The Fractional Discrete Nonlinear Schrodinger Equation

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Fractional Calculus

L' Hopital, Leibniz,....

Is there an operator H such that $H^2 f(x) = f'(x)$? i.e., a "half derivative"

$$f(x) = x^k, \text{ then } \frac{d^n}{dx^n} x^k$$

$$= \frac{k!}{(k-n)!} x^{k-n} = \frac{\Gamma(k+1)}{\Gamma(k-n+1)} x^{k-n}$$

Take n (integer) → a (real)

Fractional Calculus

$$\frac{d^a}{dx^a}x^k = \frac{\Gamma(k+1)}{\Gamma(k-a+1)}x^{k-a}$$

Example k=1, a=1/2

$$\frac{d^{1/2}}{dx^{1/2}}x = \frac{\Gamma(2)}{\Gamma(3/2)}\sqrt{x} = \frac{2}{\sqrt{\pi}}\sqrt{x}$$

y se cumple $\frac{dx}{dx} = \frac{d^{1/2}}{dx^{1/2}}\left(\frac{d^{1/2}x}{dx^{1/2}}\right) =$

$$= \frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}x^{1/2} = \frac{2}{\sqrt{\pi}}\frac{\Gamma(3/2)}{\Gamma(1)} = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1$$

Fractional Calculus

So, in principle for $f(x) = \sum_n c_n\, x^n$, one could compute $(d^a/dx^a) f(x)$ by deriving each term in the series

There are some problems, however

$$(d^a/dx^a)1 = (d^a x^0/dx^a) = (1/\Gamma(1-a))\; x^{-a} \neq 0$$

weird

We could have computed in different order

$$(d^a/dx^a)1 = (d^{a-1}/dx^{a-1})(d/dx)1 = 0$$

better

How do we interpret this?

Fractional Calculus

How about the fractional integral ?

$$I_x^1 f(x) = \int_0^x f(s)\ ds$$

Laplace

$$\mathcal{L}\{I_x^1\ f(x)\} = (1/s)\ \mathcal{L}\{f(x)\}$$

$$\mathcal{L}\{I_x^n\ f(x)\} = (1/s^n)\ \mathcal{L}\{f(x)\}$$

$$n \rightarrow a$$

$$\mathcal{L}\{I_x^a\ f(x)\} = (1/s^a)\ \mathcal{L}\{f(x)\}$$

convolution

$$I_x^a f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(s)}{(x-s)^{1-\alpha}}\ ds$$

Now we can define a fractional derivative

Fractional Calculus

$$\frac{d^\alpha}{dx^\alpha} f(x) = \left(\frac{d^m}{dx^m}\right) \; I_x^{m-\alpha} \; f(x)$$

$$m - 1 < \alpha < m$$

$$= \frac{d^m}{dx^m} \left[\frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f(s)}{(x-s)^{1+\alpha-m}} \; ds \right]$$

Riemann-Liouville form

$$\frac{d^\alpha}{dx^\alpha} f(x) = I_x^{m-\alpha} \left(\frac{d^m}{dx^m}\right) \;\; f(x)$$

$$= \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(s)}{(x-s)^{1+\alpha-m}} \; ds$$

Caputo form

Fractional Laplacian

Tight-binding in 1D

$$i\frac{dC_n}{dt} + V(C_{n+1} + C_{n-1}) + \chi\,|C_n|^2 C_n = 0$$ DNLS

$$\Delta_n C_n \equiv C_{n+1} - 2C_n + C_{n-1}$$ discrete Laplacian

$$i\frac{dC_n}{dt} + 2V C_n + V\,\Delta_n C_n + \chi\,|C_n|^2 C_n = 0.$$

Using L. Rocal et al. (2018)

$$(-\Delta_n)^s C_n = \sum_{m \neq n} K^s(n-m)(C_n - C_m), \qquad 0 < s < 1$$

Fractional Laplacian

$$K^s(m) = L_s \, \frac{\Gamma(|m| - s)}{\Gamma(|m| + 1 + s)}.$$

kernel

$$C_n(t) = \exp(i\lambda t)\ \phi_n$$

$$(-\lambda + 2V)\,\phi_n + V \sum_{m \neq n} K^s(n-m)(\phi_m - \phi_n) + \chi\,\phi_n^3 = 0$$

$$\chi = 0 \text{ and } \phi_n = A \exp(ikn)$$

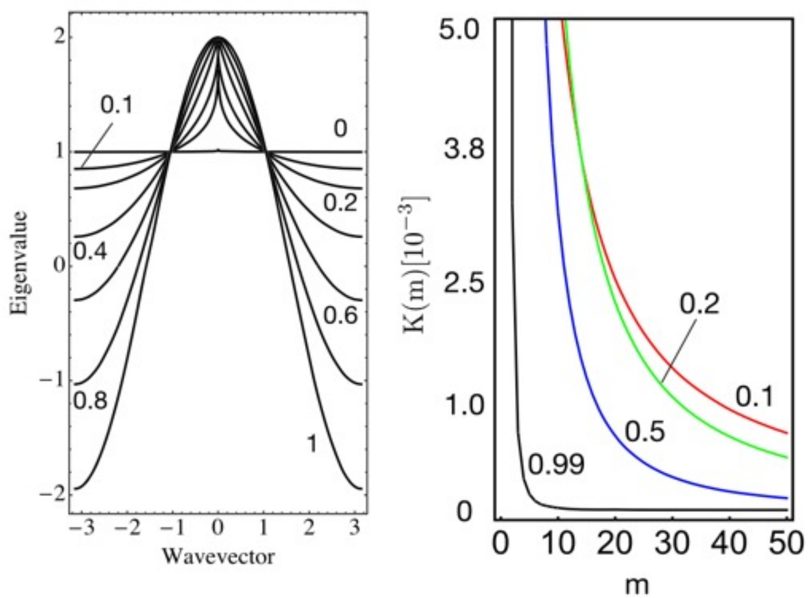
$$\lambda(k) = 2V - 4V \sum_{m=1}^{\infty} K^s(m) \sin((1/2)mk)^2$$

Dispersion relation

Dispersion Relation

$$\lambda(k) = 2 - \frac{16\,\Gamma(s+(1/2))}{\sqrt{\pi}\;\Gamma(1+s)}$$
$$\times\Big(1-\exp(-ik)\,s\,\Gamma(1+s)[\,R(1,1-s,2+s;\exp(-ik))$$
$$+\exp(2ik)\,R(1,1-s,2+s;\exp(ik))\,]\Big)$$

where $R(a,b,c;z) = {}_2F_1(a,b,c;z)/\Gamma(c)$ is the regularized hypergeometric function.



Dispersion Relation

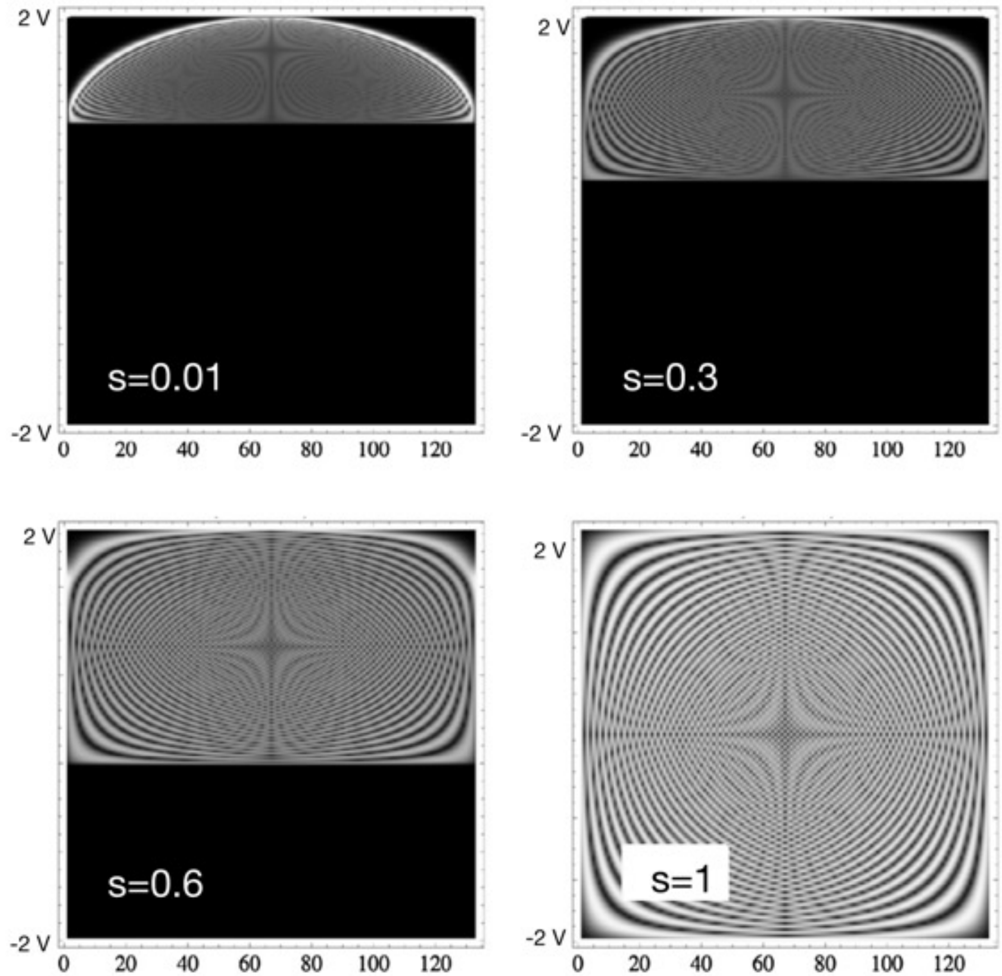
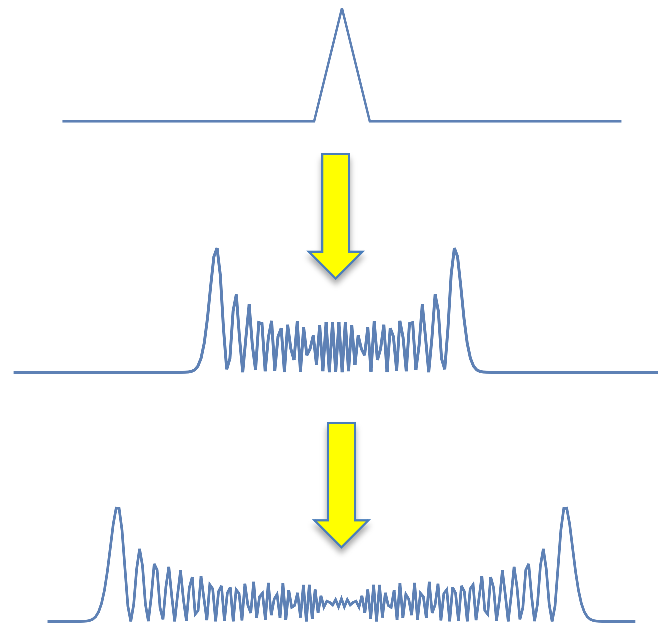


Fig. 2. Density plot of the spatial profiles $|\phi_n|^2$ of the linear modes ordered according to their eigenvalue. For $s \sim 0$ ($s \sim 1$) the bandwidth is V ($4V$) ($N = 133$).

Mean Square Displacement

$$\langle n^2 \rangle = \sum_n n^2 |\phi_n(t)|^2 / \sum_n |\phi_n(t)|^2$$



$$\langle n^2 \rangle = \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{d\lambda(k)}{dk} \right)^2 dk \right] (Vt)^2,$$

$$\frac{\langle n^2 \rangle}{(Vt)^2} = \left(\frac{1}{\pi} \right) 2^{4s-1}\, s\, \Gamma(s + (1/2))^2 \times \left\{ \frac{1}{\Gamma(1+s)^2} + \frac{8s(s-1)^2}{\Gamma(3+s)^2}\; {}_pF_q(\{a\}, \{b\}, 1) \right\}$$

where, $\{a\} = \{3, 2-s, 2-s\}$, $\{b\} = \{3+s, 3+s\}$ and ${}_pF_q(\{a\}, \{b\}; z)$ is the generalized hypergeometric function.

Nonlinear Fractional Modes

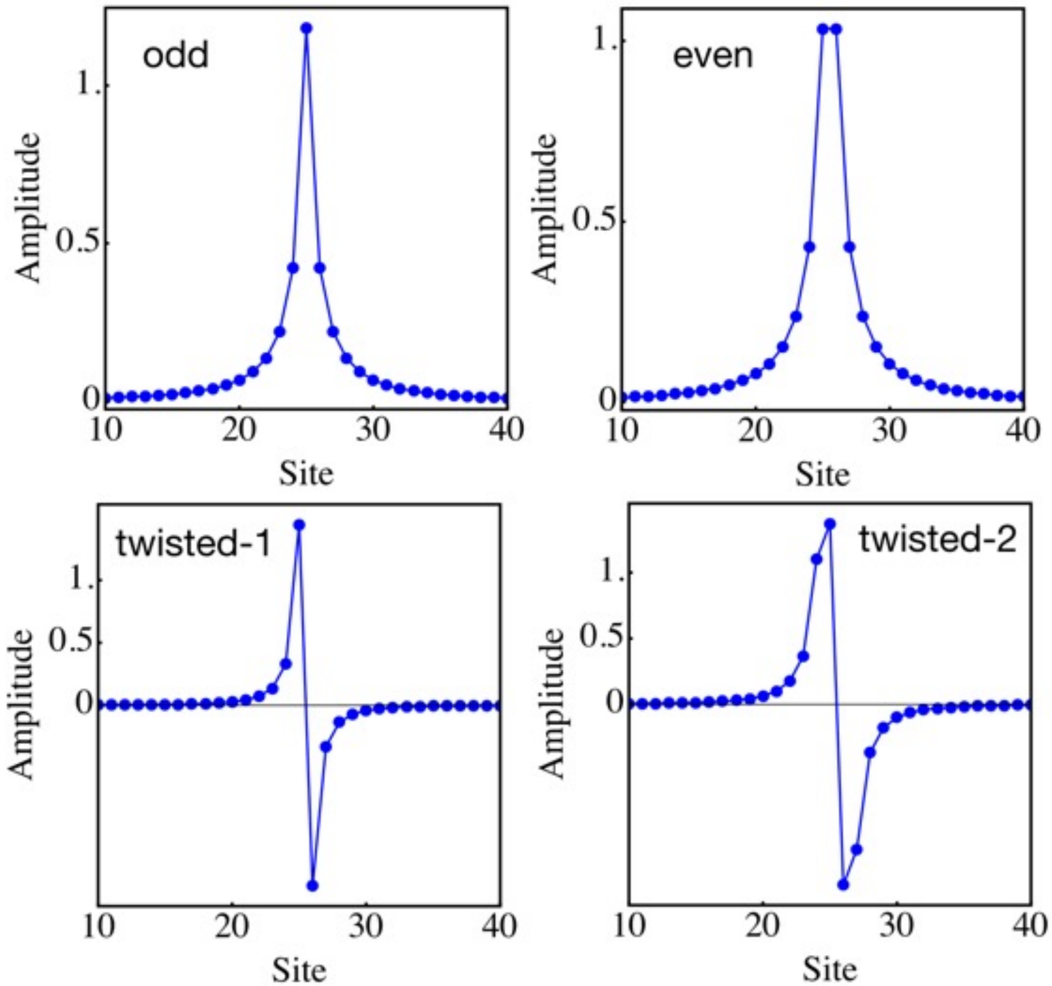
Nonlinear case: $\chi \neq 0$

$$(-\lambda + 2V)\,\phi_n + V \sum_{m \neq n} K^s(n-m)(\phi_m - \phi_n) + \chi\,\phi_n^3 = 0$$

System of N coupled nonlinear equations
Nonlocal coupling
Use multidimensional Newton-Raphson, starting from the decoupled limit
Obtain bulk and surface fractional discrete solitons

Nonlinear Fractional Modes

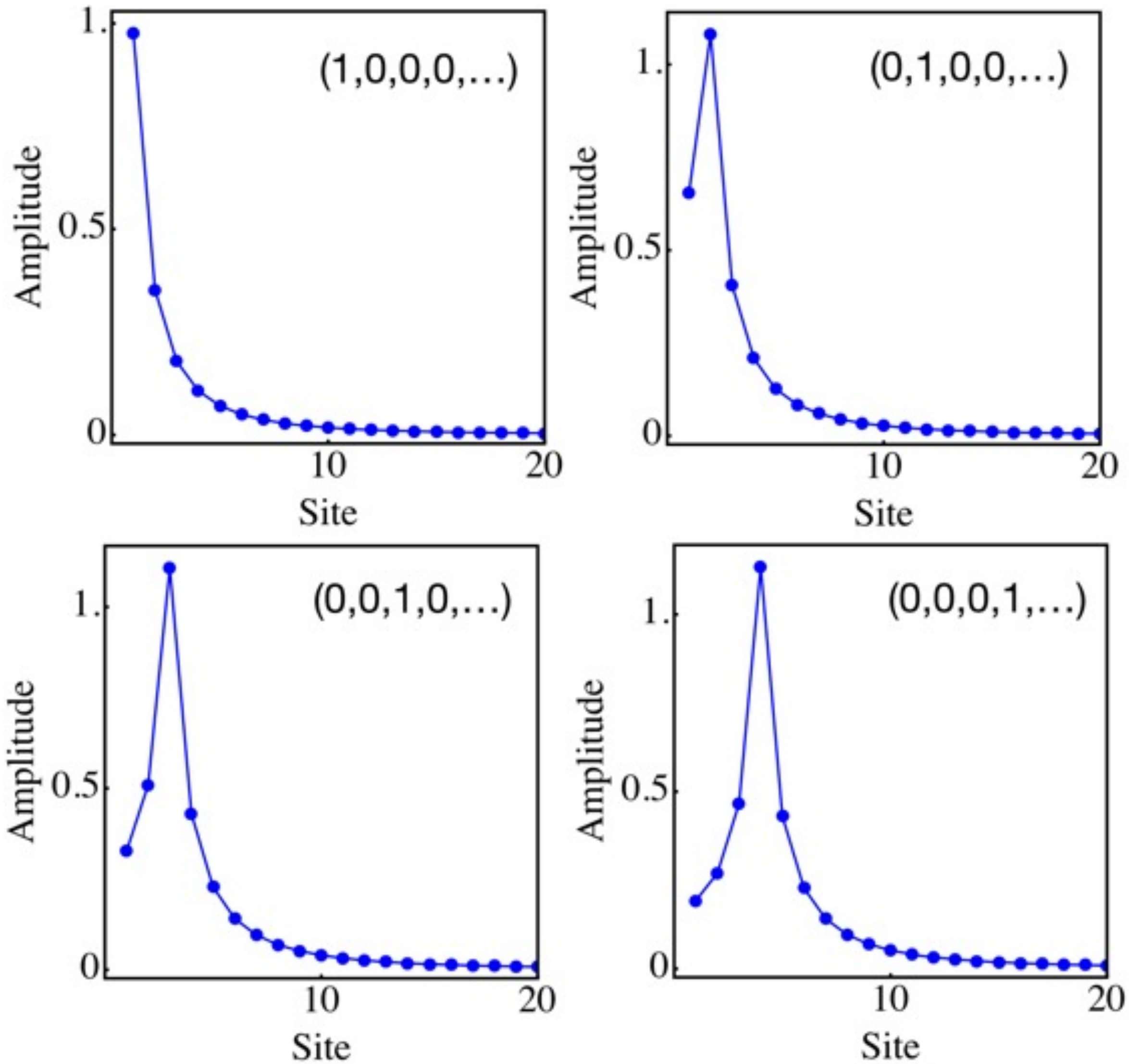
BULK



S=1/2
N=512
lambda=2.6

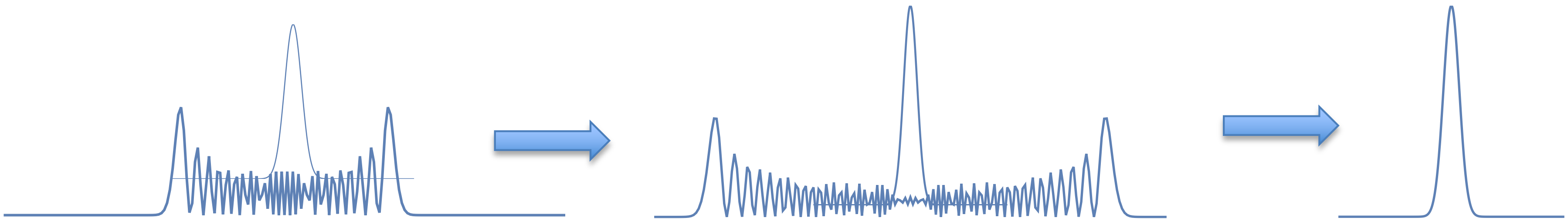
Nonlinear Fractional Modes

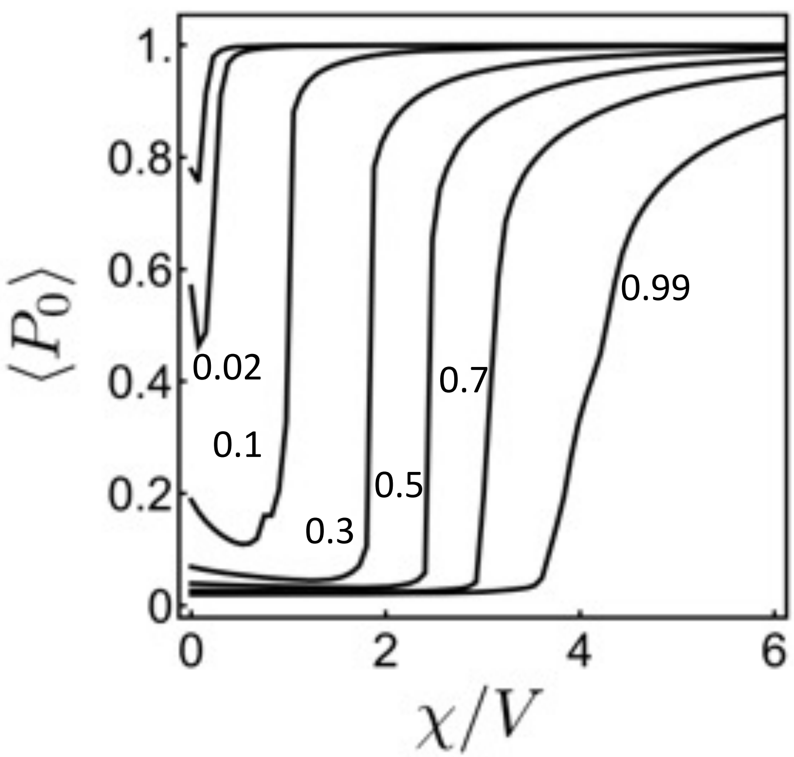
SURFACE



S=1/2
N=512
lambda=2.6

Fractional Selftrapping





$$\langle P_0 \rangle = \frac{1}{T} \int_0^T |C_0(t)|^2 dt$$

ST transition is maintained
Onset of linear trapping

SUMMARY

Effect of replacing the usual discrete laplacian by a fractional one

Linear case: linear wave spectra, intersite coupling and RMS computed in closed form. Bandwidth decreases with decreasing exponent, the RMS is always ballistic.

Nonlinear case: Computed bulk and surface modes and their stabilies as a function of fractional exponent. Selftrapping threshold shifts to lower values.

General phenomenology is more or less preserved → Discrete soliton concept robust against different mathematical extensions of Laplacian

Gracias por la atencion !!

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