① $\langle v \rangle = \dfrac{\int_0^\infty v f(v)\,dv}{\int_0^\infty f(v)\,dv} = \dfrac{\cancel{A}\int_0^\infty v^3 e^{-\alpha v^2}\,dv}{\cancel{A}\int_0^\infty v^2 e^{-\alpha v^2}\,dv}$

(i) $\int_0^\infty v^2 e^{-\alpha v^2}\,dv = -\dfrac{d}{d\alpha}\Big(\underbrace{\int_0^\infty e^{-\alpha v^2}\,dv}_{\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}}\Big) = -\frac{1}{2}\sqrt{\pi}\,\dfrac{d}{d\alpha}\,\alpha^{-1/2}$

$= \boxed{\frac{1}{4}\sqrt{\pi}\,\alpha^{-3/2}}$

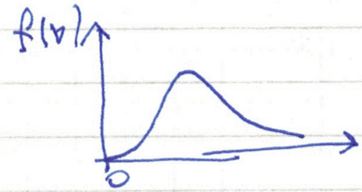
(ii) $\int_0^\infty v^3 e^{-\alpha v^2}\,dv = \int_0^\infty v^2 e^{-\alpha v^2}\,\dfrac{d(v^2)}{2} = \frac{1}{2}\int_0^\infty x e^{-\alpha x}\,dx = \frac{1}{2\alpha^2}\underbrace{\int_0^\infty u e^{-u}\,du}_{1}$

$= \boxed{1/2\alpha^2}$

$\langle v \rangle = \dfrac{(1/2\alpha^2)}{\left(\frac{\sqrt{\pi}}{4}\right)\alpha^{-3/2}} = \dfrac{4\alpha^{3/2}}{\sqrt{\pi}}\cdot\dfrac{1}{2\alpha^2} = \dfrac{2}{\sqrt{\pi}}\dfrac{1}{\sqrt{\alpha}}$

$= \dfrac{2}{\sqrt{\pi}}\sqrt{\dfrac{2KT}{m}} = \sqrt{\dfrac{8KT}{\pi m}}$ //

② $f(v) = A v^2 e^{-\alpha v^2}$



$0 = f'(v) = 2v e^{-\alpha v^2} + v^2 e^{-\alpha v^2}(-2\alpha v) = e^{-\alpha v^2}\left[2v - 2\alpha v^3\right]$

$\Rightarrow \cancel{2}\alpha v^3 = \cancel{2}v \Rightarrow \alpha v^2 = 1 \Rightarrow v = \dfrac{1}{\sqrt{\alpha}} = \sqrt{\dfrac{2KT}{m}}$ //

(3) $$\langle v^2\rangle = \frac{\int_0^\infty A v^4 e^{-\alpha v^2}\,dv}{\int_0^\infty A v^2 e^{-\alpha v^2}\,dv} = \frac{\int_0^\infty v^4 e^{-\alpha v^2}\,dv}{\frac{\sqrt{\pi}}{4}\alpha^{-3/2}}$$

$$\int_0^\infty v^4 e^{-\alpha v^2}\,dv = \frac{d^2}{d\alpha^2}\left(\int_0^\infty e^{-\alpha v^2}\,dv\right) = \frac{d^2}{d\alpha^2}\left(\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\right) = \frac{\sqrt{\pi}}{2}\frac{d^2}{d\alpha^2}\left(\alpha^{-1/2}\right)$$

$$= \frac{\sqrt{\pi}}{2}\frac{d}{d\alpha}\left(-\frac{1}{2}\alpha^{-3/2}\right) = -\frac{\sqrt{\pi}}{4}\frac{d}{d\alpha}\left(\alpha^{-3/2}\right) = -\frac{\sqrt{\pi}}{4}\cdot\left(-\frac{3}{2}\right)\alpha^{-5/2}$$

$$= \frac{3}{8}\sqrt{\pi}\,\alpha^{-5/2}$$

$$\therefore \langle v^2\rangle = \frac{(3/8)\cancel{\sqrt{\pi}}\,\alpha^{-5/2}}{\cancel{(\sqrt{\pi}/4)}\,\alpha^{-3/2}} = \frac{3}{2}\alpha^{-1} = \frac{3}{2\alpha} = \frac{3}{\cancel{2}}\frac{\cancel{2}kT}{m} =$$

$$\Rightarrow \sqrt{\langle v^2\rangle} = \sqrt{3\frac{kT}{m}}$$

$$v_{mp} < \langle v\rangle < \sqrt{\langle v^2\rangle}.$$

Integrals

$$I_n = \int_0^\infty v^n e^{-\alpha v^2} dv \qquad (1)$$

<u>$n = 2p+1$</u>

$$I_n = \int_0^\infty v^{2p} v\, e^{-\alpha v^2} dv = \frac{1}{2}\int_0^\infty (v^2)^p e^{-\alpha (v^2)} d(v^2) = \frac{1}{2}\int_0^\infty x^p e^{-\alpha x} dx$$

$$= \frac{1}{2\alpha^{p+1}} \cdot \underbrace{\int_0^\infty s^p e^{-s} ds}_{p!} = \boxed{\frac{p!}{2\alpha^{p+1}}} \qquad (2)$$

<u>$n = 2p$</u>

$$I_n = \int_0^\infty v^{2p} e^{-\alpha v^2} dv = (-1)^p \frac{d^p}{d\alpha^p} \underbrace{\int_0^\infty e^{-\alpha v^2} dv}_{\frac{1}{\sqrt{\alpha}} \int_0^\infty e^{-x^2} dx}$$

$$I \equiv \int_0^\infty e^{-x^2} dx$$

$$I^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)} dx\,dy$$

$$r^2 = x^2 + y^2$$

$$r\,dr\,d\phi = dx\,dy$$

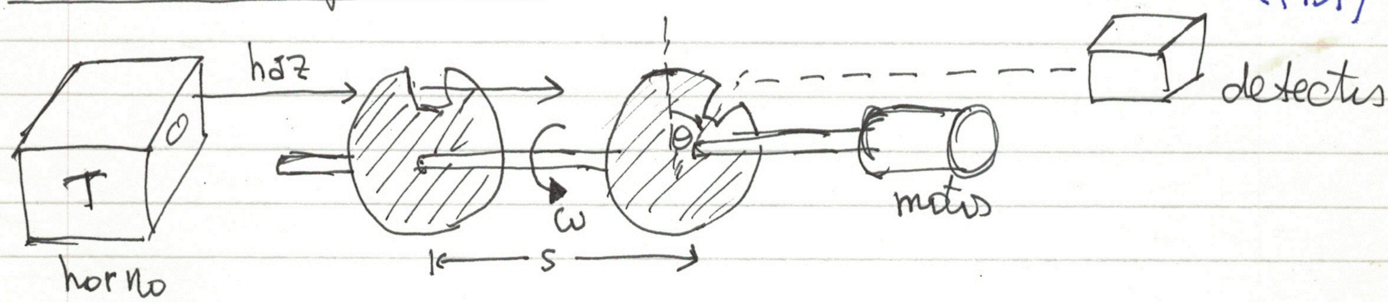
$$I^2 = \int_0^{\pi/2} d\phi \int_0^{\infty} dr\, r e^{-r^2} = \frac{\pi}{2} \int_0^{\infty} r e^{-r^2} dr =$$

$$= \frac{\pi}{4} \int_0^{\infty} e^{-r^2} dr^2 = \frac{\pi}{4} \underbrace{\int_0^{\infty} e^{-s} ds}_{1} = \frac{\pi}{4}$$

$$\boxed{I = \frac{\sqrt{\pi}}{2}}$$

$$\therefore \boxed{I_n = (-1)^p \frac{\sqrt{\pi}}{2} \frac{d^p}{d\alpha^p}\left(\alpha^{-1/2}\right)} \qquad (n = 2p).$$

comprobac. experimental (J.F. Zartman, 1931) PR 37, 383-39 (1931)



para ω dado, sólo pasan al detector aquellas moléculas que cumplan: $t = \frac{s}{v} = \frac{\theta}{\omega}$

$\Rightarrow v = \frac{s\theta}{\omega}$. Variando ω ó θ, se puede medir directam. el # de moléculas en un rango de v dado.

Los resultados concuerdan con M-B

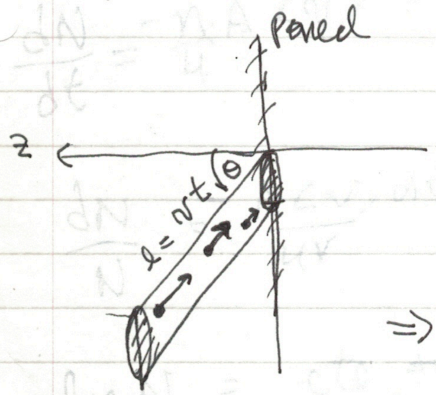
Ejercicio: todo **M-B** calculos: como fn de T:

(a) $\bar{v}$ (rapidez promedio) $= \sqrt{\frac{8k_BT}{\pi m}}$

(b) rapidez más probable v_{mp}

(c) veloc. cuadrática media: $\sqrt{\overline{v^2}}$

Calcular la tasa de escape de moléculas desde una pequeña abertura de área A, en un contenedor.



$$\Delta N = nAvt\cos\theta\, f(\vec{v})$$

$$\frac{\Delta N}{A\Delta t} = n\, v\cos\theta\, f(\vec{v})$$

$$\Rightarrow \text{flujo} = \int n v\cos\theta\, f(\vec{v})\, d^3v$$

$$= n\left(\frac{m}{2\pi kT}\right)^{3/2}\iiint e^{-\frac{mv^2}{2kT}} v^3 \cos\theta \sin\theta\, dv\, d\theta\, d\phi$$

$$= 2\pi n\left(\frac{m}{2\pi kT}\right)^{3/2}\underbrace{\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta}_{1/2}\ \underbrace{\int_0^{\infty} v^3 e^{-\frac{mv^2}{2kT}}\, dv}_{\underbrace{1/(2m/2kT)^2}_{2\left(\frac{kT}{m}\right)^2}} = n\sqrt{\frac{kT}{2\pi m}}$$

$$= \frac{n}{4}\bar{v}$$

$$\frac{dN}{dt} = -\frac{n}{4} A \langle v \rangle = -\frac{NA}{4V} \langle v \rangle$$

$$\frac{dN}{N} = -\frac{A \langle v \rangle}{4V} dt$$

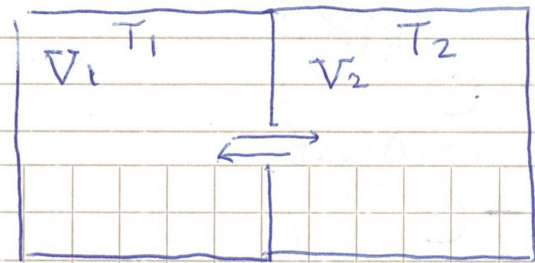
$$\ln N = cte - \frac{A \langle v \rangle t}{4V}$$

$$\boxed{N = N(0)\, e^{-\frac{A \langle v \rangle t}{4V}}}$$

$$P = P(0)\, e^{-\frac{A \langle v \rangle t}{4V}}$$

$$\boxed{P = P(0)\, e^{-\frac{A}{V}\sqrt{\frac{kT}{2\pi m}}\, t}}$$

Ej: calc. tiempo de vaciado.



$$\frac{dN_1}{dt} = -A_1\left(\frac{N_1}{V_1}\right)\frac{\langle v\rangle_1}{4} + A_2\left(\frac{N_2}{V_2}\right)\frac{\langle v\rangle_2}{4}$$

$$\frac{dN_2}{dt} = -A_2\left(\frac{N_2}{V_2}\right)\frac{\langle v\rangle_2}{4} + A_1\left(\frac{N_1}{V_1}\right)\frac{\langle v\rangle_1}{4}$$

Sup: $A_1 = A_2$, $V_1 = V_2$ y $T_1 = T_2$

$$\Rightarrow \begin{array}{l} \dot N_1 = -\alpha N_1 + \alpha N_2 \\ \dot N_2 = \alpha N_1 - \alpha N_2 \end{array} \qquad \begin{array}{l} N_1(0) = N_0 \\ N_2(0) = 0 \end{array}$$

$$\Rightarrow \dot N_1 + \dot N_2 = 0 \Rightarrow \frac{d}{dt}(N_1+N_2) = 0 \Rightarrow \boxed{N_1 + N_2 = N_0} \quad (1)$$

$$\Rightarrow N_2 = N_0 - N_1$$

$$\Rightarrow \dot N_1 = -\alpha N_1 + \alpha(N_0 - N_1)$$
$$= -2\alpha N_1 + \alpha N_0$$

$$\frac{d}{dt}N_1 = \alpha N_0 - 2\alpha N_1 \Rightarrow N_1 = \frac{N_0}{2} + A e^{-2\alpha t}$$

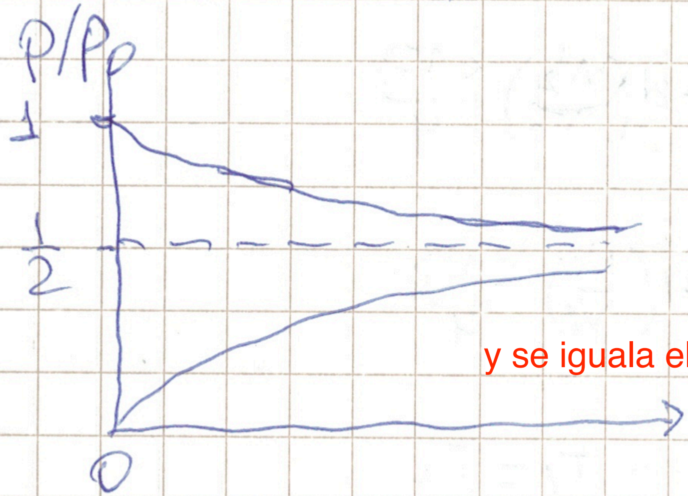
y usando $N_1(0) = N_0 \Rightarrow \boxed{N_1(t) = \frac{N_0}{2}\left(1 + e^{-2\alpha t}\right)}$

$$N_2 = N_0 - N_1 = N_0 - \frac{N_0}{2} - \frac{N_0}{2}e^{-2\alpha t} = \frac{N_0}{2} - \frac{N_0}{2}e^{-2\alpha t}$$

$$\Rightarrow \boxed{N_2(t) = \frac{N_0}{2}\left(1 - e^{-2\alpha t}\right)}$$

$$\Rightarrow P_1(t) = \frac{P_1(0)}{2}\left(1 + e^{-\frac{2A\langle v\rangle}{4V}t}\right)$$

$$P_2(t) = \frac{P_1(0)}{2}\left(1 - e^{-\frac{2A\langle v\rangle}{4V}t}\right)$$



se equilibran las presiones

y se iguala el numero de particulas en ambos containers