Momento Relativista

Idea: generalizar las leyes de Newton y las defs. de momento y energía.

Sup. que 2 cuerpos colisionan en el sist. S, en ausencia de $\vec{F}$ externas → se conserva el momentum $\vec{p} = m_1\vec{u}_1 + m_2\vec{u}_2$.

* (con Lorentz)

Sin embargo al transf. a otro sist. * en movimiento S', se encuentra que $\vec{p}$ No es conservado.
$\vec{p}\,' = m_1\vec{u}_1{}' + m_2\vec{u}_2{}' \neq \vec{p}$

→ Re-definir el momento.

(* T. de Galileo works fine)

(i) El momentum relativista debe ser conservado en todas las colisiones.

(ii) El momentum relativista debe reducirse a valor newtoniano cuando $v/c \ll 1$.

La definición correcta es (ejercicio)

$$\vec{p} = m\gamma\vec{u} = \frac{m}{\sqrt{1-\left(\frac{u}{v}\right)^2}}\vec{u} \qquad (1)$$

La fuerza relativista está definida por

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad (2)$$

Energía relativista

↓ trabajo

$$W = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} \left(\frac{dp}{dt}\right) dx$$

pero $\left(\frac{dp}{dt}\right) dx = \left(\frac{dp}{du}\frac{du}{dt}\right) dx = \left(\frac{dp}{du}\right)\frac{du}{dx}\frac{dx}{dt}\,dx$

$$= \left(\frac{dp}{du}\right) u \frac{du}{dx}\,dx = \left(\frac{dp}{du}\right) u\,du$$

ahora $\frac{dp}{du} = \frac{d}{du}\frac{mu}{\left(1-\left(\frac{u}{c}\right)^2\right)^{1/2}} = \frac{m}{\sqrt{1-\left(\frac{u}{c}\right)^2}} + \frac{mu\,\frac{(-2)u}{c^2}}{\cancel{2}\left(1-\left(\frac{u}{c}\right)^2\right)^{3/2}}$

$$= \frac{m}{\left(1-\left(\frac{u}{c}\right)^2\right)^{3/2}}$$

$$\Rightarrow \quad W = \int_0^u \frac{mu}{\left(1-\left(\frac{u}{c}\right)^2\right)^{3/2}}\,du = mc^2\left(1-\left(\frac{u}{c}\right)^2\right)^{-1/2}\Bigg|_0^u$$

$$W = \frac{mc^2}{\sqrt{1-\left(\frac{u}{c}\right)^2}} - mc^2 \quad ✓ \qquad (3)$$

$= K$ [cambio en E. cinética de la partícula]

$\frac{u}{c} \ll 1$: $\quad W \simeq mc^2\left(1+\frac{1}{2}\left(\frac{u}{c}\right)^2 \cdots\right) - mc^2 = \frac{m}{2}u^2$ ✓ $\qquad (4)$

se puede escribir $K = \gamma mc^2 - mc^2$

→ energía en reposo

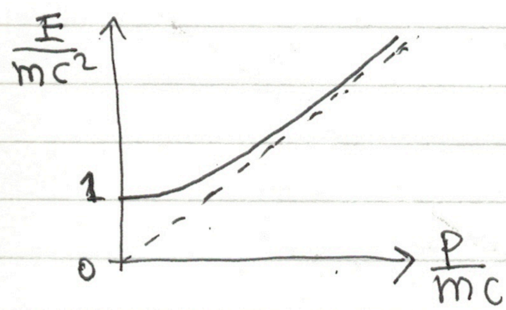
γmc^2 = energía en reposo + energía cinética

por lo tanto definimos $E = K + mc^2 = \gamma mc^2$

$$\therefore\ E = \frac{mc^2}{\sqrt{1-\left(\frac{u}{c}\right)^2}} \qquad (5)$$

De $\vec{p} = m\gamma\vec{u}$ y $E = m\gamma c^2$

se obtiene $E = \sqrt{(pc)^2 + (mc^2)^2}$ relación de dispersión



comparar con $E = \frac{p^2}{2m}$

Fotones: $m = 0 \Rightarrow E = pc$. ← relativista

$E = h\nu$ (M.C.) → $p = \frac{h}{\lambda}$

Ejercicio: Dem. que $E^2 - (pc)^2$ es invariante relativista.

$P = (E, c\vec{p})$ 4-momentum

$P \cdot P = m^2c^4$ ← evaluado en sist. en reposo.

Relación de Dispersión

$$E = m\gamma c^2 = mc^2/(1-(u/c)^2)^{1/2}$$

$$p = m\gamma u = mu/(1-(u/c)^2)^{1/2}$$

$$p^2 = m^2\gamma^2u^2, \quad \text{pero } \gamma^2u^2 = \frac{u^2}{1-(u/v)^2} = \frac{c^2(u/c)^2}{1-(u/c)^2}$$

$$= c^2\left[\frac{(u/c)^2-1+1}{1-(u/v)^2}\right] = -c^2\left[\cancel{\frac{1-(u/c)^2}{1-(u/c)^2}}\right] + \frac{c^2}{1-(u/v)^2}$$

$$= -c^2 + c^2\gamma^2$$

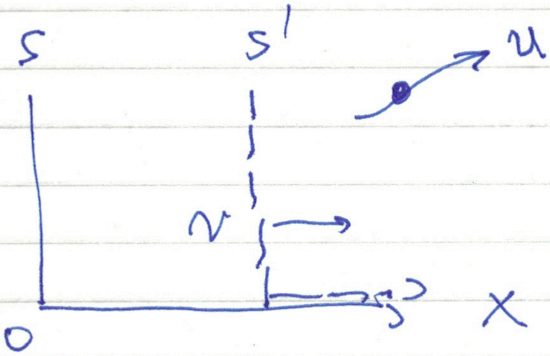
$$\Rightarrow p^2 = -m^2c^2 + m^2c^2\gamma^2 \qquad / c^2$$

$$c^2p^2 = -m^2c^4 + \underbrace{m^2\gamma^2c^4}_{E^2}$$

$$\therefore E^2 = m^2c^4 + c^2p^2$$

$$E = \sqrt{m^2c^4 + c^2p^2} \quad //$$

T. de Lorentz para energía y momentum



$$\gamma \equiv \left(1-(v/c)^2\right)^{-1/2}$$

$$u_x' = \frac{u_x - v}{1 - v u_x/c^2} \quad ; \quad u_y' = \frac{u_y/\gamma}{1 - v u_x/c^2} \qquad (1)$$

Sist. S

$$E = m_0 c^2 \gamma(u)$$
$$P_x = m_0 u_x \gamma(u)$$
$$P_y = m_0 u_y \gamma(u)$$

donde $\gamma(u) = \left(1-(u/c)^2\right)^{-1/2}$

lo mas dificil es relacionar gamma(u) con gamma(u')

Sist. S'

$$E' = m_0 c^2 \gamma(u')$$
$$P_x' = m_0 u_x' \gamma(u')$$
$$P_y' = m_0 u_y' \gamma(u')$$

con $\gamma(u') = \left(1-(u'/c)^2\right)^{-1/2}$

Dem. que:

$$E' = \gamma(v)(E - vP_x)$$
$$P_x' = \gamma\left(P_x - \frac{vE}{c^2}\right)$$
$$P_y' = P_y$$

$$E' = mc^2\gamma(u') \quad ; \quad \gamma(u') = \left(1 - \left(\frac{u'}{c}\right)^2\right)^{-1/2}$$

$$\gamma(u') = \left[1 - (u_x'^2/c^2) - (u_y'^2/c^2)\right]^{-1/2}$$

(a) $$1 - (u_x'^2/c^2) = 1 - \frac{(u_x - v)^2}{c^2(1 - vu_x/c^2)^2}$$

$$= \frac{c^2\left(1 - \frac{vu_x}{c^2}\right)^2 - (u_x - v)^2}{c^2\left(1 - \frac{vu_x}{c^2}\right)^2} =$$

$$= \frac{c^2\left(1 + \left(\frac{vu_x}{c^2}\right) - \frac{2vu_x}{c^2}\right) - u_x^2 - v^2 + 2vu_x}{c^2\left(1 - \frac{vu_x}{c^2}\right)^2}$$

$$= \frac{c^2 + c^2\left(\frac{vu_x}{c^2}\right)^2 - u_x^2 - v^2}{c^2\left(1 - \frac{vu_x}{c^2}\right)^2}$$

$$= \frac{1 + \left(\frac{vu_x}{c^2}\right)^2 - (u_x/c)^2 - (v/c)^2}{(1 - vu_x/c^2)^2}$$

$$= \frac{(1 - u_x^2/c^2)(1 - (v/c)^2)}{(1 - vu_x/c^2)^2} \qquad (1)$$

De $u_{y'} = \dfrac{u_y/\gamma(v)}{1 - v u_x/c^2}$, $\gamma(v) = (1 - v^2/c^2)^{-1/2}$

$$\Rightarrow \frac{u_{y'}^2}{c^2} = \frac{(u_y^2/c^2)\,\gamma(v)^{-2}}{\left(1 - \frac{v u_x}{c^2}\right)^2} = \frac{(u_y^2/c^2)(1 - v^2/c^2)}{(1 - v u_x/c^2)^2} \quad . \quad (2)$$

(1) + (2) $\Rightarrow$

$$1 - \frac{u'^2}{c^2} = \frac{(1 - u_x^2/c^2)(1 - v^2/c^2) - (u_y/c)^2(1 - v^2/c^2)}{(1 - v u_x/c^2)^2}$$

$$= \frac{(1 - v^2/c^2)(1 - u_x^2/c^2 - u_y^2/c^2)}{(1 - v u_x/c^2)^2} = \frac{(1 - v^2/c^2)(1 - u^2/c^2)}{(1 - v u_x/c^2)^2}$$

o sea, $\dfrac{1}{\gamma^2(u')} = \dfrac{1}{\gamma^2(v)} \cdot \dfrac{1}{\gamma^2(u)} \cdot \dfrac{1}{(1 - v u_x/c^2)^2}$

$$\Rightarrow \boxed{\gamma(u') = \gamma(v)\gamma(u)(1 - v u_x/c^2)} \qquad (3)$$

luego $E' = \gamma(u') m c^2$

$$= \gamma(v)\left[\gamma(u) m c^2 - \gamma(u) m \cancel{c^2} \frac{v u_x}{\cancel{c^2}}\right]$$

$$\therefore \boxed{E' = \gamma(v)\left[E - v P_x\right]} \qquad (4)$$

$$P_x' = \gamma(u')\, m u_x' = \gamma(v)\gamma(u)\cancel{\left(1-\frac{vu_x}{c^2}\right)} m \cdot \frac{(u_x - v)}{\cancel{\left(1-\frac{vu_x}{c^2}\right)}}$$

$$P_x' = \gamma(v)\gamma(u)\, m\, (u_x - v)$$

∅ Seq $$P_x' = \gamma(v)\left(P_x - \frac{vE}{c^2}\right) \qquad (5)$$

$$P_y' = \gamma(u')\, m u_y' = \cancel{\gamma(v)}\gamma(u)\cancel{\left(1-\frac{vu_x}{c^2}\right)} m \cdot \frac{u_y/\cancel{\gamma(v)}}{\cancel{\left(1-\frac{vu_x}{c^2}\right)}}$$

$$= m\gamma(u)u_y = P_y$$

$$\therefore \quad \boxed{P_y' = P_y} \qquad (6)$$