goes on and on.

In these notes we will discuss how the knowledge about group algebras and representation theory has proved fruitful in understanding fundamental questions about complex abelian varieties, compact Riemann surfaces, and their moduli spaces.

Let us begin by roughly introducing these objects. Abelian varieties and compact Riemann surfaces are *complex manifolds*; that is, (connected, Hausdorff) topological spaces that locally look like open subsets of $\mathbb{C}^n$ for some n—plus a technical condition better explained in Figure 1—and that also share the property of being *varieties*.

The word *variety* here is used to refer to a geometric object that can be described as the set of common zeros of polynomial equations in some appropriate space. Since we are considering compact Riemann surfaces and complex abelian varieties, the natural ambient space for them is $\mathbb{P}^n(\mathbb{C})$, the *(complex) projective n-space*, where (complex) lines in $\mathbb{C}^{n+1} \setminus \{(0, \dots, 0)\}$ through the origin are collapsed to points.

Formally, the complex projective n-space is defined as the quotient of the complex space $\mathbb{C}^{n+1} \setminus \{(0, \dots, 0)\}$ by the equivalence relation where $u = (u_0, \dots, u_n)$ and $v = (v_0, \dots, v_n) \in \mathbb{C}^{n+1} \setminus \{(0, \dots, 0)\}$ are related if and only if there is a scalar $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ such that $u = \lambda v$. The usual notation is $\mathbb{P}^n(\mathbb{C}) := (\mathbb{C}^{n+1} \setminus \{(0, \dots, 0)\})/\mathbb{C}^*$.

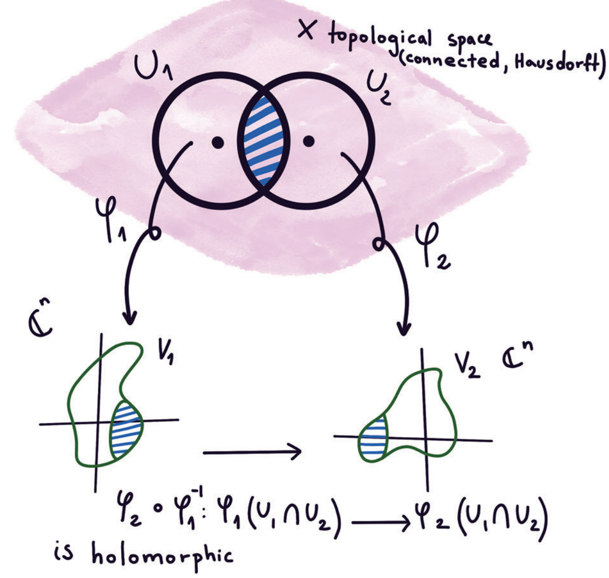

Figure 1. X has an open covering $\{U_j\}_{j\in I}$ and homeomorphisms $\{\varphi_j : U_j \to V_j \subset \mathbb{C}^n\}_{j\in I}$ such that for $i, j \in I$, one has either $U_i \cap U_j = \emptyset$, or φ_i and φ_j are required to satisfy the condition displayed in this picture.

Rubí E. Rodríguez is a professor in the Math Department at Universidad de La Frontera, located in Temuco, Chile. She is the principal researcher for the Fondecyt grant 1230708. Her email address is `rubi.rodriguez@ufrontera.cl`.

Anita M. Rojas is a professor in the Math Department at Universidad de Chile, located in Santiago, Chile. She is the principal researcher for the Fondecyt grant 1230034. Her email address is `anirojas@uchile.cl`.

Communicated by Notices *Associate Editor Han-Bom Moon.*

For permission to reprint this article, please contact: `reprint-permission@ams.org`.

DOI: https://doi.org/10.1090/noti2950

Remark 0.1. The projective n-space $\mathbb{P}^n(F)$ can be defined for any field F, using a similar equivalence relation. When F is a finite field, an interesting exercise is to compute the number of points in $\mathbb{P}^n(F)$ in terms of those in F.

For $n = 1$, we talk about *the complex projective line* $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$. It is a compact topological space of genus zero. In fact, as (one-dimensional) complex manifolds, $\mathbb{P}^1$ is isomorphic to the Riemann sphere $\widehat{\mathbb{C}} := \mathbb{C} \cup \infty$, the complex plane compactified by adding one point.

For $n = 2$, we have the projective space $\mathbb{P}^2$. Its points are denoted as $[x_0 : x_1 : x_2]$, and they correspond to the lines $\{\lambda(x_0, x_1, x_2) : \lambda \in \mathbb{C}^*\}$. The set of zeros of a homogeneous polynomial $F(x_0, x_1, x_2)$ in three variables is well defined; a plane projective algebraic curve is the set of zeros in $\mathbb{P}^2$ of a homogeneous polynomial. There is a rich theory of algebraic curves, so here begins a new approach to studying compact Riemann surfaces: Every compact Riemann surface corresponds to a plane projective algebraic curve, hence a variety, with a finite number of controlled singularities. In fact, the result is stronger: every compact Riemann surface can be represented as a smooth algebraic curve defined in some projective space.

On the other hand, an abelian variety of dimension g is a complex torus: the quotient of a complex vector space of dimension g by a discrete subgroup of maximal rank, hence a compact manifold (of dimension g), and an abelian group, that can be embedded in a projective space.

In what follows, we explain these concepts more rigorously, define their *parameters* or *moduli spaces*, explain how they are related, and illustrate current research and open questions in this field.

1. Compact Riemann Surfaces

In this section we elaborate a bit more the given definition for complex n-dimensional manifolds in the one-dimensional case; hence we consider Figure 1 with $n = 1$.

Let S be a connected, Hausdorff, topological space locally homeomorphic to $\mathbb{C}$. A complex atlas on S is a collection $\mathfrak{S} = \{(U_j, \varphi_j : U_j \to V_j \subset \mathbb{C})\}_{j \in I}$, where $S = \bigcup_{j \in I} U_j$, each U_j is an open set in S and each φ_j is a homeomorphism, and either $U_i \cap U_j = \emptyset$ or the change of coordinates $\varphi_j \circ \varphi_i^{-1}$ are holomorphic.

For every atlas $\mathfrak{S}$ there is a unique maximal atlas $\mathcal{E}$ containing it, called a *complex structure* on S.

Formally, a Riemann surface is a pair $(S, \mathfrak{S})$, or, equivalently, a pair $(S, \mathcal{E})$. Usually the complex structure (or the complex atlas) is omitted from the notation (if it is clear from the context), and one just says that a Riemann surface is a complex manifold of dimension one.

The Uniformization Theorem [FK92], a deep result in this area, states that the only simply connected Riemann surfaces (up to isomorphism) are the Riemann sphere $\widehat{\mathbb{C}}$, the complex plane $\mathbb{C}$ and the upper half-plane $\mathbb{H}_1$ in $\mathbb{C}$ (which is isomorphic, as Riemann surfaces, to the open unit disc Δ in $\mathbb{C}$).

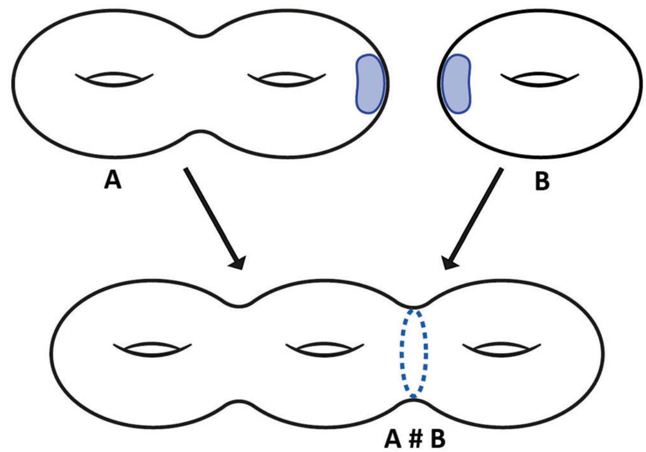

Figure 2. Topological construction of a compact Riemann surface of genus 3.

The Cauchy-Riemann relations for holomorphic functions imply that the change of coordinates $\varphi_j \circ \varphi_i^{-1}$ are $\mathcal{C}^\infty$ (when considered as real functions in $\mathbb{R}^2$), and their Jacobian determinant is positive. Therefore, every Riemann surface is an orientable real surface.

Throughout this work, we will consider compact Riemann surfaces. Since Riemann surfaces are orientable differentiable real surfaces, topologically a compact Riemann surface of genus $g \geq 1$ is a connected sum of g one-dimensional complex tori, see Figure 2. Genus $g = 0$ corresponds to the Riemann sphere, and its theory is slightly different from what we want to expose here, so we leave it out, and we will write about Riemann surfaces of genus $g \geq 1$. Genus $g = 1$ is, again, a bit different from the rest; for instance, since genus one Riemann surfaces are also (abelian) groups, but this is not the case for higher genera.

Genus one Riemann surfaces. We will see that each Riemann surface S of genus 1 is an elliptic curve, a one-dimensional complex torus and an abelian variety of dimension one. Since we are taking a *complex variables* approach to our exposition, we start with the first construction of genus one Riemann surfaces.

Definition 1.1. Let w_1 and w_2 be two $\mathbb{R}$-linearly independent complex numbers, and consider the lattice $L = \{nw_1 + mw_2 : n, m \in \mathbb{Z}\}$ generated by $\{w_1, w_2\}$. The quotient $T = \mathbb{C}/L$ is the complex torus of lattice L (see Figure 3).

Observe that the sum of complex numbers descends to an operation on T that makes it an abelian group; also the notion of analytic function between two tori is well defined (by considering the map in coordinates, see Figure 4). Thus the following definitions make sense: a homomorphism $f : T_1 \to T_2$ between two tori is an analytic

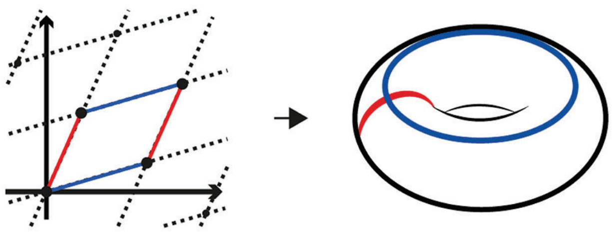

Figure 3. Complex torus of dimension one.

map which is also a group homomorphism. Two tori are isomorphic if there is a bijective homomorphism between them.

Since $\mathbb{C}$ is simply connected and $f(0_{T_1}) = 0_{T_2}$, it follows that a homomorphism $f : T_1 \to T_2$ from $T_1 = \mathbb{C}/L_1$ to $T_2 = \mathbb{C}/L_2$ gives rise to two maps:

$$\rho_a(f) : \mathbb{C} \to \mathbb{C} \quad \text{and} \quad \rho_r(f) : L_1 \to L_2, \tag{1.1}$$

called the analytic and rational representation of f, respectively; they are $\mathbb{C}$-linear and $\mathbb{Z}$-linear maps respectively. If f is an isomorphism, then $\rho_a(f)$ and $\rho_r(f)$ are invertible maps.

Hence (modulo isomorphisms) it is enough to consider lattices of the form $L_\tau = \{n + m\tau : n, m \in \mathbb{Z}\}$ with $\tau \in \mathbb{H}_1$ a complex number with positive imaginary part, and two such tori $T_1 = \mathbb{C}/L_{\tau_1}$ and $T_2 = \mathbb{C}/L_{\tau_2}$ are isomorphic if and only if

$$\tau_2 = \frac{a\tau_1 + b}{c\tau_1 + d} \quad \text{with} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

Since a torus $T = \mathbb{C}/L_\tau$ is compact, any holomorphic function from T to $\mathbb{C}$ is constant; nevertheless, it has non-constant meromorphic functions. In fact, its field of meromorphic functions is generated by two functions, the so-called *Weierstrass p-function* $\wp(z)$, which depends on L_τ, and its derivative $\wp'(z)$. Both functions are related by a cubic equation, giving a way of describing T as a plane algebraic curve, namely *an elliptic curve*. Therefore, every Riemann surface of genus one is an elliptic curve, hence an abelian variety: a complex torus that can be embedded in a projective space.

Higher genus. As mentioned above, topologically a compact Riemann surface of genus $g \geq 2$ is a connected sum of g one-dimensional complex tori, see Figure 2. The classical theory of uniformization of (arbitrary) Riemann surfaces, see for instance [FK92, §IV.5], allows us to understand compact Riemann surfaces of genus $g \geq 2$ as (one-dimensional) complex manifolds.

In fact, a compact Riemann surface of genus $g \geq 2$ is the quotient of the complex upper-half plane $\mathbb{H}_1$ by a discrete torsion-free cocompact subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$. Here cocompact means that the quotient $\mathbb{H}_1/\Gamma$ is compact. The subgroup Γ is called a *surface Fuchsian group*.

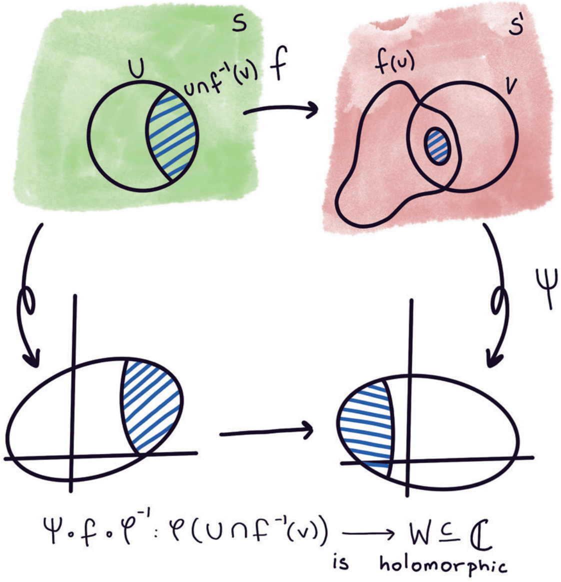

Figure 4. A morphism $f : S \to S'$ in local charts.

Since we will study group actions on (compact) Riemann surfaces and their quotients, we need to enlarge the type of Fuchsian groups under consideration to groups including torsion elements.

In general, the action of a cocompact Fuchsian group Δ on $\mathbb{H}_1$ (not necessarily torsion-free) is (partially) captured by its *signature* $s(\Delta) = (h; m_1, \dots, m_r)$, where h denotes the genus of $\mathbb{H}_1/\Delta$, $m_1, \dots, m_r$ are the ramification indices in the associated projection $\mathbb{H}_1 \to \mathbb{H}_1/\Delta$, and r is the number of *branch points* in $\mathbb{H}_1/\Delta$. Observe that the m_j are the orders of the (nontrivial) stabilizers of elements in Δ fixing points in $\mathbb{H}_1$. It is known that a Fuchsian group Δ of signature $s(\Delta) = (h; m_1, \dots, m_r)$ has a *canonical presentation* of the form

$$\Delta = \langle \alpha_1, \beta_1, \dots, \alpha_h, \beta_h, x_1, \dots, x_r : \tag{1.2}$$
$$x_1^{m_1} = \cdots = x_r^{m_r} = \prod_{j=1}^{h} [\alpha_j, \beta_j] \prod_{i=1}^{r} x_i = 1\rangle,$$

where $[\alpha_j, \beta_j]$ stands for their commutator. A $2h + r$ tuple of generators of Δ satisfying the presentation (1.2) is called *a tuple of canonical generators* (with respect to $s(\Delta)$).

We refer to the beautiful work [Oh22] for a clear discussion, with illuminating figures, about these kind of surfaces.

Isomorphisms between Riemann surfaces. A continuous function $f : S \to S'$ between two Riemann surfaces is said to be *holomorphic* if for every chart (U, φ) in S and (V, ψ) in S' with $U \cap f^{-1}(V) \neq \emptyset$ the map $\psi \circ f \circ \varphi^{-1}|_{\varphi(U \cap f^{-1}(V))}$ is holomorphic (Fig. 4).

A bijective holomorphic map is an isomorphism, and if in addition $S = S'$, we say that f is an automorphism of S; the group of automorphisms of S is denoted by $\mathrm{Aut}(S)$. We say that a (finite abstract) group H acts on a Riemann surface S if there is a monomorphism $\mu : H \to \mathrm{Aut}(S)$; in this case we write $H \leq \mathrm{Aut}(S)$. The action of a group H as automorphisms of a compact Riemann surface S is (partially) grasped by the *signature* of the cover $S \to S/H$, as follows.

Definition 1.2. Let S be a compact Riemann surface and let $H \leq \mathrm{Aut}(S)$.

H acts on S with signature $s(H) = (h; m_1, \dots, m_r)$ if the quotient Riemann surface S/H has genus h and the quotient map $S \to S/H$ ramifies over r points with ramification indices $m_1, \dots, m_r$; that is, the stabilizer in H of each *ramification point* in the fiber of a branch point $q_j \in S/H$ is of order m_j.

If Γ is a surface Fuchsian group such that $S \cong \mathbb{H}_1/\Gamma$, the condition on H acting on S with signature $s(H) = (h; m_1, \dots, m_r)$ is equivalent to the existence of a Fuchsian group Δ together with a group epimorphism $\theta : \Delta \to H$ such that $\ker(\theta) = \Gamma$ and $s(H) = s(\Delta)$. In addition, θ is called a surface-kernel epimorphism and the image under θ of a tuple of canonical generators (with respect to $s(\Delta)$) of Δ is called a generating vector for H.

Note that the sequence $1 \to \Gamma \to \Delta \xrightarrow{\theta} H \to 1$ is exact and that $S/H \cong \mathbb{H}_1/\Delta$.

A curve of genus g is called *hyperelliptic* if the cyclic group $\mathbf{C}_2$ of order 2 acts on it with signature $(0; 2, \dots, 2)$ (with 2g+2 branch points).

Part of the geometry of the action of a group H on S is captured in the signature $s(H)$, a bit more in the corresponding generating vector. There is a big community of geometers studying group actions on compact Riemann surfaces, classifying them up to equivalence, generating computer algorithms to work with them, etc. See [LR22] and the references given there, for more on group actions, covers, examples, and applications.

From compact Riemann surfaces to algebraic curves. To see that every compact Riemann surface S of genus $g \geq 2$ is also a variety, take a divisor D on S (a formal sum of points in S), and define $L(D)$ as the space of meromorphic functions f on S whose order at poles are not worse than the corresponding coefficient in D. This vector space is finitely generated, say $L(D) = \langle f_1, f_2, \dots, f_n \rangle$, so (when nontrivial) it provides a map $\psi_D : S \to \mathbb{P}^{n-1}$ given by $x \in S \mapsto [f_1(x) : \cdots : f_n(x)]$.

The Riemann-Roch theorem, a deep result that relates the degree of D (roughly its number of points), the dimension of $L(D)$, the canonical divisor and the genus of S, implies that when the degree of D is $2g + 1$ then $\dim L(D) = g + 2$, and therefore ψ_D is an immersion of S into $\mathbb{P}^{g+1}$. Then one still has to show that the image is a variety; this may be done is various ways. One is to appeal to another deep result, Chow's theorem.

Therefore every compact Riemann surface S of genus $g \geq 2$ is also a variety, an algebraic curve, but it is no longer a group. Nevertheless, Abel and Jacobi instead constructed a g-dimensional complex torus associated to S, its *Jacobian variety* (see page 7). Next, we introduce the basics of complex tori and abelian varieties in higher dimension.

2. Higher Dimension

Instead of staying in (complex) dimension one and going from one-dimensional complex tori (that is, Riemann surfaces of genus one) to Riemann surfaces of higher genera, another way to move forward conceptually is to increase the dimension and stay in the world of complex tori, see [Rod14] and the references given there.

Definition 2.1. A complex torus $T = V/L$ of dimension g is the quotient of a g-dimensional complex vector space V by a discrete subgroup of maximal rank L; that is, a lattice L in V. A complex torus which is also a projective variety is called an abelian variety.

Analogously to what was done for one-dimensional complex tori, where we defined elliptic curves as the quotient of $\mathbb{C}$ by a lattice, one can build larger-dimensional tori $T = V/L$ in a very concrete way. Choose bases $\{e_1, \dots, e_g\}$ and $\{\lambda_1, \dots, \lambda_{2g}\}$ of V and L respectively, and write each λ_j in terms of the e_i; that is, $\lambda_j = \sum_{i=1}^{g} \lambda_{ij} e_i$. Then the matrix $\Pi = (\lambda_{ij}) \in M(g \times 2g, \mathbb{C})$ encodes the relation between the real and the complex coordinate functions of its lattice and of its vector space respectively. It is called *a period matrix for T*, and it captures geometric information about T.

Since L is a lattice, the rank of Π is g, hence we can normalize to see that there are bases for V and L for which a period matrix for T is of the form $\Pi = (I_g \ \tau)$, with I_g the $g \times g$ identity matrix and τ a complex $g \times g$ matrix with nonsingular imaginary part. This generalizes the simplification made for elliptic curves on page 3, where $\Pi = (1 \, \tau)$ with $\tau \in \mathbb{H}_1$.

In order for a complex torus to be an abelian variety, it needs to have enough meromorphic functions to be embedded into a projective space. There are several characterizations of when this happens; the following is the classical one, in terms of polarizations.

Definition 2.2. A polarization on a torus $T = V/L$ is a nondegenerate real skew-symmetric form E on V such that $E(iu, iv) = E(u, v)$ for all $u, v \in V$, and $E(L \times L) \subset \mathbb{Z}$, where i denotes a complex number with $i^2 = -1$.

An abelian variety is a complex torus T that *admits a polarization*, and a polarized abelian variety $A = (T, E)$ of dimension g is a pair consisting of a complex torus $T = V/L$ and a polarization E on T.

In terms of a period matrix for the torus T, the *Riemann relations* give necessary and sufficient conditions for T to be an abelian variety [BL04, IV. §2].

Theorem 2.3 (Riemann Relations). *Let T be a complex torus and Π a period matrix for it. Then T is an abelian variety if and only if there exists a nondegenerate skew-symmetric $2g\times 2g$ integral matrix E such that $\Pi E^{-1}\Pi^t = 0$ and $-i\,\Pi E^{-1}\overline{\Pi}^t$ is positive definite, where ^{t} represents the transposed matrix.*

It is important to highlight that not every torus is an abelian variety.

For a polarized abelian variety $A = (T, E)$ of dimension g, with $T = V/L$, Frobenius algorithm [Lan82, VI. §3] gives a way of finding a basis for L with respect to which E is given by

$$E = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \in M_{2g}(\mathbb{Z}),$$

where $D = \mathrm{diag}(d_1, \dots, d_g)$ with d_i natural numbers such that $d_i | d_{i+1}$. Such a basis for L is called *symplectic*, and the tuple $(d_1, \dots, d_g)$ is the *type of the polarization E*. If $d_g = 1$, A is called a *principally polarized abelian variety*, *ppav* for short.

If $A = V/L$ is an abelian variety, then it follows from Theorem 2.3 that there is a basis for V and a symplectic basis for L with respect to which a period matrix for A has the form $\Pi = (D\ Z)$, where D is the diagonal matrix reflecting the type of the polarization E, and Z is a matrix in the Siegel upper space $\mathbb{H}_g = \{M \in M_g(\mathbb{C}) : M^t = M \text{ and } \mathrm{Im}(M) \text{ positive definite}\}$. The matrix Z is called *a Riemann matrix* for T.

Example 1. Every compact Riemann surface of genus one satisfies the Riemann relations. Let us take $T = \mathbb{C}/L$ with $L = \langle 1, \tau \rangle_{\mathbb{Z}}$, $\tau \in \mathbb{H}_1$. The real skew-symmetric form $E(a_1 + b_1\tau, a_2 + b_2\tau) = a_1 b_2 - a_2 b_1$ satisfies the requirements in Def. 2.2. Notice that we recover, in a different way as in the previous section, the fact that every Riemann surface of genus one is a variety.

To define *homomorphisms* between abelian varieties, first consider their underlying complex tori structure. Hence homomorphisms between them correspond to the natural generalization to what was defined for dimension one. A homomorphism $f : T_1 \to T_2$ between two complex tori $T_j = V_j/L_j$ of dimensions g_1 and g_2 respectively, is an analytic map that is also a group homomorphism. If T is any torus, a homomorphism $f : T \to T$ is called an endomorphism of T, with $\mathrm{End}(T)$ denoting the ring of endomorphisms of T.

As in the one-dimensional case, see (1.1), a homomorphism $f : T_1 \to T_2$ gives rise to two representations $\rho_a(f) : V_1 \to V_2$ and $\rho_r(f) : L_1 \to L_2$. In terms of the period matrices Π_1 and Π_2 for T_1 and T_2 respectively, $\rho_a(f)$ is given by a matrix $M \in M(g_2 \times g_1, \mathbb{C})$ and $\rho_r(f)$ by a matrix $R \in M(2g_2 \times 2g_1, \mathbb{Z})$ satisfying the *Hurwitz* equation

$$M\Pi_1 = \Pi_2 R. \tag{2.1}$$

Interesting homomorphisms, which play a role in the problems we want to describe, are the *isogenies*. A homomorphism $f : T_1 \to T_2$ is an *isogeny* if it is surjective with finite kernel, or, equivalently, if it is surjective and $g_1 = g_2$. The *exponent* and the *degree* of the isogeny f are the exponent and the order of its kernel, respectively. The degree corresponds to $|\det \rho_r(f)|$. For every isogeny $f : T_1 \to T_2$ of exponent e, there is a unique (up to isomorphism) isogeny $g : T_2 \to T_1$ such that $f \circ g$ and $g \circ f$ are multiplication by e on T_2 and T_1 respectively. Hence isogenies are the units of the endomorphism algebra $\mathrm{End}_{\mathbb{Q}}(T) = \mathrm{End}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Therefore, they establish an equivalence relation among complex tori. So, most classification theorems about complex tori and abelian varieties are given up to isogeny.

In the following section, we go back to the last lines of Section 1; how to associate to every Riemann surface of genus $g \geq 2$ an abelian variety.

Jacobian varieties. Let S be a compact Riemann surface of genus g; consider the g-dimensional vector space $V = H^{1,0}(S, \mathbb{C})^*$: the dual of the space of holomorphic differential 1-forms, and the lattice $L = H_1(S, \mathbb{Z})$, where the injection of L in V is given by

$$H_1(S, \mathbb{Z}) \to H^{1,0}(S, \mathbb{C})^*$$
$$\alpha \mapsto (\omega \mapsto \int_\alpha \omega).$$

Then the complex torus $JS = H^1(S, \mathbb{C})^*/H_1(S, \mathbb{Z})$ is an abelian variety, since it admits a canonical principal polarization given by extending the geometric intersection number in the lattice L. This ppav JS is called *the Jacobian variety associated to S*. According to Torelli's theorem, two Jacobian varieties are isomorphic if and only if the corresponding Riemann surfaces are. Therefore, there is a bijective correspondence between Jacobian varieties and Riemann surfaces.

Remark 2.4. By checking the Riemann relations (Thm. 2.3), one can verify whether a period matrix $\Pi = (D\,Z)$ of a complex torus T corresponds to an abelian variety. Unfortunately, there is no practical method to identify when a matrix Z in $\mathbb{H}_g$ corresponds to a Riemann matrix of a Jacobian variety. This is known as the Schottky problem.

Some interesting properties for abelian varieties. Among the several properties of interest about abelian varieties, we focus on one about its geometry (*completely decomposable*) and another about its endomorphism algebra (*with complex multiplication*).

Let $T = V/L$ be a torus and $W \leq V$ be a subspace such that $W \cap L$ is a lattice in W; then $T_W = W/(W \cap L)$ defines a subtorus of T. Images and (the connected component containing 0) of kernels of homomorphisms between tori are examples of subtori. A complex torus $T \neq \{0\}$ is *simple* if its only subtori are itself and $\{0\}$. Since any subtorus $B \subset A$ of an abelian variety (A, E) is an abelian variety by restricting the polarization on A to B, $(B, E|_B)$ is an *abelian subvariety* of A; thus an abelian variety is called *simple* if and only if its only abelian subvarieties are itself and the trivial one.

Clearly, one-dimensional abelian varieties are simple. Determining whether a given abelian variety is simple or not is not easy. There are several approaches to studying this property. For instance, see [ALR17] where a criterion in terms of a period matrix is given, as well as examples of simple abelian varieties and references to go deeper in this subject.

The algebra $\mathrm{End}_{\mathbb{Q}}(T) = \mathrm{End}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ of a simple torus T is a skew-field of finite dimension over $\mathbb{Q}$, and any finite dimensional algebra is isomorphic to $\mathrm{End}_{\mathbb{Q}}(T)$ for some torus T. The case is different for abelian varieties, as their endomorphism algebras are semisimple. This result is a consequence of Poincaré's Reducibility Theorem, which says that every subvariety $(B, E|_B)$ of an abelian variety (A, E) has a complementary abelian subvariety $(C, E|_C)$: $B \cap C$ is finite and the addition map $s : B \times C \to A$ is an isogeny. This leads to the fundamental decomposition theorem of this subject.

Theorem 2.5 (Poincaré's complete reducibility). *Let (A, E) be an abelian variety, then there are simple abelian subvarieties $X_1, \ldots, X_t$ of A not isogenous to each other, and positive integers $k_1, \ldots, k_t$, such that A is isogenous to $X_1^{k_1} \times \cdots \times X_t^{k_t}$. This decomposition is unique up to permutation and isogenies of the factors.*

An abelian variety is *completely decomposable* if it is isogenous to a product of elliptic curves. A simple abelian variety A has *complex multiplication* if $\mathrm{End}_{\mathbb{Q}}(A)$ is a number field K of degree $2 \dim A$ over $\mathbb{Q}$. In this case K is a CM-field; that is, a totally imaginary degree two extension of a totally real field. If A is not simple, then it is said to be of CM-type if all of its simple factors (Theorem 2.5) have complex multiplication. We mention some open questions regarding abelian varieties with these properties in Section 5.

In what follows, we denote a polarized abelian variety (pav) by a single letter A, omitting the notation of the polarization.

3. Moduli Spaces

The natural next step after defining certain *objects* of study—for us, compact Riemann surfaces and principally polarized abelian varieties—and their notion of *equivalence*, here isomorphisms, is the *classification problem*: the objects are classified *up to equivalence*, and the set of equivalence classes is their *moduli space*.

It turns out that these moduli spaces, though defined as sets, can be endowed of a geometric (or algebraic) structure. The loci containing objects with nontrivial automorphisms complicate every approach.

In this work, we are interested in $\mathcal{M}_g$, the moduli space of compact Riemann surfaces of genus g, and $\mathcal{A}_g$, the moduli space of principally polarized abelian varieties of dimension g; both can be given an orbifold structure. Also of interest are some loci in them, such as

- their *singular locus*, which corresponds to the (isomorphism classes of) objects with nontrivial automorphisms (for $g \geq 4$ in case of $\mathcal{M}_g$).
- the *Jacobian (or Torelli) locus* $\mathcal{T}_g \subset \mathcal{A}_g$, which corresponds to the image by the injective *Torelli* map $\mathcal{J} : \mathcal{M}_g \to \mathcal{A}_g$ defined by assigning to each Riemann surface S its Jacobian variety JS.
- families in $\mathcal{T}_g$ with complex multiplication (or of CM-type), or completely decomposable.

The moduli space $\mathcal{A}_g$. Let A_1 and A_2 be two pav. A bijective homomorphism $f : A_1 \to A_2$ between the underlying tori is an isomorphism between pav if it preserves the polarization. In such a case, when bases are chosen so that $\Pi_1 = (D\, Z_1)$ and $\Pi_2 = (D\, Z_2)$ are the period matrices of A_1 and A_2 respectively, then an isomorphism f preserves the polarization if and only if $\rho_r(f) \in \mathrm{Sp}^D(2g, \mathbb{Z})$, where

$$\mathrm{Sp}^D(2g, \mathbb{Z}) = \{N \in M(2g \times 2g, \mathbb{Z}) : N^t \cdot J_D \cdot N = J_D\},$$

with $J_D := \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$ and N^t denoting the transpose of N. For the principally polarized case, where $D = I$, the usual notation is $\mathrm{Sp}(2g, \mathbb{Z})$, the symplectic group.

It follows from the Hurwitz's equation (2.1) that $\mathrm{Sp}^D(2g, \mathbb{Z})$ acts on $\mathbb{H}_g$ and the quotient is the moduli space of pav of type D. Therefore, for *ppavs* one has

$$\mathcal{A}_g = \mathbb{H}_g / \mathrm{Sp}(2g, \mathbb{Z}); \tag{3.1}$$

this is a complex analytic space of dimension $\frac{g(g+1)}{2}$.

In particular, an automorphism of a ppav given by a matrix $Z \in \mathbb{H}_g$ is given by an element of the symplectic group fixing Z.

The moduli space $\mathcal{M}_g$. The construction of $\mathcal{M}_g$ as a complex analytic space requires more definitions. Roughly speaking, it arises as the quotient of the *Teichmüller space T_g*, capturing all complex structures on a compact topological space of genus g, by the *mapping class group*, which captures when these structures define *isomorphic* Riemann surfaces. It is a classical result that $\mathcal{M}_g$ has dimension $3g - 3$, and that for $g \geq 4$ its singular locus agrees with the branch locus of the canonical projection $T_g \to \mathcal{M}_g$; that is, the locus

[Igu60, BL04]	[Bro91]	$s(G)$	G
I	2.f	$(0; 2^5)$	$\mathbf{C}_2^2$
II	2.n	$(0; 2^3, 4)$	$\mathbf{D}_4$
VI	2.o	$(0; 2, 5, 10)$	$\mathbf{C}_{10}$
III	2.s	$(0; 2^3, 3)$	$\mathbf{D}_6$
IV	2.w	$(0; 2, 4, 6)$	$\mathbf{C}_3 \rtimes \mathbf{D}_4$
V	2.aa	$(0; 2, 3, 8)$	$\mathbf{GL}(2, 3)$

Table 1. Nontrivial group actions on genus 2 (other than C_2).

of Riemann surfaces of genus g with nontrivial automorphisms.

In consequence, to study compact Riemann surfaces with automorphisms turns out to be of interest. In [Bro90] *the equisymmetric locus* $\mathcal{M}_g(H, s, v)$ is introduced: it consists of the points in $\mathcal{M}_g$ representing isomorphism classes of Riemann surfaces endowed with an action of the abstract group H with fixed signature s and fixed generating vector v, see Section 1 for these definitions. These loci are closed irreducible (not necessarily smooth) subvarieties of $\mathcal{M}_g$, and sometimes they fail to be normal.

Some results about the geometry of these loci are known for low genera. For instance, in [Igu60, §8], [BL04, §11.7] or [Bro91] a complete list of the full automorphism groups for genus 2 is given (larger than just the hyperelliptic involution). We collect these data in Table 1, where $\mathbf{C}_n$ and $\mathbf{D}_n$ are the cyclic group of order n and the Dihedral group of order $2n$ respectively.

Notice that, as pointed out in [Igu60, §8], all cases except VI are specializations of case I, and VI corresponds to the only isolated singularity of $\mathcal{M}_2$. A picture is drawn in Fig. 5.

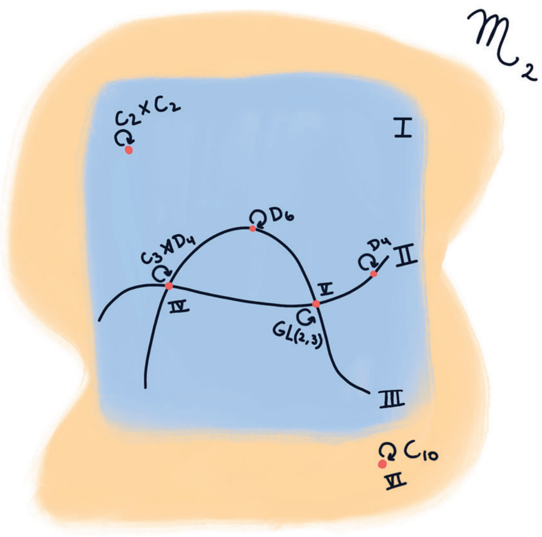

Figure 5. Families with nontrivial automorphisms in $\mathcal{M}_2$.

4. The Group Algebra Decomposition (GAD)

We have sketched the *Torelli* map $\mathcal{J} : \mathcal{M}_g \to \mathcal{A}_g$ defined by associating the corresponding Jacobian variety, which is injective by Torelli's theorem. We have also described the singular locus for both moduli spaces. We will now look at them more closely.

It is a classic result, see for instance [Wol02, Lemma 6] and the references given there, that the automorphisms of a surface S and its Jacobian JS are the same in the hyperelliptic case, and $[\mathrm{Aut}(JS) : \mathrm{Aut}(S)] = 2$ if S is not hyperelliptic. Besides, once one has a group action, all the tools from algebra, including Representation Theory of associative algebras and finite groups, come into play.

Let A be a ppav with the action of a finite group G; that is, there is a monomorphism $\rho : G \to \mathrm{Aut}(A)$. It induces a morphism $\rho : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(A)$ of the rational group algebra $\mathbb{Q}[G]$ into the endomorphism algebra $\mathrm{End}_{\mathbb{Q}}(A)$. Any element $\alpha \in \mathbb{Q}[G]$ defines an abelian subvariety $\alpha(A) := \mathrm{Im}(m\rho(\alpha)) \subset A$, where m is some positive integer chosen so that $m\rho(\alpha) \in \mathrm{End}(A)$. This definition does not depend on the chosen integer m.

Consider the decomposition of $\mathbb{Q}[G] = Q_1 \times \cdots \times Q_r$ into a product of simple $\mathbb{Q}$-algebras Q_i; here r is the number of $\mathbb{Q}$-irreducible representations of the group (up to equivalence). Each factor Q_i is determined by one of the rational irreducible representations $\{W_1, \dots, W_r\}$, in the sense that Q_i is generated by the central idempotent e_i corresponding to W_i.

The corresponding decomposition of $1 \in \mathbb{Q}[G]$ as $1 = e_1 + \cdots + e_r$, induces an isogeny

$$e_1(A) \times \cdots \times e_r(A) \to A \tag{4.1}$$

given by addition. The components $A_i := e_i(A)$ are G-stable abelian subvarieties of A with $\mathrm{Hom}_G(A_i, A_j) = 0$ for $i \neq j$. Note that it is not necessarily true that $\mathrm{Hom}(A_i, A_j) = 0$ for $i \neq j$. This decomposition is called the *isotypical decomposition* of the complex abelian variety A with G-action.

The isotypical components A_i can be decomposed further, using the decomposition of Q_i into a product of (isomorphic) minimal left ideals.

Let W_i be the irreducible rational representation of G corresponding to e_i, U_i one of the irreducible $\mathbb{C}$-representations associated to W_i (that is, a component of $W_i \otimes \mathbb{C}$), and m_i the Schur index of U_i. Set $n_i = \frac{\dim U_i}{m_i}$; then there is a set of primitive idempotents $\{f_{i1}, \cdots, f_{in_i}\}$ in $Q_i \subset \mathbb{Q}[G]$ such that $e_i = f_{i1} + \cdots + f_{in_i}$. Moreover, for each fixed i the abelian subvarieties $f_{ij}(A)$ are mutually isogenous for $j = 1, \dots, n_i$. Call any one of these isogenous factors B_i. Then $B_i^{n_i} \to e_i(A)$ is an isogeny for every $i = 1, \dots, r$. Replacing the factors in (4.1) we get a *group* *algebra decomposition* (GAD) of the G-abelian variety A

$$B_1^{n_1} \times \cdots \times B_r^{n_r} \to A. \tag{4.2}$$

Note that, whereas (4.1) is uniquely determined, (4.2) is not. It depends on the choice of the f_{ij} as well as on the choice of the B_i. See [LR04, CR06] for details.

While the dimensions of the factors will remain fixed regardless of these choices, their induced polarization and the kernel of the isogeny in (4.2) may change. The factors in (4.2) are called *primitive factors*, since they are images of primitive idempotents; they can be simple or not, depending on A and on the action of G. Moreover, they are not, in general, principally polarized.

In the case of $A = JS$ being a Jacobian variety, the geometric information about the group acting on the corresponding Riemann surface S allows us to describe the isotypical and the primitive factors on these decompositions of JS in terms of abelian varieties constructed from Riemann surfaces obtained by factoring S by appropriate subgroups of G. The dimensions of the primitive factors (hence also the isotypical ones) in terms of the *geometric signature* (data providing information somehow between the signature and the generating vector) is described in [Roj07], and the induced polarization on the isotypical factors (under some conditions) in [LR12, LRR14]. For a complete exposition about decomposition of Jacobian (and Prym) varieties, key results as well as several group actions and applications, see [LR22].

Although some facts about the isotypical and the GAD are known, other crucial aspects still remain unclear; such as whether the primitive factors are simple, or the existence of isogenies between them, or whether (or when) these decomposition coincide with the Poincaré decomposition of the variety.

So these decompositions open the door to use algebra, and computer programs, to tackle questions about abelian varieties, Riemann surfaces and their moduli spaces. We want to end this article by listing some of the currently open questions regarding these subjects.

5. Some Open Questions and Problems

We expect that by understanding GAD better, it could become a powerful tool towards answering questions such as

- [ES93]: Is it true that for every $g > 0$, there exists a a genus$-g$ curve whose Jacobian variety is completely decomposable? If not, are there only finitely many such g, and what is the largest? Does there exist a curve of genus $g > 3$ whose Jacobian variety is strictly completely decomposable; that is, isomorphic (rather than just isogenous) to a product of elliptic curves)?
- [MO13, Question 6.7]: For which $g \geq 2$ does there exist a positive-dimensional subvariety Z of the Torelli locus such that the abelian variety corresponding to the generic geometric point of Z is isogenous to a product of elliptic curves? The expectation is that for $g >> 0$ no such subvariety exists. This is closely related to Coleman's conjecture.
- [Col87]: (Coleman's conjecture) Given $g \geq 4$, there are only finitely many nonsingular projective genus g curves C over $\mathbb{C}$, up to isomorphism, such that the Jacobian JC is a CM abelian variety. This is known to be false up to genus 7, but remains open for higher values. All the known examples [MO13, FPP16] arise from group actions.
- [Bea14]: Construction of curves whose Jacobian variety has maximal Picard number $\rho = h^{1,1}$. We recall [Bea14, Prop. 3] that an abelian variety A of dimension g is *ρ-maximal* if it is isogeous to E^g, with E an elliptic curve with complex multiplication. Moreover, this is the case if and only if A is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication, and if and only if the rank over the integers of $\mathrm{End}(A)$ is $2g^2$. In [Bea14] it is pointed out that few examples of curves with such Jacobian varieties are known.
- [Bro90] To describe the equisymmetric stratification by the subvarieties $\mathcal{M}_g(H, s, v)$ of the singular locus of $\mathcal{M}_g$, and, using the Torelli map, to give structure theorems for the Jacobian varieties associated to each stratum.
- To describe or characterize the non-normal subvarieties $\mathcal{M}_g(H, s, v)$ for $g \geq 4$.

ACKNOWLEDGMENT. We would like to thank the anonymous referees for a number of valuable suggestions which certainly improved our exposition.

References

[ALR17] R. Auffarth, H. Lange, and A. M. Rojas, *A criterion for an abelian variety to be non-simple*, J. Pure Appl. Algebra **221** (2017), no. 8, 1906–1925, DOI 10.1016/j.jpaa.2016.10.020. MR3623174

[Bea14] Arnaud Beauville, *Some surfaces with maximal Picard number* (English, with English and French summaries), J. Éc. polytech. Math. **1** (2014), 101–116, DOI 10.5802/jep.5. MR3322784

[BL04] Christina Birkenhake and Herbert Lange, *Complex abelian varieties*, 2nd ed., Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 302, Springer-Verlag, Berlin, 2004, DOI 10.1007/978-3-662-06307-1. MR2062673

[Bro90] S. Allen Broughton, *The equisymmetric stratification of the moduli space and the Krull dimension of mapping class groups*, Topology Appl. **37** (1990), no. 2, 101–113, DOI 10.1016/0166-8641(90)90055-7. MR1080344

[Bro91] S. Allen Broughton, *Classifying finite group actions on surfaces of low genus*, J. Pure Appl. Algebra **69** (1991), no. 3, 233–270, DOI 10.1016/0022-4049(91)90021-S. MR1090743

[CR06] Angel Carocca and Rubí E. Rodríguez, *Jacobians with group actions and rational idempotents*, J. Algebra **306** (2006), no. 2, 322–343, DOI 10.1016/j.jalgebra.2006.07.027. MR2271338

[Col87] Robert F. Coleman, *Torsion points on curves*, Galois representations and arithmetic algebraic geometry (Kyoto, 1985/Tokyo, 1986), Adv. Stud. Pure Math., vol. 12, North-Holland, Amsterdam, 1987, pp. 235–247, DOI 10.2969/aspm/01210235. MR948246

[ES93] Torsten Ekedahl and Jean-Pierre Serre, *Exemples de courbes algébriques à jacobienne complètement décomposable* (French, with English and French summaries), C. R. Acad. Sci. Paris Sér. I Math. **317** (1993), no. 5, 509–513. MR1239039

[FK92] H. M. Farkas and I. Kra, *Riemann surfaces*, 2nd ed., Graduate Texts in Mathematics, vol. 71, Springer-Verlag, New York, 1992, DOI 10.1007/978-1-4612-2034-3. MR1139765

[FPP16] Paola Frediani, Matteo Penegini, and Paola Porru, *Shimura varieties in the Torelli locus via Galois coverings of elliptic curves*, Geom. Dedicata **181** (2016), 177–192, DOI 10.1007/s10711-015-0118-0. MR3475744

[Igu60] Jun-ichi Igusa, *Arithmetic variety of moduli for genus two*, Ann. of Math. (2) **72** (1960), 612–649, DOI 10.2307/1970233. MR114819

[Lan82] Serge Lang, *Introduction to algebraic and abelian functions*, 2nd ed., Graduate Texts in Mathematics, vol. 89, Springer-Verlag, New York-Berlin, 1982. MR681120

[LR04] H. Lange and S. Recillas, *Abelian varieties with group action*, J. Reine Angew. Math. **575** (2004), 135–155, DOI 10.1515/crll.2004.076. MR2097550

[LR22] Herbert Lange and Rubí E. Rodríguez, *Decomposition of Jacobians by Prym varieties*, Lecture Notes in Mathematics, vol. 2310, Springer, Cham, 2022, DOI 10.1007/978-3-031-10145-8. MR4559690

[LRR14] Herbert Lange, Rubí E. Rodríguez, and Anita M. Rojas, *Polarizations on abelian subvarieties of principally polarized abelian varieties with dihedral group actions*, Math. Z. **276** (2014), no. 1-2, 397–420, DOI 10.1007/s00209-013-1206-1. MR3150211

[LR12] Herbert Lange and Anita M. Rojas, *Polarizations of isotypical components of Jacobians with group action*, Arch. Math. (Basel) **98** (2012), no. 6, 513–526, DOI 10.1007/s00013-012-0400-4. MR2935657

[MO13] Ben Moonen and Frans Oort, *The Torelli locus and special subvarieties*, Handbook of moduli. Vol. II, Adv. Lect. Math. (ALM), vol. 25, Int. Press, Somerville, MA, 2013, pp. 549–594. MR3184184

[Oh22] Hee Oh, *Euclidean traveller in hyperbolic worlds*, Notices Amer. Math. Soc. **69** (2022), no. 11, 1888–1897. MR4524309

[Rod14] Rubí E. Rodríguez, *Abelian varieties and group actions*, Riemann and Klein surfaces, automorphisms, symmetries and moduli spaces, Contemp. Math., vol. 629, Amer. Math. Soc., Providence, RI, 2014, pp. 299–314, DOI 10.1090/conm/629/12561. MR3289648

[Roj07] Anita M. Rojas, *Group actions on Jacobian varieties*, Rev. Mat. Iberoam. **23** (2007), no. 2, 397–420, DOI 10.4171/RMI/500. MR2371432

[Wol02] Jürgen Wolfart, *Regular dessins, endomorphisms of Jacobians, and transcendence*, A panorama of number theory or the view from Baker's garden (Zürich, 1999), Cambridge Univ. Press, Cambridge, 2002, pp. 107–120, DOI 10.1017/CBO9780511542961.008. MR1975447

Rubí E. Rodríguez

Anita M. Rojas

Credits

Figures 1, 4, and 5 are courtesy of Anita M. Rojas's eldest daughter Lía Da Silva.

Figures 2 and 3 are courtesy of Anita M. Rojas's stepson Cristóbal Da Silva.

Photos of the authors are courtesy of Geir Da Silva.