Topological Dynamical Systems

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Contents

1 Group actions

Classically, topological dynamics is understood as the study of group (and semigroup) actions on topological spaces. It is an important chapter of modern mathematics originating from physics and the theory of differential equations, and its theoretical and practical outreach need not be outlined here. The point of view we adopt is that of the abstract theory, as exposed in references as [1, 3, 4, 6].

Basic to topological dynamics in the classical sense is the idea of global symmetry. However, many interesting systems only present *local (or partial) forms of symmetry*. Partial symmetry is treated using concepts as *groupoids*, *partial group actions* or *inverse semigroup actions*.

As a general convention, all the topological spaces (including the groupoids) are Hausdorff. Local compactness is required only when needed.

A topological group is a groupoid G with a topology such that the inversion $a \mapsto a^{-1}$ and multiplication $(a, b) \mapsto ab$ are continuous. The unit is denoted by e.

1.1 The framework

Definition 1.1. *A group action* (or *a dynamical system*) $(\mathsf{G}, \theta, \Sigma)$ consists in a topological group G, a topological space Σ and the continuous action map

$$\theta : \mathsf{G}\times\Sigma \ni (a, \sigma) \mapsto \theta_a(\sigma) \equiv a\bullet_\theta\sigma \in \Sigma \tag{1.1}$$

satisfying for $a, b \in \mathsf{G}$ and $\sigma \in \Sigma$:

1. $\mathsf{e}\bullet_\theta\sigma = \sigma, \forall\, \sigma \in \Sigma$,
2. $(ab)\bullet_\theta\sigma = a\bullet_\theta(b\bullet_\theta\sigma)$.

If the action θ is understood, we will write $a \bullet \sigma$ instead of $a \bullet_\theta \sigma$.

Exercise 1.2.

- Show that each θ_a is a homeomorphism.
- Write down the axioms of a dynamical system using the notation θ_a or the notation $\theta(a, \sigma)$.
- This was "a left action". Define "right actions". Try to pass from right actions to left actions. Define "bi-transformation groups" (the left and the right action should commute; this will look as a sort of associativity).

Example 1.3. The restricted action to an invariant subset $\Sigma_0 \subset \Sigma$.

Example 1.4. The topological groupoid G also acts on itself, with $a \bullet b := ab$.

Example 1.5. How to spend half of your life studying dynamical systems without mentioning the word "group"? Consider a homeomorphism $T : \Sigma \to \Sigma$. When $T^3 := T \circ T \circ T$ appears, you do not agree to cal it θ_3 and when T^{-1} appears you do not accept to call it θ_{-1}. However: *a single homeomorphism means a $\mathbb{Z}$-dynamical system.* Use words as "cascade", "discrete time", etc.

Example 1.6. Autonomous systems of differential equations

$$\dot{x} = F(x)\,, \quad \mathbb{R}^n \supset D \xrightarrow{F} \mathbb{R}^n$$

define *flows* (*continuous time dynamical systems*), for which $\mathsf{G} = \mathbb{R}$. Suitable conditions on (D, F) should be imposed to guarantee that solutions exist, are unique, depend continuous on initial conditions and remain in D.

Example 1.7. *Products of dynamical systems.* Let $\big(\mathsf{G}_1,\theta_1,\Sigma_1\big)$ and $\big(\mathsf{G}_2,\theta_2,\Sigma_2\big)$ two group actions. Then $\big(\mathsf{G}_1\times\mathsf{G}_2,\theta_1\times\theta_2,\Sigma_1\times\Sigma_2\big)$ is a group action with

$$\big(\theta_1\times\theta_2\big)_{(a_1,a_2)}(\sigma_1,\sigma_2):=\big(\theta_{1,a_1}(\sigma_1),\theta_{2,a_2}(\sigma_2)\big)\,.$$

Example 1.8. *Diagonal products of dynamical systems.* Let $\big(\mathsf{G},\theta_1,\Sigma_1\big)$ and $\big(\mathsf{G},\theta_2,\Sigma_2\big)$ two group actions *with the same group*. Then $\big(\mathsf{G},\theta_1\times\theta_2,\Sigma_1\times\Sigma_2\big)$ is a group action with

$$\big(\theta_1\times\theta_2\big)_a(\sigma_1,\sigma_2):=\big(\theta_{1,a}(\sigma_1),\theta_{2,a}(\sigma_2)\big)\,.$$

Example 1.9. *Quotient dynamical systems.* An equivalence relation $R\subset\Sigma\times\Sigma$ is *invariant* under the action θ of G if

$$a\in\mathsf{G}\,,\ (\sigma,\tau)\in R\ \Rightarrow\ (a\bullet\sigma,a\bullet\tau)\in R\,.$$

Set $p:\Sigma\to\Sigma/R$ be the canonical map. One defines an action of the discrete group G on Σ/R by

$$\tilde\theta_a\big(p(\sigma)\big)\equiv a\tilde\bullet\sigma:=p(a\bullet\sigma)\,.$$

Under certain assumptions, it is also continuous for the initial topology on G ((1) R open, (2) Σ compact and Hausdorff and R closed).

Remark 1.10. The group action is called *effective* if $\theta_a=\theta_b\ \Leftrightarrow\ a=b$. This means that $\theta_{\mathsf{G}}\to\mathrm{Homeo}(\Sigma)$ is injective. Factor out the kernel, to get an effective group action $\big(\mathsf{G}/\ker(\theta),\hat\theta,\Sigma\big)$.

For $a\in\mathsf{G}$, $\mathsf{A},\mathsf{B}\subset\mathsf{G}$, $M\subset\Sigma$ we use the notations

$$\mathsf{AB}:=\big\{ab\,\big|\,a\in\mathsf{A}\,,\,b\in\mathsf{B}\big\}\,, \tag{1.2}$$

$$a\bullet M:=\big\{a\bullet\sigma\,\big|\,\sigma\in M\big\}\,,$$

$$\mathsf{A}\bullet M:=\big\{a\bullet\sigma\,\big|\,a\in\mathsf{A}\,,\sigma\in M\big\}=\bigcup_{a\in\mathsf{A}}a\bullet M.$$

Remark 1.11. For topological group actions the product $\mathsf{A}M$ is open provided that only the subset M is open. In addition, if A,B are subsets of the group, AB is open whenever at least one of the subsets is so. If $\mathsf{A}\subset\mathsf{G}$ and $M\subset\Sigma$ are compact, $a\bullet M\subset\Sigma$ is compact.

Definition 1.12.

- We are going to use *orbits* $\mathcal{O}_\sigma:=\mathsf{G}\bullet\sigma$ and *orbit closures* (or *quasi-orbits*) $\overline{\mathcal{O}}_\sigma$.
- *The orbit equivalence relation* will be denoted by $\sim$. So $\sigma\sim\tau$ means that $\tau\in\mathcal{O}_\sigma$.
- A subset $M\subset\Sigma$ is called *invariant* if $a\bullet M\subset M$, for every $a\in\mathsf{G}$.
- If $N\subset\Sigma$, its *saturation*
$$\mathsf{Sat}(N)=\mathsf{G}\bullet N=\bigcap_{\substack{N\subset M\\ M\text{ invariant}}}M$$
is the smallest invariant subset of Σ containing N.

Exercise 1.13.

- Show that $\sim$ is indeed an equivalence relation, so the orbits form a partition.
- The orbit closure *do not form a partition*. Very easily one may have $\overline{\mathcal{O}}_\sigma\subsetneq\overline{\mathcal{O}}_\tau$.
- The invariant subsets form a family stable under a lot of operations.

- *The isotropy group* $\mathsf{G}_\sigma := \{a \in \mathsf{G} \,|\, a \bullet \sigma = \sigma\}$ is indeed a (closed) group. Find the connection between G_σ and G_τ if $\sigma \sim \tau$.

Proposition 1.14.
- *The saturation of an open set is also open.*
- *The interior M°, the closure $\overline{M}$ and the boundary ∂M of an invariant subset M of Σ are also invariant.*

Proof. If N is an open set, $\mathsf{Sat}(N) = \mathsf{G} \bullet N = \bigcup_{a \in \mathsf{G}} a \bullet N$ is open.

One has $(M^\circ)^c = \overline{M^c}$ and $\partial M = \overline{M} \setminus M^\circ$. Since the difference of two invariant sets is clearly invariant, it is enough to show that M° is invariant. If $\sigma \in M$ is an interior point, there exists some open set $U \subset M$ containing σ, so we have

$$a \bullet \sigma \in \mathsf{G} \bullet U \subset \mathsf{G} \bullet M = M,$$

implying that $a \bullet \sigma$ is also an interior point of M, since $\mathsf{G} \bullet U$ is open. $\square$

Definition 1.15. A *morphism* (or *homomorphism*, or *equivariant map*) of the group actions $(\mathsf{G}, \theta, \Sigma)$, $(\mathsf{G}, \theta', \Sigma')$ is a continuous function $f : \Sigma \to \Sigma'$ such that for all $\sigma \in \Sigma$, $a \in \mathsf{G}$ one has

$$f\big(\theta(a, \sigma)\big) = \theta'\big(a, f(\sigma)\big)\,. \tag{1.3}$$

An *epimorphism* is a surjective homomorphism; in such a case we say that $(\mathsf{G}, \theta', \Sigma')$ is *a factor* of $(\mathsf{G}, \theta, \Sigma)$ and that $(\mathsf{G}, \theta, \Sigma)$ is *an extension* of $(\mathsf{G}, \theta', \Sigma')$.

Writing $\bullet$ instead of θ and $\bullet'$ instead of θ', the requirement in (1.3) is

$$f(a \bullet \sigma) = a \bullet' f(\sigma)\,, \quad \forall\, (a, \sigma) \in \mathsf{G} \times \Sigma\,. \tag{1.4}$$

Exercise 1.16. Draw a commuting diagram to illustrate (1.3).

Exercise 1.17. Check that the composition of two morphisms of actions is a morphism of actions. Is there a category?

Exercise 1.18. Define morphisms from $\big(\mathsf{G}_1, \theta_1, \Sigma_1\big)$ to $\big(\mathsf{G}_2, \theta_2, \Sigma_2\big)$ *involving a change of the group*. Is there a category?

Exercise 1.19. Define various morphisms (of various types) connected with Examples 1.7 and 1.8.

Remark 1.20. A separately continuous map $f : X \times Y \to Z$ might not be jointly continuous. If

$$X \ni x \to f(x, y_0) \in Z\,, \quad Y \ni y \to f(x_0, y) \in Z$$

are continuous for every $x_0 \in X$and $y_0 \in Y$, you are not sure that f is continuous. A counterexample:

$$\mathbb{R} \times \mathbb{R} \ni (x, y) \to \frac{xy}{x^2 + y^2} \in \mathbb{R}\,.$$

If $f : X \times X \to Z$ is separately continuous, you cannot be sure that

$$F : X \to Z\,, \quad F(x) := f(x, x)$$

is continuous ! Note that F is the composition $X \xrightarrow{\delta} X \times X \xrightarrow{f} Z$, where $\delta(x) := (x, x)$.

Theorem 1.21. *(Robert Ellis, 1957) Let* G *be a locally compact Hausdorff group and* Σ *a locally compact Hausdorff space. Let* $\theta : \mathsf{G} \times \Sigma \to \Sigma$ *be a separately continuous function such that*

$$\theta(\mathsf{e}, \sigma) = \sigma\,, \quad \theta\big(a, \theta(b, \sigma)\big) = \theta(ab, \theta)\,, \quad \forall\, a, b \in \mathsf{G}\,, \sigma \in \Sigma\,.$$

Then θ *is (jointly) continuous (a topological action).*

Can you improve this result?

1.2 Recurrence sets

We fix a dynamical system (group action) $(\mathsf{G}, \theta, \Sigma)$. Notation:

$$\theta(a, \sigma) = \theta_a(\sigma) = a \bullet \sigma \,, \quad \forall\, a \in \mathsf{G} \,, \sigma \in \Sigma \,.$$

Definition 1.22. Let us introduce the function

$$\mathsf{G} \times \Sigma \ni (a, \sigma) \overset{\vartheta}{\to} (\sigma, a \bullet \sigma) \in \Sigma \times \Sigma$$

and denote by q the projection on the first variable $\mathsf{G} \times \Sigma \to \mathsf{G}$. For every $M, N \subset \Sigma$ one defines *the recurrence set*

$$\mathrm{Rec}_\theta(M, N) \equiv \mathsf{G}_M^N := q\big[\vartheta^{-1}(M \times N)\big] \,.$$

A simple inspection of the definitions reveals that

$$\begin{aligned} \mathsf{G}_M^N &= \{a \in \mathsf{G} \,|\, (a \bullet M) \cap N \neq \emptyset\} \\ &= \{a \in \mathsf{G} \,|\, \exists\, \sigma \in M \,,\, \theta_a(\sigma) \in N\} \,, \end{aligned}$$

which explains the terminology. One also uses the term "dwelling set".

Proof.

$$\begin{aligned} a \in \mathsf{G}_M^N \;&\Leftrightarrow\; a = q(b, \sigma) \,,\ \text{for some } (b, \sigma) \in \vartheta^{-1}(M \times N) \\ &\Leftrightarrow\; \exists\, \sigma \in \Sigma \ \text{such that}\ \vartheta(a, \sigma) = (\sigma, a \bullet \sigma) \in M \times N \\ &\Leftrightarrow\; \exists\, \sigma \in \Sigma \ \text{such that}\ \sigma \in M,\, a \bullet \sigma \in N \\ &\Leftrightarrow\; (a \bullet M) \cap N \neq \emptyset \,. \end{aligned}$$

□

Proposition 1.23. *If $M, N$are open, G_M^N is also open.*

Proof. Because ϑ is continuous and q is open. □

Remark 1.24. One can say a lot about the behavior of a dynamical system studying the recurrence sets. For given M, N it is important to see if G_M^N is small (empty, relatively compact) or large (not relatively compact, the complement of a relatively compact, syndetic, replete, everything). The sets M, N are also very relevant (points, open sets, neighborhoods of something, equal, very far apart, ...).

Remark 1.25. The set G_M^N is increasing in M and N.

One has $\mathsf{G}_\emptyset^\emptyset = \emptyset$ and $\mathsf{G}_\Sigma^\Sigma = \mathsf{G}$. Something more?

Remark 1.26. Note that $\mathsf{G}_M^N = \bigcup_{\sigma \in M} \mathsf{G}_\sigma^N$, where

$$\mathsf{G}_\sigma^N \equiv \mathsf{G}_{\{\sigma\}}^N = \{a \in \mathsf{G} \,|\, a \bullet \sigma \in N\} \,.$$

Example 1.27. *The stabilizer (isotropy group)* $\mathsf{G}_\sigma^\sigma = \{a \in \mathsf{G} \,|\, a \bullet \sigma = \sigma\}$ is a particular case. What is the meaning of $\mathsf{G}_\sigma^\sigma = \mathsf{G}$? What is the meaning of $\mathsf{G}_\sigma^\tau \neq \emptyset$? What is the meaning of $\mathsf{G}_\sigma^\tau = \mathsf{G}$?

The next straightforward results will be useful in the next sections.

Lemma 1.28. *If $M, N \subset \Sigma$ and $b, c \in \mathsf{G}$, then $\mathsf{G}_{b \bullet M}^{c \bullet N} = c\, \mathsf{G}_M^N b^{-1}$ and $\big(\mathsf{G}_M^N\big)^{-1} = \mathsf{G}_N^M$.*

Proof. We only show the first equality:

$$\begin{aligned} a \in \mathsf{G}_{b\bullet M}^{c\bullet N} &\Leftrightarrow \exists\, \sigma \in M\,, \tau \in N \text{ with } a \bullet (b \bullet \sigma) = c \bullet \tau \\ &\Leftrightarrow \exists\, \sigma \in M\,, \tau \in N \text{ with } \big(c^{-1} a\, b\big) \bullet \sigma = \tau \\ &\Leftrightarrow c^{-1} a\, b \in \mathsf{G}_M^N \\ &\Leftrightarrow a \in c\, \mathsf{G}_M^N b^{-1}. \end{aligned}$$

□

Lemma 1.29. *Let* $M, N \subset \Sigma$. *Then*

$$\mathsf{Sat}(M) \cap N \neq \emptyset \Leftrightarrow \mathsf{Sat}(M) \cap \mathsf{Sat}(N) \neq \emptyset \Leftrightarrow \mathsf{G}_M^N \neq \emptyset\,.$$

Proof. One has $\mathsf{Sat}(M) \cap \mathsf{Sat}(N) \neq \emptyset$ if and only if there exist $a_1, a_2 \in \mathsf{G}$, $\sigma \in M$ and $\tau \in N$ such that $a_1 \bullet \sigma = a_2 \bullet \tau$, which is equivalent to $\big(a_2^{-1} a_1\big) \bullet \sigma = \tau \in N$, so we have $a_2^{-1} a_1 \in \mathsf{G}_M^N$ and $\big(a_2^{-1} a_1\big) \bullet \sigma \in \mathsf{Sat}(M)$.

For the converse: If $\mathsf{G}_M^N \neq \emptyset$, there exists $a \in \mathsf{G}$ such that $a \bullet \sigma = \tau$, with $\sigma \in M$ and $\tau \in N$. Then $\mathsf{Sat}(M) \cap N \neq \emptyset$, from which $\mathsf{Sat}(M) \cap \mathsf{Sat}(N) \neq \emptyset$ follows. □

Exercise 1.30. What can you say about $\mathsf{G}_{M_1 \cup M_2}^{N_1 \cup N_2}$ or about $\mathsf{G}_{M_1 \cap M_2}^{N_1 \cap N_2}$? Other options?

Example 1.31. *Products of dynamical systems.* Let $\big(\mathsf{G}_1, \theta_1, \Sigma_1\big)$ and $\big(\mathsf{G}_2, \theta_2, \Sigma_2\big)$ two group actions. Then $\big(\mathsf{G}_1 \times \mathsf{G}_2, \theta_1 \times \theta_2, \Sigma_1 \times \Sigma_2\big)$ is a group action with

$$\big(\theta_1 \times \theta_2\big)_{(a_1, a_2)}(\sigma_1, \sigma_2) := \big(\theta_{1, a_1}(\sigma_1), \theta_{2, a_2}(\sigma_2)\big)\,.$$

It is easier to write

$$(a_1, a_2) \bullet (\sigma_1, \sigma_2) = \big(a_1 \bullet_1 \sigma_1, a_2 \bullet_2 \sigma_2\big)\,.$$

If $M_1, N_1 \subset \Sigma_1$ and $M_2, N_2 \subset \Sigma_2$, then

$$\big(\mathsf{G}_1 \times \mathsf{G}_2\big)_{M_1 \times M_2}^{N_1 \times N_2} = (\mathsf{G}_1)_{M_1}^{N_1} \times (\mathsf{G}_2)_{M_2}^{N_2}\,.$$

What about the orbits? Is it true that $\mathcal{O}_{(\sigma_1, \sigma_2)}^{\theta_1 \times \theta_2} = \mathcal{O}_{\sigma_1}^{\theta_1} \times \mathcal{O}_{\sigma_2}^{\theta_2}$? What about the invariant sets?

Example 1.32. *Diagonal products of dynamical systems.* Let $\big(\mathsf{G}, \theta_1, \Sigma_1\big)$ and $\big(\mathsf{G}, \theta_2, \Sigma_2\big)$ two group actions *with the same group*. Then $\big(\mathsf{G}, \theta_1 \times \theta_2, \Sigma_1 \times \Sigma_2\big)$ is a group action with

$$\big(\theta_1 \times \theta_2\big)_a(\sigma_1, \sigma_2) := \big(\theta_{1,a}(\sigma_1), \theta_{2,a}(\sigma_2)\big)$$

or, written differently,

$$a * (\sigma_1, \sigma_2) = \big(a \bullet_1 \sigma_1, a \bullet_2 \sigma_2\big)\,.$$

What about the orbits? What about the invariant sets?

If $M_1, N_1 \subset \Sigma_1$ and $M_2, N_2 \subset \Sigma_2$, then

$$\mathsf{G}_{M_1 \times M_2}^{N_1 \times N_2} = \mathsf{G}_{M_1}^{N_1} \cap \mathsf{G}_{M_2}^{N_2}\,.$$

Proof.

$$\begin{aligned} a \in \mathsf{G}_{M_1 \times M_2}^{N_1 \times N_2} &\Leftrightarrow \big[a * (M_1 \times M_2)\big] \cap \big(N_1 \times N_2\big) \neq \emptyset \\ &\Leftrightarrow \big[(a \bullet_1 M_1) \times (a \bullet_2 M_2)\big] \cap \big(N_1 \times N_2\big) \neq \emptyset \\ &\Leftrightarrow (a \bullet_1 M_1) \cap N_1 \neq \emptyset \text{ and } (a \bullet_1 M_1) \cap N_2 \neq \emptyset \\ &\Leftrightarrow a \in \mathsf{G}_{M_1}^{N_1} \text{ and } a \in \mathsf{G}_{M_2}^{N_2}\,. \end{aligned}$$

□

Definition 1.33. A *morphism* (or *equivariant map*) of the group actions $(\mathsf{G},\theta,\Sigma)$, $(\mathsf{G},\theta',\Sigma')$ is a continuous function $f:\Sigma\rightarrow\Sigma'$ such that for all $\sigma\in\Sigma$, $a\in\mathsf{G}$ one has

$$f\big(\theta(a,\sigma)\big)=\theta'\big(a,f(\sigma)\big)\,. \tag{1.5}$$

An *epimorphism* is a surjective homomorphism; in such a case we say that $(\mathsf{G},\theta',\Sigma')$ is *a factor* of $(\mathsf{G},\theta,\Sigma)$ and that $(\mathsf{G},\theta,\Sigma)$ is *an extension* of $(\mathsf{G},\theta',\Sigma')$.

Writing $\bullet$ instead of θ and $\bullet'$ instead of θ', the requirement in (1.3) is

$$f(a\bullet\sigma)=a\bullet' f(\sigma)\,,\quad \forall\,(a,\sigma)\in\mathsf{G}\times\Sigma\,. \tag{1.6}$$

Proposition 1.34. *Let* $(\mathsf{G},\theta,\Sigma)$ *and* $(\mathsf{G},\theta',\Sigma')$ *two dynamical systems and* $f:\Sigma\rightarrow\Sigma'$ *a morphism. If* $M,N\subset\Sigma$ *then*

$$\mathsf{G}_M^N\subset\mathsf{G}'^{f(N)}_{f(M)}\,. \tag{1.7}$$

When f *is injective one has equality.*

Proof. One verifies easily that the next sequence of equivalences and implications is rigorous:

$$\begin{aligned} a\in\mathsf{G}_M^N &\Leftrightarrow (a\bullet M)\cap N\neq\emptyset\\ &\Leftrightarrow f\big[(a\bullet M)\cap N\big]\neq\emptyset\\ &\Rightarrow f(a\bullet M)\cap f(N)\neq\emptyset\\ &\Leftrightarrow \big(a\bullet' f(M)\big)\cap f(N)\neq\emptyset\\ &\Leftrightarrow a\in\mathsf{G}'^{f(N)}_{f(M)}\,. \end{aligned}$$

In general one has $f(A\cap B)\subset f(A)\cap f(B)$ and the inclusion is always an equality if and only if is f is injective; this is when $\Rightarrow$ is an equivalence. $\square$

Proposition 1.35. *Let* $(\mathsf{G},\theta,\Sigma)$ *and* $(\mathsf{G},\theta',\Sigma')$ *two dynamical systems and* $f:\Sigma\rightarrow\Sigma'$ *a morphism.*

(i) The map f *sends invariant sets into invariant sets.*

(ii) Let $\sigma,\tau\in\Sigma$. *If* $\sigma\sim\tau$ *then* $f(\sigma)\sim' f(\tau)$. *When* f *is injective:* $\sigma\sim\tau\Leftrightarrow f(\sigma)\sim' f(\tau)$

(iii) Let $\sigma\in\Sigma$. *One has* $f\big(\mathcal{O}_\sigma\big)=\mathcal{O}'_{f(\sigma)}$.

Proof. (i) Suppose that $B\subset\Sigma$ is invariant; we show that $f(B)\subset\Sigma'$ is invariant. Let $\sigma\in B$, so $f(\sigma)\in f(B)$. Then for every $a\in\mathsf{G}$ we have $a\bullet' f(\sigma)=f(a\bullet\sigma)\in f(B)$.

(ii) One has

$$\sigma\sim\tau\;\Leftrightarrow\;\exists\,a\in\mathsf{G}\,,\,a\bullet\sigma=\tau\;\Rightarrow\;\exists\,a\in\mathsf{G}\,,f(a\bullet\sigma)=a\bullet' f(\sigma)=f(\tau)\;\Leftrightarrow\;f(\sigma)\sim' f(\tau)\,.$$

Suppose that f is injective. Then the middle implication is an equivalence.

Another proof uses (1.7) and the fact that $\sigma\sim\tau$ if and only if $\mathsf{G}_\sigma^\tau\neq\emptyset$,.

(iii) If $\tau\in\mathcal{O}_\sigma$, then $\tau\sim\sigma$, thus $f(\tau)\sim' f(\sigma)$, meaning that $f(\tau)\in\mathcal{O}'_{f(\sigma)}$. We showed that $f\big(\mathcal{O}_\sigma\big)\subset\mathcal{O}'_{f(\sigma)}$. But $f\big(\mathcal{O}_\sigma\big)$ is non-void and $\bullet'$-invariant, so it coincides with $\mathcal{O}'_{f(\sigma)}$. $\square$

Example 1.36. Let Σ be composed only of fixed points and $\Sigma':=\{\rho\}$ composed of a single fixed point. The constant function

$$f(\sigma):=\rho\,,\quad\forall\,\sigma\in\Sigma$$

is a morphism:

$$f(a\bullet\sigma)=\rho\quad\text{and}\quad a\bullet' f(\sigma)=a\bullet'\rho=\rho.$$

Then $f(\sigma_1)=f(\sigma_2)$, but σ_1 and σ_2 are not $\bullet$-equivalent if they are different.

Example 1.37. In general, if $g : X \to Y$ is continuous and $A \subset X$, then $g\big(\overline{A}\big) \subset \overline{g(A)}$, maybe strictly. A strict inclusion may hold even for orbits: one may have $f(\mathcal{O}) = \mathcal{O}'$, but $f\big(\overline{\mathcal{O}}\big) \subsetneq \overline{\mathcal{O}'}$.

The group $\mathbb{R} = \mathsf{G} = \Sigma$ acts upon itself by translations. There is just one orbit (transitivity). Its Alexandrov compactification $\Sigma' = \mathbb{R} \cup \{\infty\}$ is an $\mathbb{R}$-space with one orbit equal to $\mathbb{R}$ and another one only containing the fixed point ∞. The canonical injection $j : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is a morphism. But

$$j\big(\overline{\mathbb{R}}\big) = j(\mathbb{R}) = \mathbb{R} \subsetneq \overline{j(\mathbb{R})} = \mathbb{R} \cup \{\infty\}\,.$$

2 Topological transitivity

Definition 2.1. We will fix a dynamical system $(\mathsf{G}, \theta, \Sigma)$.

- If there is just one orbit, the action is *transitive*. This means that $\sigma, \tau \in \Sigma \Rightarrow \sigma \sim \tau$. Equivalently, there is no other invariant subset besides $\emptyset, \Sigma$.
- A point having a dense orbit is called *a transitive point*.
- If there is a dense orbit, i.e. if a transitive point does exist, the action is *pointwise transitive*.
- If all the orbits are dense, the action is *minimal*
  ($\Leftrightarrow$ no no-trivial invariant open sets $\Leftrightarrow$ no no-trivial invariant closed sets)
- The action is *topologically transitive* if for every $U, V \subset \Sigma$ open and non-void, $\mathsf{G}_U^V \neq \emptyset$ holds.

Proposition 2.2. *Transitive* $\Rightarrow$ *minimal* $\Rightarrow$ *pointwise transitive* $\Rightarrow$ *topologically transitive* .

Proof. The first two implications are obvious (and obviously they are not equivalence).

We verify the third one. Assume that $\mathfrak{O}_\sigma$ is a dense orbit and let $\emptyset \neq U, V$ be open sets. One has

$$a \bullet \sigma \in \mathfrak{O}_\sigma \cap U \neq \emptyset \quad \text{and} \quad b \bullet \sigma \in \mathfrak{O}_\sigma \cap V \neq \emptyset$$

for elements $a, b \in \mathsf{G}$. Since $\big(b\, a^{-1}\big)(a \bullet \sigma) = b \bullet \sigma$ we infer that $b\, a^{-1} \in \mathsf{G}_U^V \neq \emptyset$. $\square$

Remark 2.3. The subset A of the topological space X is called *nowhere dense* if the interior of its closure is void: $\mathring{\overline{A}} = \emptyset$. The set $1/\mathbb{N}$ is nowhere dense in $\mathbb{R}$. Of course, if A is nowhere dense, it has empty interior. But in $X = \mathbb{R}$, the subset $A := \mathbb{Q}$ has void interior, but it is not nowhere dense (it is dense!). In a topological vector space a subspace is dense or nowhere dense.

Exercise 2.4.

- Finite unions of nowhere dense sets is nowhere dense.
- A is nowhere dense in X if and only if A is not dense in any open subset U of X .
- A is nowhere dense if and only if it is contained in the boundary of an open set.
  Apply this to $1/\mathbb{N}$.

Theorem 2.5. *The following conditions are equivalent:*

(0) The system is topologically transitive.

(i) Each non-empty open invariant subset of Σ *is dense.*

(ii) Any two open non-void invariant subsets of Σ *have non-trivial intersection.*

(ii') Σ is not the union of two proper invariant closed subsets.

(iii) Each invariant subset of Σ *is ether dense, or nowhere dense (*topological transitivity*).*

Proof. $(0) \Rightarrow (i)$ Let $\emptyset \neq U \subset \Sigma$ open and invariant. By assumption, for every non-void open set $V \subset \Sigma$ there exists some $a \in \mathsf{G}$ making $(a \bullet U) \cap V$ non-void. But $a \bullet U \subset U$, implying $U \cap V \neq \emptyset$. Thus, U meets every other non-void open set and must be dense.

$(i) \Rightarrow (ii)$ If each non-void open invariant subset is dense, it meets every other (invariant) non-void open set.

$(ii) \Leftrightarrow (ii')$ If (ii') fails, i.e. $\Sigma = C \cup D$ with C and D proper closed invariant subsets, then $C^c \cap D^c = \emptyset$. This contradicts (ii), since C^c and D^c are open, non-void and invariant.

On the other hand, if A, B are open non-empty invariant sets such that $A \cap B = \emptyset$, then $A^c \cup B^c = \Sigma$ with A^c, B^c proper closed invariant subsets, finishing the proof of the equivalence.

$(ii) \Rightarrow (iii)$. So let us assume (ii), but let $A \subset \Sigma$ be invariant, neither dense, nor nowhere dense. Then $\big(\overline{A}\big)^\circ$ and $(A^c)^\circ = \big(\overline{A}\big)^c$ are both non-void open sets, which are invariant by Proposition 1.14. They should meet, by (ii), but this is obviously false.

$(iii) \Rightarrow (0)$ Suppose (iii) holds. Let $\emptyset \neq U, V \subset \Sigma$ open sets. $\mathsf{Sat}(U)$ is an invariant set containing U, so it cannot be nowhere dense, meaning that it is dense. Hence $\mathsf{Sat}(U) \cap V \neq \emptyset$ and we conclude by Lemma 1.29. $\square$

Proposition 2.6. *Let f be an epimorphism between the group actions $(\mathsf{G}, \bullet, \Sigma)$ and $\big(\mathsf{G}, \bullet', \Sigma'\big)$. Suppose that the action $(\mathsf{G}, \bullet, \Sigma)$ has one of the properties*

$$\mathcal{P} \in \{\textit{transitivity}, \textit{pointwise transitivity}, \textit{topological transitivity}\}$$

Then the action $\big(\mathsf{G}, \bullet', \Sigma'\big)$ also has $\mathcal{P}$.

Proof. $\mathcal{P} =$ *transitivity*. We showed that under a morphism $f\big(\mathcal{O}_\sigma\big) = \mathcal{O}'_{f(\sigma)}$. Thus, if Σ is transitive and f is also surjective:

$$\Sigma' = f(\Sigma) = f\big(\mathcal{O}_\sigma\big) = \mathcal{O}'_{f(\sigma)},$$

showing that Σ' is transitive.

$\mathcal{P} =$ *pointwise transitivity*. Under a morphism $f\big(\overline{\mathcal{O}_\sigma}\big) \subset \overline{\mathcal{O}'_{f(\sigma)}}$. If $\overline{\mathcal{O}_\sigma} = \Sigma$, so

$$\overline{\mathcal{O}'_{f(\sigma)}} \supset f\big(\overline{\mathcal{O}_\sigma}\big) = f(\Sigma) = \Sigma'.$$

It follows that $f(\sigma)$ is a transitive point of Σ'.

$\mathcal{P} = (ii)$. Let $\emptyset \neq U', V' \subset \Sigma'$ be invariant open sets. Then $f^{-1}(U')$, $f^{-1}(V')$ are open, non-void and invariant. Since $(\mathsf{G}, \bullet, \Sigma)$ satisfies (ii), one has $f^{-1}(U') \cap f^{-1}(V') \neq \emptyset$. Consequently

$$U' \cap V' = f\big[f^{-1}(U')\big] \cap f\big[f^{-1}(V')\big] \supset f\big[f^{-1}(U') \cap f^{-1}(V')\big] \neq \emptyset$$

and $\big(\mathsf{G}, \bullet', \Sigma'\big)$ also satisfies (ii). $\square$

Exercise 2.7. Give 4 (four) other proofs that topological transitivity propagates through epimorphisms.

Definition 2.8. The topological space Σ is called

- *second countable* if it has a countable base of open sets,
- *a Baire space* if any countable intersection of dense open sets is dense
  ($\Leftrightarrow$ any countable union of closed sets with empty interior has empty interior)

Remark 2.9. Second countable $\Rightarrow$ first countable (each point has a countable base of neighborhoods). A counterexample for the equivalence?

Example 2.10. Hausdorff locally compact spaces and complete metric spaces are Baire.

$\mathbb{Q} \subset \mathbb{R}$ is not a Baire space.

Proposition 2.11. *If Σ is a Baire second-countable space, then topological transitivity and pointwise transitivity are equivalent.*

Proof. Having in view Theorem 2.5, what remains is to show that (i) implies pointwise transitivity.

Since Σ is second-countable, its topology has a countable basis $\{V_n \neq \emptyset\}_{n\in\mathbb{N}}$. By defining

$$U_n = \mathsf{G} \bullet V_n \equiv \mathrm{Sat}(V_n)\,,$$

we get countably many invariant open (therefore dense, by (i)) subsets of a Baire space, so

$$U = \cap_n U_n$$

is also a dense (invariant) set.

Let $W \neq \emptyset$ be an open subset of Σ. By the definition of a basis, there exists some $V_n \subset W$. Hence we have

$$U \subset U_n = \mathsf{G} \bullet V_n \subset \mathsf{G} \bullet W.$$

Therefore, if $\sigma \in U$ then $a^{-1} \bullet \sigma \in W$ for some $a \in \mathsf{G}$. Hence σ has a dense orbit and the action is pointwise transitive. ☐

Remark 2.12. What we used in the proof of Proposition 2.11 is the fact that the intersection U is non-void. One could improve: *under the given requirements, the set of points with dense orbit is a dense G_δ-set.*

3 Recurrence of points

3.1 Limit sets

A continuous action $(\mathsf{G},\theta,\Sigma)$ *will be fixed, with* G *non-compact*. Let $\mathcal{K}(T)$ denote the family of compact subsets of the topological space T.

Definition 3.1. *The limit set of the point* $\sigma\in\Sigma$ is the closed subset of Σ

$$\mathcal{L}^\theta_\sigma\equiv\mathcal{L}_\sigma:=\bigcap_{\mathsf{K}\in\mathcal{K}(\mathsf{G})}\overline{(\mathsf{G}\backslash\mathsf{K})\bullet\sigma}\,.$$

The limit set would be void if G was allowed to be compact (which is not), but it can also be void in other situations.

We say that the net $(a_i)_{i\in I}\subset\mathsf{G}$ *diverges* if for every compact $\mathsf{K}\subset\mathsf{G}$ there exists $i_{\mathsf{K}}\in I$ such that $a_i\notin\mathsf{K}$ if $i\geq i_{\mathsf{K}}$. The existence of divergent nets hangs on our non-compact assumption.

Lemma 3.2. *The following statements for* $\sigma,\tau\in\Sigma$ *are equivalent:*

(i) τ *belongs to the limit set* $\mathcal{L}_\sigma$ *.*

(ii) For every neighborhood V *of* τ *there exists a divergent net* $(a_i)_{i\in I}$ *in* G *such that* $a_i\bullet\sigma\in V$ *for any* $i\in I$.

(ii') For every neighborhood V *of* τ*, the recurrence set* G^V_σ *is not relatively compact.*

(iii) There is a divergent net $(b_i)_{i\in I}$ *in* G *such that* $b_i\bullet\sigma\to\tau$.

Proof. $(i)\Rightarrow(ii)$ Let V be a neighborhood of τ. If τ belongs to the limit set $\mathfrak{L}_\sigma$, then $\tau\in\overline{\big(\mathsf{G}\setminus\mathsf{K}\big)\bullet\sigma}$ for every $\mathsf{K}\in\mathcal{K}(\mathsf{G})$. Hence we can choose elements $a_{\mathsf{K}}\in\mathsf{G}\setminus\mathsf{K}$ such that $a_{\mathsf{K}}\bullet\sigma\in V$. The net $(a_{\mathsf{K}})_{\mathsf{K}\in\mathcal{K}(\mathsf{G})}$ is obviously divergent.

$(iii)\Rightarrow(i)$ Let $\mathsf{K}\in\mathcal{K}\big(\mathsf{G}\big)$. Since b_i is divergent, there exists $i_{\mathsf{K}}\in I$ such that $b_i\in\mathsf{G}\setminus\mathsf{K}\,,\forall\,i\geq i_{\mathsf{K}}$. So we have that

$$\tau=\lim_{i\in I}b_i\bullet\sigma=\lim_{i\geq i_{\mathsf{K}}}b_i\bullet\sigma\in\overline{\big(\mathsf{G}\setminus\mathsf{K}\big)\bullet\sigma}\,.$$

It follows that $\tau\in\mathcal{L}_\sigma$.

$(ii)\Rightarrow(iii)$ Consider the set $\mathfrak{N}_\tau$ of neighborhoods of τ, and order it by reversing the inclusions. For each neighborhood V of τ, select some divergent net $(a_{i,V})_{i\in I}$ such that $a_{i,V}\bullet\sigma\in V$ (it can be built over the same labels). Observe that, for each $\mathsf{K}\in\mathcal{K}\big(\mathsf{G}\big)$, there exists i^V_{K} such that $a_{i,V}\notin\mathsf{K}$ for every $i\geq i^V_{\mathsf{K}}$. Define $b_{\mathsf{K},V}=a_{i^V_{\mathsf{K}},V}$, which form a net when $\mathfrak{N}_\tau\times\mathcal{K}(\mathsf{G})$ is given the product order. By construction, we get a divergent net and $\tau=\lim\limits_{\mathsf{K},V}b_{\mathsf{K},V}\bullet\sigma$. EXERCISE.

$(ii)\Leftrightarrow(ii')$ follows from the definitions. EXERCISE. ☐

Remark 3.3. Limit sets are mostly studied for actions of one of the groups $\mathbb{Z}$ or $\mathbb{R}$ on topological spaces, where one distinguishes between positive and negative limit points. Besides being non-compact $\mathbb{Z}$ and $\mathbb{R}$ have an extra feature: they have 'two ends". This feature leads to various ramifications, as distinguishing between positive and negative limit sets. One sets for instance, if $\mathsf{G}=\mathbb{R}$ and $\sigma\in\Sigma$:

$$\mathcal{L}^+_\sigma:=\bigcap_{s\in\mathbb{R}}\overline{(s,\infty)\bullet\sigma}\,.$$

It is easily shown that the following are equivalent:

- τ belongs to *the positive limit set* $\mathcal{L}_\sigma$.
- For every neighborhood V of τ, the recurrence set $\mathbb{R}_\sigma^V$ is not contained in a set of the form $(-\infty, s_0)$.
- There exists a sequence $t_n \to +\infty$ in $\mathbb{R}$ such that $t_n \bullet \sigma \to \tau$.

The next easy lemma is sometimes useful to compute limit sets.

Lemma 3.4. *Suppose that there exists a family of compact sets* $\{\mathsf{K}_\lambda\}_{\lambda\in\Lambda}$ *that exhausts* G. *That is,* Λ *is a directed set,* $\mathsf{G} = \bigcup_{\lambda\in\Lambda} \mathsf{K}_\lambda$ *and* $\mathsf{K}_{\lambda_1} \subset \mathsf{K}^\circ_{\lambda_2}$ *whenever* $\lambda_1 \le \lambda_2$. *Then the limit set* $\mathcal{L}_\sigma$ *can be computed as*

$$\mathcal{L}_\sigma = \bigcap_{\lambda\in\Lambda} \overline{(\mathsf{G}\setminus \mathsf{K}_\lambda) \bullet \sigma} \,.$$

Proof. Obviously, $\mathcal{L}_\sigma \subset \bigcap_{\lambda\in\Lambda} \overline{(\mathsf{G}\setminus \mathsf{K}_\lambda) \bullet \sigma}$. For the opposite inclusion, it is enough to find for every compact subset K of G an index $\mu \in \Lambda$ such that $\mathsf{K} \subset \mathsf{K}_\mu$. Indeed, K is covered by the family of interiors of the sets K_λ so, by compactness, it is also covered by a finite subfamily $\big\{\mathsf{K}^\circ_{\lambda_1}, \dots \mathsf{K}^\circ_{\lambda_m}\big\}$. Since Λ is directed, that index exists. So we have $\mathsf{K} \subset \mathsf{K}_\mu$ implying that

$$\overline{(\mathsf{G} \setminus \mathsf{K}_\mu) \bullet \sigma} \subset \overline{(\mathsf{G} \setminus \mathsf{K}) \bullet \sigma}.$$

The conclusion follows. ☐

Example 3.5. If $\mathsf{G} = \mathbb{R}$ for example, one may use the family $\big\{[-t, t] \,\big|\, t > 0\big\}$.

Proposition 3.6. *All the points in the orbit of* σ *have the same limit set* $\mathcal{L}_\sigma$.

Proof. For $\mathsf{K} \subset \mathsf{G}$, with computations based on the definitions

$$\big(\mathsf{G}\setminus \mathsf{K}\big) \bullet (a \bullet \sigma) = \big[\big(\mathsf{G}\setminus \mathsf{K}\big)a\big] \bullet \sigma = \big(\mathsf{G}a \setminus \mathsf{K}a\big) \bullet \sigma = (\mathsf{G}\setminus \mathsf{K}a) \bullet \sigma \,,$$

implying that $\mathcal{L}_{\,a\bullet\sigma} \subset \mathcal{L}_\sigma$ (because $\mathsf{K}a$ is compact), from which the statement follows. ☐

Proposition 3.7. *One has*

$$\overline{\mathcal{O}}_\sigma = \mathcal{O}_\sigma \cup \mathcal{L}_\sigma \,. \tag{3.1}$$

Proof. The $\supset$ inclusion in (3.1) is obvious. For each $\mathsf{K} \in \mathcal{K}(\mathsf{G})$ one can write

$$\overline{\mathcal{O}}_\sigma = \overline{\mathsf{G} \bullet \sigma} = \overline{\big(\mathsf{K} \cup \mathsf{K}^{\mathrm{c}}\big) \bullet \sigma} = (\mathsf{K} \bullet \sigma) \cup \overline{\mathsf{K}^{\mathrm{c}} \bullet \sigma} \subset \mathcal{O}_\sigma \cup \overline{\mathsf{K}^{\mathrm{c}} \bullet \sigma} \,,$$

from which $\subset$ follows. ☐

Proposition 3.8. *The closed set* $\mathcal{L}_\sigma$ *is invariant.*

Proof. Let $(a, \tau) \in \mathsf{G} \times \Sigma$ with $\tau \in \mathcal{L}_\sigma$. Using the convergence criterion (iii) of Lemma 3.2, there exists some divergent net $(b_i)_{i\in I}$ in G such that $b_i \bullet \sigma \to \tau$. One may write

$$a \bullet \tau = a \bullet \lim_i (b_i \bullet \sigma) = \lim_i a \bullet (b_i \bullet \sigma) = \lim_i (ab_i) \bullet \sigma \,.$$

But $(ab_i)_{i\in I}$ is also divergent, so $a \bullet \tau \in \mathcal{L}_\sigma$. ☐

Proposition 3.9. *If the orbit of σ is relatively compact, $\mathcal{L}_\sigma$ is non-empty and it attracts the points of the orbit $\mathcal{O}_\sigma$: for every neighborhood W of $\mathcal{L}_\sigma$, there is a compact subset K of G such that*

$$\big(\mathsf{G}\backslash\mathsf{K}\big)\bullet\sigma\subset W. \tag{3.2}$$

Proof. It is enough to show that, for a fixed open neighborhood W of $\mathcal{L}_\sigma$, there is a compact subset K of G such that (3.2) holds: if $\mathcal{L}_\sigma$ were void, the empty set would be a neighborhood, which contradicts the inclusion (G is non-compact).

For any $\mathsf{K}\in\mathcal{K}(\mathsf{G})$, we set $\Sigma_\sigma(\mathsf{K}):=\overline{\big(\mathsf{G}\setminus\mathsf{K}\big)\bullet\sigma}$; complements will refer to $\overline{\mathcal{O}}_\sigma$. Since $\mathcal{L}_\sigma\subset W\cap\overline{\mathfrak{O}}_\sigma$, the family $\big\{\Sigma_\sigma(\mathsf{K})^c\,\big|\,\mathsf{K}\in\mathcal{K}(\mathsf{G})\big\}$ is an open cover of the complement of $W\cap\overline{\mathcal{O}}_\sigma$ in the compact space $\overline{\mathcal{O}}_\sigma$. We extract a finite subcover $\big\{\Sigma_\sigma(\mathsf{K}_i)^c\,\big|\,i=1,\dots,n\big\}$. Then

$$\Big(\mathsf{G}\setminus\bigcup_{i=1}^{n}\mathsf{K}_i\Big)\bullet\sigma\subset\bigcap_{i=1}^{n}\overline{\big(\mathsf{G}\setminus\mathsf{K}_i\big)\bullet\sigma}\subset W\cap\overline{\mathcal{O}}_\sigma\subset W$$

and the proof is finished, since a finite union of compact sets is compact. □

Proposition 3.10. *For every epimorphism f between actions $(\mathsf{G},\bullet,\Sigma)$, $\big(\mathsf{G},\bullet',\Sigma'\big)$ and every point $\sigma\in\Sigma$, we have $f\big(\mathcal{L}^\theta_\sigma\big)=\mathcal{L}^{\theta'}_{f(\sigma)}$.*

Proof. By Proposition 3.2 : $\tau'=f(\tau)\in f\big(\mathcal{L}^\theta_\sigma\big)$, with $\tau\in\mathcal{L}^\theta_\sigma$, if and only if exists a divergent net $(a_i)_{i\in I}$ in G such that

$$\tau'=f\big(\lim_i a_i\bullet\sigma\big)=\lim_i f(a_i\bullet\sigma)=\lim_i a_i\bullet' f(\sigma)\,,$$

which is equivalent with $\tau'\in\mathcal{L}^{\theta'}_{f(\sigma)}$. □

3.2 Recurrent points and wandering

A continuous action $(\mathsf{G},\theta,\Sigma)$ is fixed, with G locally compact but non-compact.

Definition 3.11. A point is called *recurrent* if it is a limit point of itself.

When $\sigma\in\mathcal{L}_\sigma$ holds, we say that σ is *a recurrent point*.

We denote by $\Sigma^\theta_{\rm rec}\equiv\Sigma_{\rm rec}$ the family of all the recurrent points of the group action.

In the relation $\overline{\mathcal{O}}_\sigma=\mathcal{O}_\sigma\cup\mathcal{L}_\sigma$ the union could be disjoint or not.

Proposition 3.12. *For a point $\sigma\in\Sigma$, the following five conditions are equivalent:*

(a) $\mathcal{O}_\sigma\cap\mathcal{L}_\sigma\neq\emptyset$,

(b) $\mathcal{L}_\sigma=\overline{\mathcal{O}}_\sigma$,

(c) $\sigma\in\mathcal{L}_\sigma$ *(σ is a recurrent point)*,

(d) there is a divergent net $(a_i)_{i\in I}\subset\mathsf{G}$ such that $a_i\bullet\sigma\to\sigma$,

(e) G^U_σ is not relatively compact for any open neighborhood U of σ .

Proof. The equivalence $(c) \Leftrightarrow (d) \Leftrightarrow (e)$ is the content of Lemma 3.2 for $\tau = \sigma$.

$(b) \Rightarrow (c)$ is obvious: $\sigma \in \mathcal{O}_\sigma \subset \overline{\mathcal{O}}_\sigma = \mathcal{L}_\sigma$.

$(c) \Rightarrow (a)$ is obvious: $\sigma \in \mathcal{O}_\sigma \cap \mathcal{L}_\sigma$.

$(a) \Rightarrow (b)$ We know from Proposition 3.8 that $\mathcal{L}_\sigma$ is closed and invariant. If (a) holds, it contains the closure of the orbit $\mathcal{O}_\sigma$. But it cannot be strictly bigger, by (3.1). $\square$

Corollary 3.13. $\Sigma_{\rm rec}$ *is invariant.*

Proof. We can describe the condition $\sigma \in \Sigma_{\rm rec}$ by (b) . For a recurrent point σ and for $a \in \mathsf{G}$ we can write

$$\mathcal{L}_{a\bullet\sigma} = \mathcal{L}_\sigma = \overline{\mathcal{O}}_\sigma = \overline{\mathcal{O}}_{a\bullet\sigma} \, ,$$

so $a \bullet \sigma$ is also recurrent. $\square$

Exercise 3.14. If $f : \Sigma \to \Sigma'$ is an epimorphisms between dynamical systems $(\mathsf{G}, \theta, \Sigma)$ and $\big(\mathsf{G}, \theta', \Sigma'\big)$, one has $f\big(\Sigma^\theta_{\rm rec}\big) \subset \Sigma^{\theta'}_{\rm rec}$ (the image by an epimorphism of a θ-recurrent point is a θ'-recurrent point).

Definition 3.15. (a) The point $\sigma \in \Sigma$ is *wandering* with respect to the action $(\mathsf{G}, \theta, \Sigma)$ with non-compact group if σ has a neighborhood W such that G^W_W is relatively compact.

(b) In the opposite case, we say that σ is *non-wandering*. This means that for every neighborhood W of σ the set G^W_W is not relatively compact.

(c) We denote by $\Sigma^\theta_{\rm nw} \equiv \Sigma_{\rm nw}$ the family of all the non-wandering points. If $\Sigma^\theta_{\rm nw} = \Sigma$ one says that *the action is non-wandering*.

Proposition 3.16. *The set* $\Sigma_{\rm nw}$ *is closed.*

Proof. Let $\tau \in \overline{\Sigma}_{\rm nw}$. By definition of closure, every open neighborhood U of τ intersects $\Sigma_{\rm nw}$.

So let $\sigma \in \Sigma_{\rm nw} \cap U$. As U is a neighborhood of $\sigma \in \Sigma_{\rm nw}$, the set G^U_U is not relatively compact. Thus $\tau \in \Sigma_{\rm nw}$. $\square$

Proposition 3.17. *The set* $\Sigma_{\rm nw}$ *is invariant.*

Proof. Suppose that $\sigma \notin \Sigma_{\rm nw}$; we are going to show that $b \bullet \sigma \notin \Sigma_{\rm nw}$, for all $b \in \mathsf{G}$.

Let W be an open neighborhood of σ such that G^W_W is relatively compact. Then $b \bullet W$ is an open neighborhood of $b \bullet \sigma$. We showed in Lemma 1.28 that

$$\mathsf{G}^{b\bullet W}_{b\bullet W} = b\, \mathsf{G}^W_W b^{-1},$$

which is relatively compact. $\square$

Proposition 3.18. *One has*

$$\Sigma_{\rm rec} \subset \bigcup_{\sigma\in\Sigma} \mathcal{L}_\sigma \subset \Sigma_{\rm nw} \, . \tag{3.3}$$

Proof. The first inclusion follows from the definition of recurrent points:

$$\tau \in \Sigma_{\rm rec} \Leftrightarrow \tau \in \mathcal{L}_\tau \subset \bigcup_{\sigma\in\Sigma} \mathcal{L}_\sigma \subset \Sigma_{\rm nw} .$$

So we only need to prove that $\mathcal{L}_\sigma \subset \Sigma_{\rm nw}$. Pick $\tau \in \mathcal{L}_\sigma$ and let U be a neighborhood of τ. By Lemma 3.2, we already know that G_σ^U is not relatively compact. If $a \in \mathsf{G}_\sigma^U$, then $a \bullet \sigma \in U$; using Lemma 1.28 we get

$$\mathsf{G}_\sigma^U a^{-1} = \mathsf{G}_{a\bullet\sigma}^U \subset \mathsf{G}_U^U ,$$

showing that the later set is not relatively compact. □

Corollary 3.19. *If at least one of the orbits is relatively compact (in particular if Σ is compact), $\Sigma_{\rm nw}$ is non-void.*

Proof. This follows from (3.3) and Proposition 3.9. □

Proposition 3.20. *If Σ is compact, $\Sigma_{\rm nw}$ attracts the points of Σ: for every $\sigma \in \Sigma$ and for every neighborhood V of $\Sigma_{\rm nw}$, one has $a \bullet \sigma \in V$ for every $a \in \mathsf{G}$ outside some compact set.*

Proof. One has to show that for every neighborhood V of $\Sigma_{\rm nw}$, the set $\mathsf{G}\setminus\mathsf{G}_\sigma^V$ is relatively compact.

If V is a neighborhood of $\Sigma_{\rm nw}$, by (3.3), it is also a neighborhood of the limit set $\mathcal{L}_\sigma$. One applies Proposition 3.9 to infer that there exists a compact subset K of G such that $\big(\mathsf{G}\setminus\mathsf{K}\big) \bullet \sigma \subset V$, i. e. $\mathsf{G}\setminus\mathsf{K} \subset \mathsf{G}_\sigma^V$. Then the complement of G_σ^V in G is contained in K and the proof is finished. □

Exercise 3.21. If $f : \Sigma \to \Sigma'$ is an epimorphisms between dynamical systems $(\mathsf{G}, \theta, \Sigma)$ and $\big(\mathsf{G}, \theta', \Sigma'\big)$, one has $f\big(\Sigma_{\rm nw}^\theta\big) \subset \Sigma_{\rm nw}^{\theta'}$ (the image by an epimorphism of a θ-non-wandering point is a θ'-non-wandering point).

4 Minimality and almost periodicity

4.1 Fixed points

Let $(\mathsf{G}, \theta, \Sigma)$ be a continuous group action.

Definition 4.1. *A fixed point* is a point $\sigma \in \Sigma$ such that $a \bullet \sigma = \sigma$ for every $a \in \mathsf{G}$. This is equivalent with $\mathsf{G}_\sigma^\sigma = \mathsf{G}$. We write $\sigma \in \Sigma_{\rm fix}^\theta \equiv \Sigma_{\rm fix}$.

Proposition 4.2. *The set $\Sigma_{\rm fix}$ is closed and invariant.*

Proof. If $\sigma \in \Sigma_{\rm fix}$, then $\mathfrak{O}_\sigma = \{\sigma\}$, which makes $\Sigma_{\rm fix}$ trivially invariant.

If $\sigma \in \overline{\Sigma}_{\rm fix}$, then exists a net $(\sigma_i)_{i\in I}$ of fixed points converging to σ. Let $a \in \mathsf{G}$; by continuity

$$a \bullet \sigma = a \bullet (\lim_j \sigma_i) = \lim_i (a \bullet \sigma_i) = \lim_i \sigma_i = \sigma.$$

So σ is a fixed point for the action. $\square$

4.2 Minimal systems

Minimality is a very important property in classical topological dynamics. During this subsection, both G and Σ are assumed to be locally compact.

Definition 4.3. A closed invariant subset $M \subset \Sigma$ is called *minimal* if it does not contain proper non-void closed (open) invariant subsets.

Equivalently, M is *minimal* if all the orbits contained in M are dense in M.

The action is minimal if Σ itself is minimal.

Remark 4.4. The minimal sets are the closed invariant non-empty subsets of Σ which are minimal under such requirements.

Theorem 4.5. *If Σ is compact, there exists a minimal subset of Σ.*

Proof. Let

$$\mathscr{C}(\Sigma) := \{S \subset \Sigma \,|\, \emptyset \neq S \text{ closed and invariant}\} \neq \emptyset .$$

With the order relation

$$S \leq T \Leftrightarrow S \supset T$$

$(\mathscr{C}(\Sigma), \leq)$ is inductively ordered (each totally ordered subset has an upper bound). WHY ???

Then, by Zorn's Lemma, $(\mathscr{C}(\Sigma), \leq)$ has a maximal element (which is a minimal subset). $\square$

Example 4.6. Fixed points are minimal.

More generally, closed orbits are minimal.

Remark 4.7. Two minimal sets either coincide or are disjoint.

Proposition 4.8. *Transitivity $\Rightarrow$ minimality $\Rightarrow$ pointwise transitivity $\Rightarrow$ topological transitivity.*

Proof. The first two implications are obvious. The last one has been obtained above. $\square$

Exercise 4.9. A closed set S is nowhere dense if and only if it coincides with its boundary.

Proposition 4.10. *Minimal sets are either clopen (closed and open) or nowhere dense ($(\overline{M})^\circ = \emptyset$).*

Proof. The boundary of an invariant set is invariant (Proposition 1.14).

So, if M is minimal, the boundary $\partial M \subset M$ is closed and invariant, therefore it should be void (i.e. M is open) or coincide with M (meaning that M is nowhere dense). □

Corollary 4.11. *Suppose that Σ is connected and not minimal. Any minimal subsystem is nowhere dense.*

Exercise 4.12. A function $\varphi : \Sigma \to \mathbb{R}$ is called *invariant* with respect to the group action if $\varphi(\sigma) = \varphi(\tau)$ whenever $\sigma \overset{\theta}{\sim} \tau$. If the action is minimal and φ is continuous at least at one point, it has to be constant.

We proceed now to characterize minimality.

Proposition 4.13. *Let $\emptyset \neq M \subset \Sigma$ be closed and invariant. Then M is minimal if and only if for every $U \subset \Sigma$ open, with $U \cap M \neq \emptyset$, one has $M = \mathrm{Sat}(U \cap M)$.*

Proof. **if:** Assume that $\emptyset \neq N \subset M$ is closed and invariant; then $U = N^c = \Sigma \setminus N$ is open. If we show that $N^c \cap M = \emptyset$ one gets $M = N$, i.e. the minimality of M. But $N^c \cap M = \emptyset$ follows if we check that $M \not\subset \mathrm{Sat}(N^c \cap M)$, by assumption. This would follow from $N \cap \mathrm{Sat}(N^c \cap M) = \emptyset$. But $\sigma \in N \cap \mathrm{Sat}(N^c \cap M)$ means that $\sigma \in N$ and it is in the orbit of some element of N^c. This is impossible since N is invariant and N^c is its complement.

only if: Assuming now that M is minimal, for every $\sigma \in M$ one has $M = \mathfrak{O}_\sigma = \overline{\mathfrak{O}}_\sigma$. If $U \cap M \neq \emptyset$, U being open, we have $U \cap \mathfrak{O}_\sigma \neq \emptyset$. But this means that σ is in the orbit of some point that belongs to $U \cap \mathfrak{O}_\sigma \subset U \cap M$, so $x \in \mathrm{Sat}(U \cap M)$. □

Proposition 4.14. *If Σ is compact and minimal and G is not compact, the action is non-wandering.*

Proof. We know from Propositions 3.16 and 3.17 and Corollary 3.19 that Σ_{nw} is a non-void closed invariant set. By minimality, it coincides with Σ. □

We finish with results on the behavior of minimality under epimorphisms, in both directions.

Proposition 4.15. *Let $f : \Sigma \to \Sigma'$ be an epimorphism between the actions $(\mathsf{G}, \theta, \Sigma)$ and $\big(\mathsf{G}, \theta', \Sigma'\big)$. If $M \subset \Sigma$ is minimal and $f(M)$ is closed in Σ', then $f(M)$ is minimal.*

Proof. This is dealt with easily by Proposition 1.35: if $\sigma \in M$ then

$$f(M) = f\big(\overline{\mathfrak{O}}_\sigma\big) \subset \overline{\mathfrak{O}}'_{f(\sigma)} \subset \overline{f(M)} = f(M)\,,$$

so the orbit of $f(\sigma)$ is dense in the closed set $f(M)$. □

Proposition 4.16. *Let $f : \Sigma \to \Sigma'$ be an epimorphism between the actions $(\mathsf{G}, \theta, \Sigma)$ and $\big(\mathsf{G}, \theta', \Sigma'\big)$. Suppose that Σ is compact (hence Σ' is also compact). If $M' \subset \Sigma'$ is minimal, there exists $M \subset \Sigma$ minimal such that $f(M) = M'$.*

Proof. The inverse image $f^{-1}(M')$ is non-void closed and $\bullet$-invariant. By Zorn's Lemma, it contains a minimal (and compact) subsystem M. The direct image $f(M) \subset M'$ is non-void closed and $\bullet'$-invariant, so it must coincide with M'. □

Question: Is $f^{-1}(M')$ minimal ?

4.3 Periodic and almost periodic points

Although it is not always necessary, in this subsection we prefer to assume that in the continuous action $(\mathsf{G}, \theta, \Sigma)$ *the group* G *is locally compact but non-compact*.

Definition 4.17. The subset A of G is called *syndetic* if $\mathsf{KA} = \mathsf{G}$ for a compact subset K of G.

Remark 4.18. (a) Besides this left syndeticity, there is also an obvious notion of *right syndeticity*. They coincide (at least if) G is Abelian. A is left (right) syndetic if and only if A^{-1} is right (left) syndetic. A subgroup is left syndetic if and only if it is right syndetic.

(b) If the group is compact (but it is not!) everybody is syndetic.

Exercise 4.19. (a) If $\mathsf{A} \subset \mathsf{B}$ and A is syndetic then B is syndetic.

(b) A relatively compact set is not syndetic.

(c) The complement of a relatively compact set is syndetic.

(d) A closed subgroup H is syndetic iff the quotient space G/H is compact. But one is not only interested in subgroups.

(e) The set A is syndetic if and only if there exists a compact set $\mathsf{L} \subset \mathsf{G}$ such that $(\mathsf{L}a) \cap \mathsf{A} \neq \emptyset$ for every $a \in \mathsf{G}$.

Is $\mathbb{Z} + 0,13$ syndetic in $\mathbb{R}$?

Is $\{\pm n^2 \,|\, n \in \mathbb{N}\}$ syndetic in $\mathbb{R}$? Is $\{\pm n^2 \,|\, n \in \mathbb{N}\} + [-1/9, 1/8]$ syndetic in $\mathbb{R}$?

Is $\{\pm n^2 + [-n, n] \,|\, n \in \mathbb{N}\}$ syndetic in $\mathbb{R}$?

(f) If $\mathsf{G} = \mathbb{Z}$ or $\mathsf{G} = \mathbb{R}$, a subset of G is syndetic if and only if it is *relatively dense*, that is, it does not have arbitrarily large gaps (intervals not intersecting the set).

Definition 4.20. (a) We say that $\sigma \in \Sigma$ is *periodic*, and we write $\sigma \in \Sigma_{\rm per}$, if G_σ^σ is syndetic.

(b) The point σ is called *weakly periodic* (we write $\sigma \in \Sigma_{\rm wper}$) if the subgroup G_σ^σ is not compact.

(c) The point $\sigma \in \Sigma$ is said to be *almost periodic* if G_σ^U is syndetic for every neighborhood U of σ in Σ. We denote by $\Sigma_{\rm alper}$ the set of all the almost periodic points. If $\Sigma_{\rm alper} = \Sigma$, the action is *pointwise almost periodic*.

Proposition 4.21. *One has*

$$\Sigma_{\rm fix} \overset{(1)}{\subset} \Sigma_{\rm per} \overset{(2)}{\subset} \Sigma_{\rm wper} \cap \Sigma_{\rm alper} \overset{(3)}{\subset} \Sigma_{\rm wper} \cup \Sigma_{\rm alper} \overset{(4)}{\subset} \Sigma_{\rm rec} \overset{(5)}{\subset} \bigcup_{\sigma \in \Sigma} \mathfrak{L}_\sigma \overset{(6)}{\subset} \Sigma_{\rm nw}\,. \tag{4.1}$$

Proof. The inclusions (1) and (3) are obvious. We proved (5) and (6) previously, in Proposition 3.18.

To deduce (2) from the definitions, note that $\mathsf{G}_\sigma^\sigma \subset \mathsf{G}_\sigma^U$ if $\sigma \subset U$ and that a syndetic set is not compact (since G not compact).

The inclusion (4) also follows easily from the definitions, by the same type of arguments: use Proposition 8.37 (e) to describe recurrent points. ☐

Example 4.22. Let $\mathsf{G} = \mathbb{R}^2$ act (transitively) on $\Sigma = \mathbb{R}$ by $(a, b) \bullet x := x + a$. Then $(\mathbb{R}^2)_0^0 = \{0\} \times \mathbb{R}$ is not compact, so 0 (and not only!) is weakly periodic. It is not periodic. It is not almost periodic, since $(\mathbb{R}^2)_0^{(-1,1)} = (-1, 1) \times \mathbb{R}$ is not syndetic in $\mathbb{R}^2$.

Proposition 4.23. *The sets* $\Sigma_{\rm per}, \Sigma_{\rm wper}$ *are invariant.*

Proof. By Lemma 1.28, one has $\mathsf{G}_{a\bullet\sigma}^{a\bullet\sigma} = a\, \mathsf{G}_\sigma^\sigma\, a^{-1}$, so $\Sigma_{\rm wper}$ is invariant.

We focus now on $\Sigma_{\rm per}$; let $\sigma \in \Sigma_{\rm per}$ and $a \in \mathsf{G}$. For some compact set K , we have

$$\mathsf{K}\, a^{-1} \mathsf{G}_{a\bullet\sigma}^{a\bullet\sigma}\, a = \mathsf{K}\, \mathsf{G}_\sigma^\sigma = \mathsf{G} = \mathsf{G}\, a\,.$$

Hence

$$(\mathsf{K} a^{-1})\, \mathsf{G}_{a\bullet\sigma}^{a\bullet\sigma} = \mathsf{G}\,,$$

meaning that $\mathsf{G}_{a\bullet\sigma}^{a\bullet\sigma}$ is syndetic in G , and thus $a \bullet \sigma \in \Sigma_{\rm per}$ holds. □

Exercise 4.24. Is $\Sigma_{\rm aper}$ invariant?

Remark 4.25. The sets of periodic, weakly periodic or almost periodic points might fail to be closed.

4.4 Compact orbits

We connect now periodicity with the type of the orbit.

Theorem 4.26. *The periodic points are precisely those having a compact orbit.*

Exercise 4.27. What happens to periodic points under an epimorphism? Two solutions, please!

Proposition 4.28. *Every periodic point* σ *has a compact orbit.*

Proof. If $\sigma \in \Sigma_{\rm per}$, then $\mathsf{K}\, \mathsf{G}_\sigma^\sigma = \mathsf{G}$ for some compact set $\mathsf{K} \subset \mathsf{G}$.

Any net $(\sigma_i) \subset \mathfrak{O}_\sigma$ can be written as $\sigma_i = (b_i a_i) \bullet \sigma$ for some nets $(b_i) \subset \mathsf{K}$, $(a_i) \subset \mathsf{G}_\sigma^\sigma$. By compacity, extract a subnet $(b_j) \subset (b_i)$ such that $b_j \to b \in \mathsf{K}$ and get

$$\sigma_j := (b_j a_j) \bullet \sigma = b_j \bullet (a_j \bullet \sigma) = b_j \bullet \sigma \to a \bullet \sigma \in \mathfrak{O}_\sigma\,.$$

We conclude that $\mathfrak{O}_\sigma$ is compact. □

Proof. Let us indicate a second proof.

For $\sigma \in \Sigma$, let us define the continuous surjective function

$$\alpha^\sigma : \mathsf{G} \to \mathfrak{O}_\sigma \subset \Sigma\,, \quad \alpha^\sigma(a) := \theta_a(\sigma) \equiv a \bullet \sigma\,.$$

If $\sigma \in \Sigma_{\rm per}$, then $\mathsf{K}\, \mathsf{G}_\sigma^\sigma = \mathsf{G}$ for some compact set $\mathsf{K} \subset \mathsf{G}$. By using the definition of G_σ^σ , one gets (?)

$$\mathfrak{O}_\sigma = \alpha^\sigma(\mathsf{G}) = \alpha^\sigma(\mathsf{K})\,,$$

which is compact, as a direct continuous image of a compact set. □

Proposition 4.29. *If the point $\sigma \in \Sigma$ has a compact orbit and the conditions of Corollary 4.39 hold, then G^σ_σ is syndetic in G, so σ is periodic.*

Proof. Let $\mathsf{N}_a \subset \mathsf{G}$ be a relatively compact, open neighborhood of a, for every $a \in \mathsf{G}$. Observe that

$$\mathsf{G} = \mathsf{G}\,\mathsf{G}^\sigma_\sigma = \Big(\bigcup_{a\in\mathsf{G}} \mathsf{N}_a\Big)\,\mathsf{G}^\sigma_\sigma\,.$$

So

$$\mathfrak{O}_\sigma = \mathsf{G}\bullet\sigma = \Big(\bigcup_{a\in\mathsf{G}} \mathsf{N}_a\Big)\mathsf{G}^\sigma_\sigma\bullet\sigma = \bigcup_{a\in\mathsf{G}} \mathsf{N}_a\bullet\sigma\,.$$

As we have a group action, The set $\mathsf{N}_a\bullet\sigma = \alpha^\sigma(\mathsf{N}_a)$ is open and by compactness, there exists a finite index set $\mathsf{F} = \{a_1,\dots,a_n\}$ such that

$$\mathfrak{O}_\sigma = \bigcup_{i=1}^{n} \mathsf{N}_{a_i}\bullet\sigma \subset \Big(\bigcup_{i=1}^{n} \overline{\mathsf{N}}_{a_i}\Big)\bullet\sigma\,.$$

Define $\mathsf{K} = \bigcup_{i=1}^n \overline{\mathsf{N}}_{a_i}$ and notice that for every $a\bullet\sigma \in \mathfrak{O}_\sigma$, there exists $b \in \mathsf{K}$ such that

$$a\bullet\sigma = b\bullet\sigma \Rightarrow (b^{-1}a)\bullet\sigma = \sigma\,,$$

meaning that $b^{-1}a \in \mathsf{G}^\sigma_\sigma$ and $a = b\big(b^{-1}a\big) \in \mathsf{K}\,\mathsf{G}^\sigma_\sigma$. As K is compact, we conclude. □

Definition 4.30. Let $(\mathsf{G}, \theta \equiv \bullet, \Sigma)$ be a dynamical system. To any $\sigma \in \Sigma$ one associates its *orbit*

$$\mathfrak{O}_\sigma := \mathsf{G}\bullet\sigma\,,$$

its *isotropy group*

$$\mathsf{G}^\sigma_\sigma := \big\{a \in \mathsf{G}\,\big|\, a\bullet\sigma = \sigma\big\}$$

its *quotient map*

$$q_\sigma : \mathsf{G} \to \mathsf{G}/\mathsf{G}^\sigma_\sigma$$

and its *orbit map*

$$\alpha_\sigma : \mathsf{G} \to \mathfrak{O}_\sigma \subset \Sigma\,, \quad \alpha_\sigma(a) := \theta_a(\sigma) \equiv a\bullet\sigma\,.$$

Remark 4.31. Since

$$\alpha_\sigma(a) = \alpha_\sigma(b) \;\Leftrightarrow\; a\bullet\sigma = b\bullet\sigma \;\Leftrightarrow\; \big(b^{-1}a\big)\bullet\sigma = \sigma \;\Leftrightarrow\; b^{-1}a \in \mathsf{G}^\sigma_\sigma\,,$$

one has the commutative diagram

$$\begin{tikzcd} \mathsf{G} \arrow[r, "\alpha_\sigma"] \arrow[d, "q_\sigma"'] & \mathfrak{O}_\sigma \\ \mathsf{G}/\mathsf{G}^\sigma_\sigma \arrow[ur, "\beta_\sigma"'] & \end{tikzcd}$$

where β_σ is a continuous bijection.

Definition 4.32. The function $f : X \to Y$ between two topological spaces is

- *open* if $f(A) \subset Y$ is open as soon as $A \subset X$ is open.

- *open at* $x_0 \in X$ if for every neighborhood N of x_0, the image $f(N)$ is open in $f(X) \subset Y$ in the relative topology.

Proposition 4.33. *The following are equivalent:*

(i) The map $\beta_\sigma : \mathsf{G}/\mathsf{G}_\sigma^\sigma \to \mathfrak{O}_\sigma$ *is a homeomorphism.*

(ii) The map $\alpha_\sigma : \mathsf{G} \to \mathfrak{O}_\sigma$ *is open.*

(iii) The map $\alpha_\sigma : \mathsf{G} \to \mathfrak{O}_\sigma$ *is open at* e.

Proof. Exercise in topology. Use the fact that q_σ is open. □

Definition 4.34. The topological space X is called

- σ-compact if it is the countable union of compact subsets.
- *a Baire space* if any countable intersection of dense open sets is dense
  (⇔ any countable union of closed sets with empty interior has empty interior).

Example 4.35. If X is discrete, σ-compact ⇔ countable.

Example 4.36. Hausdorff locally compact spaces and complete metric spaces are Baire.
$\mathbb{Q} \subset \mathbb{R}$ is not a Baire space.

Theorem 4.37. *If* G *is a locally compact* σ*-compact topological group and* $\mathfrak{O}_\sigma$ *is a Baire space in the relative topology, then* β_σ *is a homeomorphism.*

Proof. We are going to prove (iii) in Proposition 4.33, which is equivalent with (i).

Claim: *Let* U *be a neighborhood of* e *and let* V *another neighborhood of* e *such that* $V^{-1}V \subset U$. *Then* $V \bullet \sigma$ *has non-empty interior in* $\mathfrak{O}_\sigma$.

We first show that *the Claim finishes the proof*:

Let $b \in V$ such that
$$b \bullet \sigma \in (V \bullet \sigma)^\circ \subset V \bullet \sigma .$$
Then $b^{-1} \bullet (V \bullet \sigma) = \left(b^{-1}V\right) \bullet \sigma$ is a neighborhood of $\sigma = b^{-1} \bullet (b \bullet \sigma)$ in $\mathfrak{O}_\sigma$. Since
$$\left(b^{-1}V\right) \bullet \sigma \subset \left(V^{-1}V\right) \bullet \sigma \subset U \bullet \sigma ,$$
then $U \bullet \sigma = \alpha_\sigma(U)$ is also a neighborhood of σ in $\mathfrak{O}_\sigma$.

We now *prove the Claim*. Let W be a symmetric neighborhood of e with compact $\overline{W} \subset V$.

Exercise There exists a countable subset $A \subset \mathsf{G}$ such that
$$\mathsf{G} = AW = A\overline{W} = \bigcup_{a \in A} a\overline{W} .$$
(By σ-compactness, write $\mathsf{G} = \bigcup_{n \in \mathbb{N}} K_n$; then cover each K_n by a finite number of translates of $\overline{W}$.)

The set $\left(A\overline{W}\right) \bullet \sigma$ is compact, thus closed in $\mathfrak{O}_\sigma$. By the Baire property, there is $c \in A$ such that $\left(c\overline{W}\right) \bullet \sigma$ has non-empty interior in $\mathfrak{O}_\sigma$. Since θ_c is a homeomorphism of the orbit $\mathfrak{O}_\sigma$, then $\overline{W} \bullet \sigma$ has non-empty interior in $\mathfrak{O}_\sigma$. But $\overline{W} \bullet \sigma \subset V \bullet \sigma$, hence $V \bullet \sigma$ also has non-empty interior in $\mathfrak{O}_\sigma$. □

Example 4.38. A subset $\mathfrak{O}$ in a (Hausdorff) locally compact space Σ is locally compact in the relative topology if and only if it is *locally closed*, i.e. it satisfies one of the next equivalent properties:

- It is the intersection between an open and a closed set.
- It is open in its closure.

Corollary 4.39. *Let $(\mathsf{G},\theta,\Sigma)$ be a dynamical system, where G is a Hausdorff, locally compact σ-compact group. Let $\sigma\in\Sigma$ and assume one of the following conditions:*

- *Σ is a complete metric space and the orbit $\mathfrak{O}_\sigma$ is closed,*
- *Σ is locally compact and the orbit $\mathfrak{O}_\sigma$ is locally closed.*

Then $\beta_\sigma:\mathsf{G}/\mathsf{G}^\sigma_\sigma\to\mathfrak{O}_\sigma$ is a homeomorphism.

4.5 Connection between almost periodicity and minimality

$(\mathsf{G},\theta,\Sigma)$ is still a dynamical system, with $\mathsf{G}\,,\Sigma$ (Hausdorff and) locally compact.

Theorem 4.40. *The point σ is almost periodic if and only if $\overline{\mathfrak{O}}_\sigma$ is minimal and compact.*

Proof. Suppose that σ is almost periodic; we show first that its orbit closure $\overline{\mathfrak{O}}_\sigma$ is compact.

Let U_0 be a compact neighborhood of σ. Using the assumptions, for some compact set K one has

$$\mathfrak{O}_\sigma=\mathsf{G}\bullet\sigma=\big(\mathsf{K}\,\mathsf{G}_\sigma^{U_0}\big)\bullet\sigma=\mathsf{K}\bullet\big(\mathsf{G}_\sigma^{U_0}\bullet\sigma\big)\subset\mathsf{K}\bullet U_0=\text{compact}\,,$$

so $\overline{\mathfrak{O}}_\sigma$ is compact.

If $\overline{\mathfrak{O}}_\sigma$ is not minimal, it strictly contains a minimal (and compact) set M. The point σ does not belong to M, so there are disjoint open sets $U,V\subset\Sigma$ such that $\sigma\in U$ and $M\subset V$. For an arbitrary compact set $\mathsf{K}\subset\mathsf{G}$ we will now show that $\mathsf{K}\,\mathsf{G}_\sigma^U\neq\mathsf{G}$, implying that in fact σ is not almost periodic.

The set M being invariant, $\mathsf{K}^{-1}\bullet M\subset M$ holds. Let then W be a neighborhood of M with $\mathsf{K}^{-1}\bullet W\subset V$. Since $M\subset\overline{\mathfrak{O}}_\sigma=\overline{\mathsf{G}\bullet\sigma}$, there exists $b\in\mathsf{G}$ such that $b\bullet\sigma\in W$. Then

$$\big(\mathsf{K}^{-1}b\big)\bullet\sigma=\mathsf{K}^{-1}\bullet(b\bullet\sigma)\subset\mathsf{K}^{-1}\bullet W\subset V,$$

and thus $\big(\mathsf{K}^{-1}b\big)\bullet\sigma$ is disjoint from U, meaning that $\mathsf{K}^{-1}b$ is disjoint from G_σ^U. This shows that

$$\mathsf{G}\ni b\notin\mathsf{K}\,\mathsf{G}_\sigma^U\neq\mathsf{G}\,,$$

finishing the proof.

For the converse, suppose now that $\overline{\mathfrak{O}}_\sigma=\overline{\mathsf{G}\bullet\sigma}$ is minimal and compact. Let U be an open neighborhood of σ. We prove that G_σ^U is syndetic.

For each $a\in\mathsf{G}$, choose an open neighborhood N_a of a with compact closure. The sets $\mathsf{G}\bullet U$ and $\mathsf{N}_a\bullet U$ are open in Σ. By minimality one has

$$\overline{\mathfrak{O}}_\sigma\subset\mathsf{G}\bullet U=\bigcup_{a\in\mathsf{G}}a\bullet U\subset\bigcup_{a\in\mathsf{G}}\mathsf{N}_a\bullet U.$$

By compactness of $\overline{\mathfrak{O}}_\sigma$ applied to the open cover above, for a finite set $\mathsf{F}=\{a_1,\dots,a_k\}\subset\mathsf{G}$ we get

$$\overline{\mathfrak{O}}_\sigma\subset\bigcup_{i=1}^k\mathsf{N}_{a_i}\bullet U\subset\bigcup_{i=1}^k\overline{\mathsf{N}_{a_i}}\bullet U=\Big(\bigcup_{i=1}^k\overline{\mathsf{N}_{a_i}}\Big)\bullet U=:\mathsf{K}\bullet U\,.$$

If $b \in \mathsf{G}$ then $b \bullet \sigma \in \mathsf{K} \bullet U$ and then one has $b \bullet \sigma \in c \bullet U$ for some $c \in \mathsf{K}$. Then

$$c^{-1} \bullet (b \bullet \sigma) = \big(c^{-1} b\big) \bullet \sigma \in U.$$

This means that $c^{-1} b \in \mathsf{G}_\sigma^U$ or, equivalently, that

$$b \in c\, \mathsf{G}_\sigma^U \subset \mathsf{K} \mathsf{G}_\sigma^U .$$

Since b is arbitrary one gets $\mathsf{G} = \mathsf{K}\, \mathsf{G}_\sigma^U$, where $\mathsf{K} := \bigcup_{i=1}^k \overline{\mathsf{N}_{a_i}}$ is compact. We checked that G_σ^U is syndetic in G, so σ is almost periodic. □

Remark 4.41. If Σ is a compact space, Zorn's Lemma implies that it has a minimal subset $M \subset \Sigma$. Since every $x \in M$ is almost periodic, by Theorem 7.8, almost periodic points always exist.

Corollary 4.42. *The set $\Sigma_{\rm alper}$ is invariant.*

Proof. The second part of Theorem 7.8 guarantees that, if $\sigma \in \Sigma_{\rm alper}$, then $\mathfrak{O}_\sigma \subset \Sigma_{\rm alper}$. □

Corollary 4.43. *Suppose that Σ is compact and the action is minimal. Then G_σ^V is syndetic for every $\sigma \in \Sigma$ and every open non-void subset V of Σ (and not just for neighborhoods of σ).*

Proof. By minimality, there exists $z \in \mathsf{G}$ such that V is an open neighborhood of $z \bullet \sigma$. Hence $\mathsf{G}_{z\bullet\sigma}^V$ is syndetic by Theorem 7.8; it can be written as $\mathsf{K}\, \mathsf{G}_{z\bullet\sigma}^V = \mathsf{G}$ for some compact subset K of G. Then, using Lemma 1.28

$$\mathsf{G} = \mathsf{G}\, z = \mathsf{K}\, \mathsf{G}_{z\bullet\sigma}^V\, z = \mathsf{K}\, \mathsf{G}_\sigma^V ,$$

meaning that G_σ^V is also syndetic. □

We now lift almost periodic points through epimorphisms.

Proposition 4.44. *Let $f : \Sigma \to \Sigma'$ be an epimorphism between the actions $(\mathsf{G}, \theta, \Sigma)$ and $\big(\mathsf{G}, \theta', \Sigma'\big)$. Suppose that Σ is compact (hence Σ' is also compact). If $\sigma' \in \Sigma'$ is almost periodic, then $\sigma' = f(\sigma)$ for some almost periodic point σ of Σ.*

You can write: $\sigma' \in \Sigma'_{\rm alper} \Rightarrow f^{-1}(\sigma) \cap \Sigma_{\rm alper} \neq \emptyset$.

Proof. As σ' is almost periodic, $M' := \overline{\mathfrak{O}}'_{\sigma'}$ is minimal, so we can find some (compact) minimal subset $M \subset \Sigma$, such that $f(M) = M'$ and $\sigma' = f(\sigma)$ for some $\sigma \in M$. By Theorem 7.8, any $\sigma \in M$ is almost periodic. □

Definition 4.45. (a) If $\Sigma_{\rm alper} = \Sigma$ (all the points are almost periodic), we say that the action is *pointwise almost periodic*.

(b) We say that the action is *semisimple* if all the orbit closures are minimal (equivalently: the orbit closures form a partition of Σ).

Corollary 4.46. *(a) If all the orbits are closed, the action is semisimple.*

(b) A pointwise almost periodic action is semisimple.

(c) If all the orbits are compact, the action is pointwise almost periodic.

Proof. The statements are obvious or they follow easily from Theorem 7.8. □

5 Mixing

Once again $(\mathsf{G}, \theta, \Sigma)$ is a topological dynamical system with non-compact group G .

Definition 5.1.
- The recurrence set corresponding to $U, V \subset \Sigma$

$$\mathsf{G}_U^V = \{a \in \mathsf{G} \,|\, (a \bullet U) \cap V \neq \emptyset\} = \{a \in \mathsf{G} \mid \exists\, \sigma \in U,\, \theta_a(\sigma) \in V\} \subset \mathsf{G}\,.$$

- The action is *topologically transitive* if for every $U, V \subset \Sigma$ open and non-void, $\mathsf{G}_U^V \neq \emptyset$ holds.
- We say that the action is *non-wandering* if for every open non-void $W \subset \Sigma$ the set G_W^W is not relatively compact.

Let us set $\mathrm{Top}(X)$ for the topology (all the open sets) of a topological space X as well as

$$\mathrm{Top}^*(X) := \mathrm{Top}(X) \setminus \{\emptyset\}\,.$$

Definition 5.2.
- The action $(\mathsf{G}, \theta, \Sigma)$ is *weakly mixing* whenever for every $U, U', V, V' \in \mathrm{Top}^*(\Sigma)$ non-empty open sets, one has $\mathsf{G}_U^V \cap \mathsf{G}_{U'}^{V'} \neq \emptyset$.
- It is *strongly mixing* if the complement of G_U^V is relatively compact for every $U, V \in \mathrm{Top}^*(\Sigma)$.

Proposition 5.3. *Strongly mixing implies weakly mixing, which implies topological transitivity.*

Proof. The first assertion follows from the equality

$$\big(\mathsf{G}_U^V \cap \mathsf{G}_{U'}^{V'}\big)^c = \big(\mathsf{G}_U^V\big)^c \cup \big(\mathsf{G}_{U'}^{V'}\big)^c$$

and the fact that the union of two relatively compact sets is relatively compact, so $\big(\mathsf{G}_U^V \cap \mathsf{G}_{U'}^{V'}\big)^c \neq \mathsf{G}$.

For the second one just take $U = U'$ and $V = V'$. □

Exercise 5.4. Strongly mixing implies non-wandering.

Example 5.5. The Bernoulli shift is strongly mixing.

Proposition 5.6. *The action* $(\mathsf{G}, \theta, \Sigma)$ *is weakly mixing if and only if the (diagonal) product system* $(\mathsf{G}, \Theta \equiv \diamond, \Sigma \times \Sigma)$ *given by*

$$\Theta_g\big(\sigma_1, \sigma_2\big) := \big(\theta_g(\sigma_1), \theta_g(\sigma_2)\big)$$

is topologically transitive.

Proof. With respect with the product action one has

$$\begin{aligned} g \in \mathsf{G}_{U \times U'}^{V \times V'} &\Leftrightarrow\; g \diamond \big(U \times U'\big) = (g \bullet U) \times (g \bullet U') \subset V \times V' \\ &\Leftrightarrow\; g \bullet U \subset V \text{ and } g \bullet U' \subset V' \\ &\Leftrightarrow\; g \in \mathsf{G}_U^V \cap \mathsf{G}_{U'}^{V'} \end{aligned}$$

We have shown that $\mathsf{G}_{U \times U'}^{V \times V'} = \mathsf{G}_U^V \cap \mathsf{G}_{U'}^{V'}$, so they are simultaneously non-void.

Then recall the topology of $\Sigma \times \Sigma$. □

Proposition 5.7. *Let* $(\mathsf{G},\theta,\Sigma)\xrightarrow{f}\big(\mathsf{G},\theta',\Sigma'\big)$ *be an epimorphism*

- *If* $(\mathsf{G},\theta,\Sigma)$ *is weakly mixing,* $\big(\mathsf{G},\theta',\Sigma'\big)$ *is weakly mixing.*
- *If* $(\mathsf{G},\theta,\Sigma)$ *is strongly mixing,* $\big(\mathsf{G},\theta',\Sigma'\big)$ *is strongly mixing.*

Proof. For $U,V\subset\Sigma$ we showed that $\mathsf{G}_U^V\subset\mathsf{G}_{f(U)}^{f(V)}$.

I will do "strongly mixing". "Weakly mixing" is an EXERCISE.

Let $U',V'\in\mathrm{Top}^*(\Sigma')$ and set $U:=f^{-1}(U')\,,V:=f^{-1}(V')\in\mathrm{Top}^*(\Sigma)$. One has

$$U'\supset f\big[f^{-1}(U')\big]=f(U)\,,\quad V'\supset f\big[f^{-1}(V')\big]=f(V)\,,$$

so we can write

$$\mathsf{G}_{f^{-1}(U')}^{f^{-1}(V')}\subset\mathsf{G}_{f[f^{-1}(U')]}^{f[f^{-1}(V')]}\subset\mathsf{G}_{U'}^{V'}\,.$$

If $\mathsf{G}_{f^{-1}(U')}^{f^{-1}(V')}$ is the complement of a relatively compact set, then $\mathsf{G}_{U'}^{V'}$ is also the complement of a relatively compact set.

WHERE DID I USE SURJECTIVITY ? IS IT POSSIBLE TO WEAKEN IT ? $\square$

Definition 5.8. Let Y be a set and $\mathscr{B},\mathscr{F}\subset 2^Y$ two non-void families of subsets of Y.

- We say that $\mathscr{F}$ is a *filter* if $\emptyset\notin\mathscr{F}$,

$$A,B\in\mathscr{B}\;\Rightarrow\;A\cap B\in\mathscr{F}\,,$$

$$\mathscr{F}\ni C\subset D\;\Rightarrow\;D\in\mathscr{F}\,.$$

- We say that $\mathscr{B}$ is a *filter base* if $\emptyset\notin\mathscr{B}$ and

$$A,B\in\mathscr{B}\;\Rightarrow\;\exists\,C\in\mathscr{B}\,,\;C\subset A\cap B\,.$$

Exercise 5.9. *The filter* $\mathscr{F}(\mathscr{B})$ *generated by the filter base* $\mathscr{B}$:

$$\mathscr{F}(\mathscr{B}):=\big\{F\in 2^Y\;\big|\;\exists\,B\in\mathscr{B}\,,\;B\subset F\big\}$$

is a filter.

Proposition 5.10. *Suppose that* G *is Abelian.*

Then $(\mathsf{G},\theta,\Sigma)$ *is weakly mixing if and only if*

$$\mathscr{B}:=\big\{\mathsf{G}_U^V\;\big|\;U,V\in\mathrm{Top}'(\Sigma)\big\}$$

is a filter base in G.

We showed in Proposition 1.23 that each G_U^V is an open subset of G.

Proof. $\Leftarrow$ is obvious from
$$\emptyset \neq \mathsf{G}_U^V \subset \mathsf{G}_{U_1}^{U_2} \cup \mathsf{G}_{V_1}^{V_2}\,,$$
where you start with $U_1, V_1, U_2, V_2 \in \mathrm{Top}'(\Sigma)$.

$\Rightarrow$ Let $U_1, V_1, U_2, V_2 \in \mathrm{Top}'(\Sigma)$. There exists $g \in \mathsf{G}_{U_1}^{U_2} \cup \mathsf{G}_{V_1}^{V_2}$. We are going to prove that

$$\mathsf{G}_{U_1\cap(g^{-1}\bullet U_2)}^{V_1\cap(g^{-1}\bullet V_2)} \subset \mathsf{G}_{U_1}^{V_1} \cap \mathsf{G}_{U_2}^{V_2}\,.$$

If $h \in \mathsf{G}_{U_1\cap(g^{-1}\bullet U_2)}^{V_1\cap(g^{-1}\bullet V_2)}$, then

$$\begin{aligned}
\emptyset \neq h \bullet \big[U_1 \cap (g^{-1} \bullet U_2)\big] \,\cap\, \big[V_1 \cap (g^{-1} \bullet V_2)\big] \\
\subset \big[(h \bullet U_1) \cap V_1\big] \,\cap\, \big[(hg^{-1}) \bullet U_2 \,\cap\, g^{-1} \bullet V_2\big] \\
= \big[(h \bullet U_1) \cap V_1\big] \,\cap\, g^{-1} \bullet \big[(h \bullet U_2) \,\cap\, V_2\big]\,,
\end{aligned}$$

which implies that
$$(h \bullet U_1) \cap V_1 \neq \emptyset \neq (h \bullet U_2) \,\cap\, V_2\,,$$
i.e. $h \in \mathsf{G}_{U_1}^{V_1} \cap \mathsf{G}_{U_2}^{V_2}$. □

Of course, for any $n \in \mathbb{N}$

$$\theta_g^n\big(\sigma_1, \dots, \sigma_n\big) := \big(\theta_g(\sigma_1), \dots, \theta_g(\sigma_n)\big)$$

defines a continuous action of G on $\Sigma^n := \Sigma \times \cdots \times \Sigma$.

Corollary 5.11. *If $(\mathsf{G}, \theta, \Sigma)$ is weakly mixing and G is Abelian, $\big(\mathsf{G}, \theta^n, \Sigma^n\big)$ is also weakly mixing, for every positive integer n .*

Proof. One first shows that θ^k is topologically transitive, for every $k \in \mathbb{N}$. EXERCICE.

One has $\theta^n \times \theta^n \cong \theta^{2n}$ (canonical isomorphism of dynamical systems). Setting $k := 2n$:

$$\theta^{2n} \text{ topologically transitive } \Leftrightarrow\, \theta^n \text{ weakly mixing.}$$

□

Proposition 5.12. *Suppose that G is Abelian.*

Then $(\mathsf{G}, \theta, \Sigma)$ is weakly mixing if and only if G_U^V is thick for every $U, V \in \mathrm{Top}'(\Sigma)$.

Proof. □

6 Generalized morphisms of groupoid actions

Definition 6.1. Let $(\mathsf{G},\theta,\Sigma)$, $(\mathsf{G}',\theta',\Sigma')$ be two dynamical systems.

- *A generalized morphism of actions* is a pair (Ψ,f) , where $\Psi:\mathsf{G}\to\mathsf{G}'$ a continuous group morphism and $f:\Sigma\to\Sigma'$ is a continuous function, such that
$$f\big[\theta_a(\sigma)\big]=\theta'_{\Psi(a)}\big[f(\sigma)\big]\,,\quad\forall\,a\in\mathsf{G}\,,\sigma\in\Sigma\,. \tag{6.1}$$
- If both Ψ and f are surjective, we say that (Ψ,f) is *an epimorphism*, $(\mathsf{G}',\theta',\Sigma')$ is *a factor* of $(\mathsf{G},\theta,\Sigma)$ and $(\mathsf{G},\theta,\Sigma)$ is *an extension* of $(\mathsf{G}',\theta',\Sigma')$.
- If $\mathsf{G}=\mathsf{G}'$ and $\Psi=\mathrm{id}_{\mathsf{G}}$, we recover an old friend.

Remark 6.2. The relation (6.1) may be written as
$$f\circ\theta_a=\theta'_{\Psi(a)}\circ f\,,$$
(so it is easy to illustrate it by a commuting diagram) or even as
$$f(a\bullet\sigma)=\Psi(a)\bullet' f(\sigma)\,. \tag{6.2}$$

Remark 6.3. Let $(\mathsf{G}',\theta',\Sigma')$ be an action. Suppose that G is a topological group and $\Psi:\mathsf{G}\to\mathsf{G}'$ is a continuous group morphism. Then we can make G act on Σ' by defining
$$\theta_a(\sigma'):=\theta'_{\Psi(a)}(\sigma')\,.$$
In this case, Ψ gives birth to a morphism $(\Psi,\mathrm{id}_{\Sigma'})$ between the actions $(\mathsf{G},\theta,\Sigma)$ and $(\mathsf{G}',\theta',\Sigma')$.

Exercise 6.4. If (Ψ_1,f_1) is a morphism from $(\mathsf{G},\theta,\Sigma)$ to $(\mathsf{G}',\theta',\Sigma')$ and (Ψ_2,f_2) is a morphism from $(\mathsf{G}',\theta',\Sigma')$ to $(\mathsf{G}'',\theta'',\Sigma'')$, then $(\Psi_2,f_2)\circ(\Psi_1,f_1):=(\Psi_2\circ\Psi_1,f_2\circ f_1)$ is a morphism from $(\mathsf{G},\theta,\Sigma)$ to $(\mathsf{G}'',\theta'',\Sigma'')$.

Actually, with this morphisms, the group actions form a category.

Proposition 6.5. *Let (Ψ,f) be a morphism between the actions $(\mathsf{G},\theta,\Sigma)$ and $(\mathsf{G}',\theta',\Sigma')$. If $M,N\subset\Sigma$ then*
$$\Psi\big(\mathsf{G}_M^N\big)\subset\big(\mathsf{G}'\big)_{f(M)}^{f(N)}\,. \tag{6.3}$$
If Ψ is surjective and f is injective, in (6.3) *one obtains the equality. Compare with Proposition 1.35.*

Proof. Let $g\in\mathsf{G}_M^N$ and $\sigma\in M$ such that $g\bullet\sigma\in N$. Then
$$\Psi(a)\bullet'f(\sigma)=f(a\bullet\sigma)\in f(N)\,,$$
which shows (6.3).

On the other hand, pick $a'\in\mathsf{G}'$ and $\sigma\in M$ such that $a'\bullet' f(\sigma)\in f(N)$. If Ψ is surjective, this $a'\in\big(\mathsf{G}'\big)_{f(M)}^{f(N)}$ can be written as $a'=\Psi(a)$. Then
$$a'\bullet'f(\sigma)=\Psi(a)\bullet'f(\sigma)=f(a\bullet\sigma)\in f(N)\,.$$
By injectivity of f this implies that $a\bullet\sigma\in N$, implying in its turn that $a\in\mathsf{G}_M^N$, so the equality in (6.3) follows. ☐

Exercise 6.6. Examine the effect of a generalized morphism (maybe of a certain type) on orbits, invariant sets, dynamical properties .....

7 Actions on circles and tori

7.1 Actions of $\mathbb{R}$ on circles (continuous rotations)

First we take $\mathsf{G} := \mathbb{R} \ni s, t$ and $\Sigma := \mathbb{T} := \mathbb{R}/\mathbb{Z} \ni [x], [y]$ or equivalently $\Sigma := \mathbb{S} \ni z, z', \zeta, e^{2\pi i s}$.

Definition 7.1. Let $\vartheta \in \mathbb{R}$; we define the continuous action (CHECK!)

$$\theta : \mathbb{R} \to \mathrm{Homeo}(\mathbb{T}) \,, \quad \theta_t([x]) \equiv t \bullet [x] := [x + \vartheta t] \,.$$

Equivalently:

$$\theta : \mathbb{R} \to \mathrm{Homeo}(\mathbb{S}) \,, \quad \theta_t(z) := z e^{2\pi i \vartheta t} \,.$$

Example 7.2. If $\vartheta = 0$, one has the trivial action (only fixed points)

$$\theta_t([x]) = [x] \,, \quad \forall\, t \in \mathbb{R} \,, \, [x] \in \mathbb{T} \,.$$

Example 7.3. If $\vartheta \neq 0$, one has one orbit (transitive system) with isotropy group

$$\mathbb{R}_0^0 = \vartheta^{-1}\mathbb{Z} = \mathbb{R}_{[x]}^{[x]} \quad \text{for any} [x] \in \mathbb{T} \,.$$

This corresponds to our definition of periodicity (the isotropy group is syndetic: $\vartheta^{-1}\mathbb{Z} + \left[0, \vartheta^{-1}\right] = \mathbb{R}$). The orbit is compact and homeomorphic to $\mathbb{R}/\mathbb{Z}$.

Exercise 7.4. The actions θ^1 and θ^2 are isomorphic if and only if the real numbers ϑ^1 and ϑ^2 are equal.

What is the most general form of a closed subgroup of $\mathbb{R}$? Define the period of a periodic $\mathbb{R}$-orbit. What are the periods in our case? What should happen with the periods of two isomorphis actions? The rotation speed is important here! You can also use the isotropy subgroup!

For translations of $\mathbb{R}$ upon itself, the velocity is NOT important. For $0 \neq \beta \neq 0$ find an isomorphism between

$$t \bullet_\beta x := t + \beta x \quad \text{and} \quad t \bullet_\gamma x := t + \gamma x$$

7.2 Actions of $\mathbb{Z}$ on circles (discrete rotations)

For $\vartheta \in \mathbb{R}$, the map $\theta_1 ; \mathbb{T} \to \mathbb{T}$ is an automorphism. By iteration

$$\theta_{\pm m} := \left(\theta_1^{-1}\right)^m \quad \forall m \in \mathbb{N}$$

one gets an action of $\mathbb{Z}$ on $\mathbb{T}$. Anyhow, it is a restriction of the previous θ from $\mathbb{R}$ to $\mathbb{Z}$ and

$$\theta_n([x]) := [x + n\vartheta] \,, \quad \forall\, n \in \mathbb{Z} \,, \, [x] \in \mathbb{T} \,.$$

Proposition 7.5. *Let* $[a] \in \mathbb{T}$ *the function*

$$f_{[a]} : \mathbb{T} \to \mathbb{T} \,, \quad f_{[a]}([x]) := [a] + [x] \equiv [a + x]$$

is an automorphism of the dynamical system $(\mathbb{Z}, \theta, \mathbb{T})$.

Proof. Obviously, $f_{[a]}$ is a homeomorphisms.

I will do the following computation for $n \in \mathbb{Z}$, but it would be enough to do it for $n = 1$:

$$f_{[a]}\big(\theta_n([x])\big) = f_{[a]}\big([x + n\vartheta]\big) = [a + x + n\vartheta] = \theta_n\big([a + x]\big) = \theta_n\big[f_{[a]}([x])\big] \,,$$

so $f_{[a]} \circ \theta_n = \theta_n \circ f_{[a]}$ for every $n \in \mathbb{Z}$. ☐

Remark 7.6. Consequently, the orbit of $[0]$ is mapped over the orbit of $[a]$ for every $[a] \in \mathbb{T}$. All the points of $\mathbb{T}$ have similar properties under the action θ.

Definition 7.7. (a) We say that $\sigma \in \Sigma$ is *periodic*, and we write $\sigma \in \Sigma_{\rm per}$, if G_σ^σ is syndetic.

(b) The point $\sigma \in \Sigma$ is said to be *almost periodic* if G_σ^U is syndetic for every neighborhood U of σ in Σ. We denote by $\Sigma_{\rm alper}$ the set of all the almost periodic points.

(c) If $\Sigma_{\rm alper} = \Sigma$, the action is *pointwise almost periodic*.

Theorem 7.8. *The point σ is almost periodic if and only if $\overline{\mathfrak{O}}_\sigma$ is minimal and compact.*

Proposition 7.9. *In $(\mathbb{Z}, \theta, \mathbb{T})$ all the points are almost periodic.*

Proof. Since $\mathbb{T}$ is compact, it has a minimal subsystem. So there is a point with (compact) minimal orbit closure; it will be almost periodic. Then the other points will also have minimal orbit closures, and will be almost periodic. □

Exercise 7.10. Make a direct proof, please !

Remark 7.11. We showed before that

$$\Sigma_{\rm fix} \subset \Sigma_{\rm per} \subset \Sigma_{\rm alper} \subset \Sigma_{\rm rec} \subset \bigcup_{\sigma \in \Sigma} \mathfrak{L}_\sigma \subset \Sigma_{\rm nw}\,. \tag{7.1}$$

In our case, the chain starts with $\Sigma_{\rm alper}$.

If (and only if) $\vartheta \in \mathbb{Q} \setminus \{0\}$, it starts with $\Sigma_{\rm per}$.

If $\vartheta \in \mathbb{R}$, it is full.

Exercise 7.12. The point σ is called *weakly periodic* (we write $\sigma \in \Sigma_{\rm wper}$) if the subgroup G_σ^σ is not compact. What happens in our case ?

Exercise 7.13. If $\vartheta = p/q \in \mathbb{Q}$ (irreducible fraction, with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$), all the points of $\mathbb{T}$ are periodic with period q.

Theorem 7.14. *The discrete dynamical system $(\mathbb{Z}, \theta, \mathbb{T})$ is minimal if and only if ϑ is irrational.*

So for $\vartheta \notin \mathbb{Q}$, all the points are almost periodic without being periodic.

Proof. "Only if" is clear. In the setting of Exercise 7.13 the orbit will be finite, composed of q points.

"If". It is known that there are four types of subgroups of $\mathbb{R}$:

(i) $\{0\}$, (ii) $\mathbb{R}$, (iii) $a\mathbb{Z}$ for some $a > 0$ and (iv) dense subgroups.

Let us denote by $p : \mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$ the quotient map and $\mathcal{O}_{[0]} \subset \mathbb{T}$ the orbit of the point $[0]$. Then

$$\begin{aligned}
\mathsf{H} := {} & p^{-1}\big(\mathcal{O}_{[0]}\big) \\
= {} & \{x \in \mathbb{R} \mid p(x) = [x] = x + \mathbb{Z} \in \mathcal{O}_{[0]}\} \\
= {} & \{x \in \mathbb{R} \mid x + \mathbb{Z} = n\vartheta + \mathbb{Z} \text{ for some } n \in \mathbb{Z}\} \\
= {} & \{m + n\vartheta \mid m, n \in \mathbb{Z}\}
\end{aligned}$$

is a subgroup of $\mathbb{R}$. This is a mixture of rational and irrational numbers, so one cannot have $\mathsf{H} \subset a\mathbb{Z}$ for some positive real number a. □

Exercise 7.15. "If" would deserve a different proof:

Let $k \in \mathbb{N}$; then $\mathbb{T}$ can be covered by k closed intervals of length $1/k$. Show that each of this interval contains two of the points of the (countable) positive orbit

$$\mathcal{O}^{+}_{[0]} = \{[n\vartheta] \mid n \in \mathbb{N}\}$$

(these points are different).

Show that the orbit $\mathcal{O}_{[0]}$ is $1/k$-dense in $\mathbb{T}$. Note that k is arbitrary.

Exercise 7.16. Recall that

Strongly mixing $\Rightarrow$ weaklyly mixing $\Rightarrow$ topologically transitive $\Leftarrow$ minimal.

For $\vartheta \notin \mathbb{Q}$ our system is minimal. Show that it is not weakly mixing.

Definition 7.17. A dynamical system $(\mathsf{G}, \theta, \mathbb{Z})$ is called *coalescent* if each of its endomorphisms is an automorphism.

Proposition 7.18. *If $\vartheta \notin \mathbb{Q}$, then $(\mathbb{Z}, \theta, \mathbb{T})$ is coalescent.*

Proof. Let $f : \mathbb{T} \to \mathbb{T}$ be a homeomorphism. One has by equivariance

$$f\big([n\vartheta]\big) = f\big([0] + [n\vartheta]\big) = f\big([0]\big) + [n\vartheta]\,, \quad \forall\, n \in \mathbb{N}\,.$$

It follows that on the orbit of [0] the endomorphism is just rotation by $f\big([0]\big)$.

If $\vartheta \notin \mathbb{Q}$, this orbit is dense, so f is just rotation by $f\big([0]\big)$.

This guy is an automorphism. □

7.3 Isomorphisms between actions on circles

In this subsection $\vartheta, \vartheta^1, \vartheta^2$ are real numbers.

Proposition 7.19. *First we consider the (transitive) dynamical systems $(\mathbb{R}, \theta, \mathbb{R})$, where*

$$\theta_t(x) := x + \vartheta t\,, \quad \forall t, x \in \mathbb{R}\,.$$

If $\vartheta^1 \neq 0 \neq \vartheta^2$, the actions $\big(\mathbb{R}, \theta^1, \mathbb{R}\big)$ and $(\mathbb{R}, \theta^2, \mathbb{R})$ are isomorphic through

$$f : \mathbb{R} \to \mathbb{R}\,, \quad f(x) := \frac{\vartheta^2}{\vartheta^1}\, x\,.$$

Proof.

$$\theta^2_t[f(x)] = f(x) + \vartheta^2 t = \frac{\vartheta^2}{\vartheta^1}\, x + \vartheta^2 t = \frac{\vartheta^2}{\vartheta^1}\left(x + \vartheta^1 t\right) = f\left[\theta^1_t(x)\right].$$

□

Proposition 7.20. *We now consider the continuous actions on the torus (circle) $(\mathbb{R}, \theta, \mathbb{T})$ given by*

$$\theta_t([x]) := [x + \vartheta t]\,, \quad \forall\, t \in \mathbb{R}\,,\ [x] \in \mathbb{T} := \mathbb{R}/\mathbb{Z}\,.$$

The actions $\big(\mathbb{R}, \theta^1, \mathbb{T}\big)$ and $\big(\mathbb{R}, \theta^2, \mathbb{T}\big)$ are isomorphic if and only if the real numbers ϑ^1 and ϑ^2 are equal.

Proof. What is the most general form of a closed subgroup of $\mathbb{R}$? Define the period of a periodic $\mathbb{R}$-orbit. What are the periods in our case? What should happen with the periods of two isomorphis actions? The rotation speed is important here! You can also use the isotropy subgroup! $\square$

Definition 7.21. *The discrete actions on the torus* $(\mathbb{Z}, \theta, \mathbb{T})$:

$$\theta_t([x]) := [x + \vartheta n] , \quad \forall\, n \in \mathbb{Z} \,,\, [x] \in \mathbb{T} := \mathbb{R}/\mathbb{Z} \,.$$

Remark 7.22. We know that θ is minimal if and only if $\vartheta \notin \mathbb{Q}$. So if the two parameters ϑ^1, ϑ^2 are one rational and the other one irrational, the actions cannot be isomorphic.

Remark 7.23. We know that if $\vartheta \in \mathbb{Q}$ with $\vartheta = p/q$ (irreducible), then all the orbits are periodic, with period q. So if two rational parameters ϑ^1, ϑ^2 have different denominators, the actions cannot be isomorphic.

There are still many undetermined cases.

Theorem 7.24. *Let* $\vartheta^1, \vartheta^2 \in \mathbb{R}$. *Then* $\big(\mathbb{Z}, \theta^1, \mathbb{T}\big)$ *and* $\big(\mathbb{Z}, \theta^2, \mathbb{T}\big)$ *are isomorphic if and only if*

$$\big[\vartheta^1\big] = \big[\pm\vartheta^2\big] \,.$$

Proof. **If.**

Use $\mathrm{id}_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is the sign is $+$ and $[x] \to [-x]$ if the sign is $-$.

Actually, if $\big[\vartheta^1\big] = \big[\vartheta^2\big]$ (i. e. $\vartheta^2 = \vartheta^1 + k$ for some integer k) the two actions are identical:

$$\theta^2_n\big([x]\big) = \big[x + n\vartheta^2\big] = \big[x + n\vartheta^1 + nk\big] = \big[x + n\vartheta^1\big] = \theta^1_n\big([x]\big) \,.$$

$\square$

Lemma 7.25. *Any continuous function* $f : \mathbb{T} \to \mathbb{T}$ *has a lifting to a continuous function* $F : \mathbb{R} \to \mathbb{R}$.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "F"] \arrow[d, "p"'] & \mathbb{R} \arrow[d, "p"'] \\ \mathbb{T} \arrow[r, "f"] & \mathbb{T} \end{tikzcd} \qquad f \circ p = p \circ F \,.$$

The two functions are simultaneously homeomorphisms.

Proof. Because p is a universal covering map. $\square$

Lemma 7.26. *Two liftings* $F, G : \mathbb{R} \to \mathbb{R}$ *of the same continuous map* $f : \mathbb{T} \to \mathbb{T}$ *differ only by a constant* $m \in \mathbb{Z}$.

Proof. Since $p \circ F = f \circ p = p \circ G$, one has

$$p\big(F(x)\big) = [F(x)] = [G(x)] = p\big(G(x)\big) \,, \quad \forall\, x \in \mathbb{R} \,,$$

which means that

$$F(x) - G(x) \in \mathbb{Z} \,, \quad \forall\, x \in \mathbb{R} \,.$$

But $F - G$ is continuous, so it has no other choice than being constant. $\square$

Lemma 7.27. *Let $F : \mathbb{R} \to \mathbb{R}$ be a lifting of the continuous map $f : \mathbb{T} \to \mathbb{T}$.*

(a) Then there exists some $M \in \mathbb{Z}$ such that $F - M\,\mathrm{id}_{\mathbb{R}}$ is periodic with period 1. This means that

$$F(x) = Mx + P(x)\,, \quad \text{with} \quad P(x+1) = P(x)\,, \quad \forall\, x \in \mathbb{R}\,.$$

(b) If f is a homeomorphism, one has $M = \pm 1$, i. e.

$$F = \pm\mathrm{id}_{\mathbb{R}} + \text{periodic function.}$$

Proof. (a) For every $x \in \mathbb{R}$ one has

$$[F(x+1)] = f\big([x+1]\big) = f\big([x]\big) = [F(x)]\,.$$

This and the continuity of F imply that for some integer constant M one has

$$F(x+1) = M + F(x)\,, \quad \forall\, x \in \mathbb{R}\,.$$

Then you get the desired periodicity:

$$\big(F - M\,\mathrm{id}_{\mathbb{R}}\big)(x+1) = F(x+1) - Mx - M = F(x) - Mx = \big(F - M\,\mathrm{id}_{\mathbb{R}}\big)(x)\,.$$

(b) The situation $M = 0$ is impossible; a periodic function is not a homeomorphism. We are going to show that $M \geq 2$ is impossible. The case $M \leq -2$ is treated similarly, and this is enough.

If $M > 1$, then

$$F(1) = F(0) + M > F(0) + 1 > F(0)\,.$$

So there exists $y \in (0,1)$ such that $F(y) = F(0) + 1$. It follows that

$$f\big([y]\big) = [F(y)] = [F(0)] = f\big([0]\big)\,,$$

contradicting the injectivity of f. $\square$

Remark 7.28.

- If $F - \mathrm{id}_{\mathbb{R}}$ is periodic, then

$$F(x+1) = x + 1 + P(x+1) = x + 1 + P(x) = F(x) + 1$$

for every $x \in \mathbb{R}$. Then F is increasing and we say that f is *orientation preserving*.

- If $F - \mathrm{id}_{\mathbb{R}}$ is periodic, then $F(x+1) = F(x) - 1$ for every $x \in \mathbb{R}$. Then F is decreasing and we say that f is *orientation reversing*.

Proof. **Only if.**

Let $\big(\mathbb{Z}, \theta^1, \mathbb{T}\big) \xrightarrow{f} \big(\mathbb{Z}, \theta^1, \mathbb{T}\big)$ be an isomorphism. It is enough to take $\vartheta^1, \vartheta^2 \in [0,1)$ (why?). We must show that $\vartheta^1 = \vartheta^2$ or that $\vartheta^1 = 1 - \vartheta^2$. I cite from [3]:

If both flows rotate in the same direction and $\vartheta^1 \neq \vartheta^2$, then points of one flow, going faster, would overtake points in the other flow, and repeatedly so, which should be impossible for isomorphic flows.

By Lemmas 7.25 and 7.26, the homeomorphism $f : \mathbb{T} \to \mathbb{T}$ has a lifting to a homeomorphism $F : \mathbb{R} \to \mathbb{R}$, unique up to an additive constant belonging to $\mathbb{Z}$.

Since

$$\Theta^i_1 : \mathbb{R} \to \mathbb{R}\,, \quad \Theta^i_1(x) := x + \vartheta^i$$

is a lifting of θ_1^i , for $i = 1, 2$, then

$$F \circ \Theta_1^1 \circ F^{-1} : \mathbb{R} \to \mathbb{R}\,, \quad \big(F \circ \Theta_1^1 \circ F^{-1}\big)(x) = F\big[F^{-1}(x) + \vartheta^1\big]$$

is a lifting of θ_1^2 :

$$\begin{aligned} p \circ \big(F \circ \Theta_1^1 \circ F^{-1}\big) &= f \circ p \circ \Theta_1^1 \circ F^{-1} \\ &= f \circ \theta_1^1 \circ p \circ F^{-1} \\ &\overset{??}{=} f \circ \theta_1^1 \circ f^{-1} \circ p \\ &\overset{??}{=} \theta_1^2 \circ p \end{aligned}$$

But Θ_1^2 is also a lifting of θ_1^2 , so there exists $k \in \mathbb{Z}$ such that

$$F \circ \Theta_1^1 \circ F^{-1} = \Theta_1^2 + k\,\mathrm{id}_{\mathbb{R}}\,.$$

By iterating n times and setting $\Theta_n^i := \big(\Theta^i\big)^n$, one gets

$$F \circ \Theta_n^1 \circ F^{-1} = \Theta_n^2 + nk\,\mathrm{id}_{\mathbb{R}}\,,$$

which reads explicitly

$$F\big[F^{-1}(x) + n\vartheta^1\big] = x + n\vartheta^2 + nk\,, \quad \forall\, x \in \mathbb{R}\,,\, n \in \mathbb{Z}\,. \tag{7.2}$$

A. Suppose that f *is orientation-preserving*, i.e. that F is increasing. By Lemma 7.27 , $F - \mathrm{id}_{\mathbb{R}}$ is periodic with period 1 , and also continuous when restricted to $[0, 1]$, so it is bounded:

$$|F(y) - y| \le C < \infty\,, \quad \forall\, y \in \mathbb{R}\,.$$

By (7.2) with $y := F^{-1}(x) + n\vartheta^1$, we get

$$\big|x + n\vartheta^2 + nk - F^{-1}(x) - n\vartheta^1\big| \le C\,, \quad \forall\, x \in \mathbb{R}\,,\, n \in \mathbb{Z}\,.$$

For fixed x the l. h. s. can remain bounded only when $\vartheta^2 + k - \vartheta^1 = 0$, meaning here that $\vartheta^1 = \vartheta^2$.

B. Suppose that f *is orientation-reversing*, i.e. that F is decreasing. Then one gets an orientation preserving isomorphism $-f$, having $-F$ as a lifting. It will follow similarly that $y \to |F(y) + y|$ is bounded, hence $\vartheta^2 + k + \vartheta^1 = 0$, meaning here that $\vartheta^1 + \vartheta^2 = 0$. □

Remark 7.29. Let us set

$$\mathfrak{A}_\vartheta := C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$$

for the crossed product C^*-algebras associated to the C^*-dynamical system $\big(\mathbb{Z}, \theta, C(\mathbb{T})\big)$ obtained from the topological dynamical system $(\mathbb{Z}, \theta, \mathbb{T})$:

$$\theta_n(\varphi) := \varphi \circ \theta_{-n}\,, \quad \forall\, n \in \mathbb{Z}\,,\, \varphi \in C(\mathbb{T})\,.$$

It is called *the rotation algebra with parameter* $\vartheta \in \mathbb{R}$. It admits many other equivalent descriptions and it plays a very important role.

If $\vartheta \in \mathbb{Q}$, there are closed invariant subsets in $\mathbb{T}$, to which we associate proper θ-invariant ideals $\mathcal{J}$ in $C(T)$ and then proper invariant ideals $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ in $\mathfrak{A}_\vartheta$. If $\vartheta \notin \mathbb{Q}$, then $\mathfrak{A}_\vartheta$ is simple!

There is an isomorphism $\mathfrak{A}_{\vartheta^1} \cong \mathfrak{A}_{\vartheta^2}$ if and only if $\big[\vartheta^1\big] = \big[\pm\,\vartheta^2\big]$.

8 Shift systems

Let $S := \{0, 1, \ldots, s-1\}$ a finite set (seen as a topological space with the discrete topology). We could call it *an alphabet* and its elements are *letters*.

The group will be $\mathbb{Z}$. We work with the Hausdorff compact space

$$\Sigma \equiv \Sigma_S := S^{\mathbb{Z}} = \prod_{i\in\mathbb{Z}} S = C(\mathbb{Z}, S) = \big\{x := (x_i)_{i\in\mathbb{Z}} \,\big|\, x_i \in S\big\}$$

with the product topology.

The action will be obtained iterating the homeomorphism (*the shift*)

$$\theta_1 \equiv \tau : \Sigma \to \Sigma\,, \quad \tau(x)_i := x_{i+1} \quad \forall\, i \in \mathbb{Z}\,.$$

So we get

$$\theta : \mathbb{Z} \to \mathrm{Homeo}(\Sigma)\,, \quad \theta_n(x)_i := \tau^n(x)_i = x_{i+n} = (n \bullet x)_i\,.$$

Exercise 8.1. Show that $\big(\mathbb{Z}, \theta, \Sigma_S\big)$ is indeed a topological dynamical system. It is called *the symbolic dynamical system (flow) on s symbols (letters)*. Maybe use the canonical projections

$$\pi_i : S^{\mathbb{Z}} \to S\,, \quad \pi_i(x) := x_i\,.$$

Compute $\pi_i \circ \tau$.

Notation, terminology, remarks

- The homeomorphism τ pushes the bi-sided infinite sequence x one place to the *left*.
- *Word of length* k = element of $S^k := S \times \ldots \times S$ (k times). Word = *block*.
- *Finite word* = word of any length = element of $S^\star := \sqcup_{k\in\mathbb{N}} S^k$, where $S^0 := \{\emptyset\}$. In writing finite and infinite words (sequences), we usually skip commas and brackets:

  $$(y_1, y_2, \ldots, y_k) \equiv y_1 y_2 \ldots y_k\,, \quad (x_i)_{i\in\mathbb{N}} \equiv \ldots x_{-2}x_{-1}x_0x_1x_2\ldots$$

  when you see something as $\ldots ab\dot{c}de\ldots$, this means that c is the zero component. Note that $\tau(\ldots ab\dot{c}de\ldots) = \ldots abc\dot{d}e\ldots$
- Actually $S^\star \equiv \mathbb{F}_+(S)$ is *the free monoid on s elements*, with the multiplication (concatenation)

  $$x_0x_1\ldots x_{k-1} \cdot y_0y_1\ldots y_{l-1} := x_0x_1\ldots x_{k-1}y_0y_1\ldots y_{l-1}$$
- The non-empty finite word b *is contained (appears, occurs)* in the finite word c if $c = abd$ for some other finite words a, d (which are allowed to be void). If $a = \emptyset$, we say that c begins with b. If $d = \emptyset$, we say that c ends with b.
- How many times does 01 occur in 0113010013 ?
- If $x \in \Sigma = S^{\mathbb{Z}}$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ one sets $x_{[k,k+n]} := x_k \ldots x_{k+n} \in S^n$. Using this, you can say *where* a certain finite word appears in an infinite word.
- A word of odd length $2p+1$ *occurs as a central block in* x if it occurs at position $-p$ (i. e. $b_{p+1} = x_0$).
- To simplify (to a certain extent), we use sometimes ambiguous or incomplete notations or terminology.

Theorem 8.2. *Let $x \in \Sigma$.*

(i) $y \in \overline{\mathfrak{O}}_x$ if and only if any word occuring in y also occurs in x (an infinity of times!!)

(ii) x has a dense orbit in Σ if and only if every finite word occurs in x.

(iii) x is recurrent if and only if every word which occurs in x does so at places j with arbitrarily large j.

(iv) x is almost periodic if and only if every finite word u that appears in x appears with bounded gaps, i. e.

$$\{j \in \mathbb{Z} \,|\, u \text{ appears in } x \text{ at place } j\}$$

is syndetic in $\mathbb{Z}$).

(v) x is periodic if and only if there exists $p \in \mathbb{N}$ such that $x_{i+p} = x_i$ for evert $i \in \mathbb{Z}$ (i. e. the function $x : \mathbb{Z} \to S$ is periodic).

Definition 8.3. The Hausdorff topological space X is called 0-*dimensional* if it has a topological base formed of clopen sets (open and closed).

This is equivalent to every point $x \in X$ having a neighborhood base formed of clopen sets.

Remark 8.4. It can be shown (Brower's Theorem) that any second countable, 0-dimensional compact Hausdorff space without isolated points is homeomorphic to $\{0,1\}^{\mathbb{N}}$ and (equivalently) to the middle-third Cantor space. Such a space will be called *a Cantor set*. In particular, every countable product $\prod_{n=1}^{\infty} X_n$ with $2 \leq \#X_n < \infty$ is a Cantor set.

Definition 8.5.

- In the Hausdorff topological space X, *the connected component of a point* x is the largest connected subset $con(x)$ containing x.
- The space X is called *totally disconnected* if $con(x) = \{x\}$ for every $x \in X$.

Proposition 8.6. *(a) If X is 0-dimensional, it is totally disconnected.*

(b) For locally compact Hausdorff spaces, the two properties are equivalent.

Proof. (a) One must show that all the connected subsets A are singletons. Let $x, y \in A$, with disjoint clopen neighborhoods U and V contained in A. Then A is the union of three disjoint clopen subsets.

(b) Difficult. $\square$

Recall that $S := \{1, \dots, s\}$, the space is $\Sigma := S^{\mathbb{Z}}$ with the (correct) product topology and $\mathbb{F}_+(S) \equiv S^\star := \bigsqcup_{n\in\mathbb{N}} S^n$ is the family (a free monoid) of finite words with letters from S .

Lemma 8.7. *Let Λ be a directed set, $x^{(\lambda)}, x \in S^{\mathbb{Z}}$ for every $\lambda \in \Lambda$. One has $x^{(\lambda)} \to x$ if and only if*

$$\forall\, j \in \mathbb{Z}\,, \quad \exists\, \lambda(j) \in \Lambda \quad \text{such that} \quad x^{\lambda}_j = x_j\,,\ \forall\, \lambda \geq \lambda(j)\,.$$

Proof. Convergence in a product space is equivalent with convergence of all the j-components in S (maybe not uniformly in j) . But S is a discrete space; take $\epsilon < 1$. □

Example 8.8. Suppose that $\Lambda = \mathbb{N}$ and

$$x = \dots 1111111111111 \dots \in \{0,1\}^{\mathbb{Z}}.$$

The sequence $\left(x^{(m)}\right)_{m\in\mathbb{N}}$ with

$$x^{(m)}_j = 1 \quad \text{if} \quad |j| \leq m^2, \qquad x^{(m)}_j = 0 \quad \text{if} \quad |j| > m^2$$

converges to x . **Otros tipos?**

Exercise 8.9. If $T \subset S$, then $T^{\mathbb{Z}}$ is an subset of $S^{\mathbb{Z}}$ (select only the words with letters in the sub-alphabet T). Show that it is closed and invariant.

Definition 8.10. Let $\emptyset \neq u \in S^\star$ and $j \in \mathbb{Z}$, *the cylinder based on u at place j* is

$$C_j[u] := \left\{x \in \Sigma \,\middle|\, x_{[j,j+|u|-1]} = u\right\}.$$

If $u = u_1 \dots u_l$ if x wants to belong to $C_j[u]$, then

$$x_j = u_1\,, \dots, x_{j+l-1} = u_l\,,$$

while $x_k \in S$ is arbitrary for other values of k .

Remark 8.11. You can speak of *centered cylinders* and use the notation $C^c[u]$. You can use θ to place cylinders whenever you want: $C_j[u] = \theta_{-j}\big(C_0[u]\big)$.

More generally, shifts send cylinders into cylinders: $\theta_k\big(C_l[u]\big) = C_{l-k}[u]$. (This will show again that τ is continuous.)

Remark 8.12. If u is contained in v , then $C_j[u] \supset C_j[v]$.

Are there other possibilities?

Lemma 8.13. *Any cylinder is a basic open set in $\Sigma = S^{\mathbb{Z}} = \prod_{i\in\mathbb{Z}} S$.*

Proof. One may write $C_j[u] = \prod_{i\in\mathbb{Z}} U_i$,

where $U_i = \{u_{i-j}\}$ if $i \in \{j, j+|u|-1\}$ and $U_i = S$ otherwise. □

Lemma 8.14. *A neighborhood base of $x \in \Sigma$ is formed by all the centered cylinders*

$$\left\{C^c[x_{-j} \dots x_j] \,\middle|\, j \in \mathbb{N}\right\}.$$

Proof. If $x \in U =$ open, then

$x \in V = \prod_i V_i \subset U$ with $V_i = \{x_i\}$ for $i \in \{j, \ldots, j+n\}$ and $V_i = S$ otherwise.

This means that $V = C_j[x_j, \ldots x_j + n]$.

This cylinder containes a centered cylinder containing x (increasing the restriction). □

Exercise 8.15. Show that convergence in $S^{\mathbb{Z}}$ can be understood via Lemma 8.14.

Proposition 8.16. *Every cylinder is a clopen set.*

Thus Σ is zero-dimensional and totally disconnected. It has no isolated points.

Proof. Because each subset of S is clopen. □

Corollary 8.17. *$\Sigma = S^{\mathbb{Z}}$ is homeomorphic to the Cantor set.*

Remark 8.18. Each countable product of metric spaces is metrizable. If $\big(X_n, d_n\big)_{n\in\mathbb{Z}}$ are metric spaces,

$$\delta : \prod_n X_n \times \prod_n X_n \to \mathbb{R}_+ \,, \quad \delta(x, y) := \sum_{n\in\mathbb{Z}} 2^{-n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}$$

is a metric compatible with the product topology. It is not at all canonical (coefficients 2^{-n} to insure convergence, function $t \to \frac{t}{1+t}$ to make certain factors bounded, others).

On S one can consider the discrete metric. Being a countable product of metric spaces, Σ is metrizable.

Example 8.19. Show that $\Delta : S^{\mathbb{Z}} \times S^{\mathbb{Z}} \to \mathbb{R}_+$ given for $x \neq y$ bu

$$\Delta(x, y) := \frac{1}{1 + \min\{|j| \mid x_j \neq y_j\}}$$

is a compatible metric, with respect to which cylinders became balls:

$$C^{\mathrm{c}}\big[x_{-j} \ldots x_j\big] = \big\{y \in S^{\mathbb{Z}} \,\big|\, \Delta(x, y) \leq \big[1 + (j+1)\big]^{-1}\big\} \,.$$

Example 8.20. Show that for every two different points x, y there is $k \in \mathbb{Z}$ such that

$$\Delta\big(\theta_k(x), \theta_k(y)\big) = 1 \,.$$

One says that *the metric dynamical system* $(\mathbb{Z}, \theta, \Sigma, \Delta)$ *is expansive*.

Remark 8.21. One can show that there is no metric which respect to which τ is an isometry.

Theorem 8.22. *There are exactly s fixed points.*

Proof. Obviously, a fixed point is a constant sequence, defined by one of the letters. □

Corollary 8.23. $(\mathbb{Z}, \theta, \Sigma)$ *is not minimal.*

Definition 8.24. The point $x \in \Sigma$ is *periodic* if $\mathbb{Z}_x^x$ is a syndetic subgroup of $\mathbb{Z}$, i. e. $\mathbb{Z}/\mathbb{Z}_x^x$ is compact (finite). Actually the syndetic subgroups of $\mathbb{Z}$ have the form $q\mathbb{Z}$ for some $q \in \mathbb{N}^*$.

Theorem 8.25. *The point $x \in \Sigma$ is periodic if and only if there exists $p \in \mathbb{N}^*$ such that $x_{i+p} = x_i$ for every $i \in \mathbb{Z}$ (i. e. the function $x : \mathbb{Z} \to S$ is periodic).*

Proof. The condition $x_{i+p} = x_i$ for every $i \in \mathbb{Z}$ means that $\theta_p(x) = x$, i.e. that $p \in \mathbb{Z}_x^x$. □

Corollary 8.26. *The set Σ_{per} of all periodic points is (i) countable and (ii) dense.*

Proof. (i) The point x is periodic if and only if

$$x = \dots u\dot{u}u \dots \quad \text{for some } u \in S^\star .$$

But $S^\star$ is countable.

(ii) It is enough to show that in every (small) cylinder there is a periodic point. Let $C_j(v)$ such a cylinder, with $j \in \mathbb{Z}$ and $v \in S^\star$. The sequence

$$y := \dots v\dot{v}v \dots$$

is periodic, and $\theta_{-j}(y) \in C_j[v]$. □

Theorem 8.27. *Let $x \in \Sigma$. Then $y \in \overline{\mathfrak{O}}_x$ if and only if any word occuring in y also occurs in x (an infinity of times!!).*

We say that the dictionary $\mathcal{D}_y$ of the infinite word y is contained in the dictionary $\mathcal{D}_x$ of x.

Proof. One has $y \in \overline{\mathfrak{O}}_x$ if and only if every neighborhood of y intersects $\mathfrak{O}_x$,
equivalent to the fact that for every subword $y_j \dots y_{j+k}$ of y one has $C_j\left[y_j \dots y_{j+k}\right] \cap \mathfrak{O}_x \neq \emptyset$,
meaning that for every subword $y_j \dots y_{j+k}$ of y there exists $n \in \mathbb{Z}$ with $\theta_n(x) \in C_j\left[y_j \dots y_{j+k}\right]$,
meaning that every subword $y_j \dots y_{j+k}$ of y appears at the position $n + j$ in x, for some n. □

Exercise 8.28. Why "an infinity of times"?

Exercise 8.29. Let $S = \{0, 1\}$ and $z \in \{0, 1\}^{\mathbb{Z}}$ such that $z_n = 1$ if and only if $n = 0$. Then

$$\overline{\mathfrak{O}}_z = \mathfrak{O}_z \sqcup \mathfrak{O}_x = \mathfrak{O}_z \sqcup \{x\} ,$$

where $x := \dots 00000 \dots$.

Exercise 8.30. Let $S = \{0, 1\}$ and $z \in \{0, 1\}^{\mathbb{Z}}$ such that $z_n = 1$ if and only if $|n| = 2^m$ for some $m \in \mathbb{N}$. Then

$$\overline{\mathfrak{O}}_z = \mathfrak{O}_z \sqcup \mathfrak{O}_x \sqcup \mathfrak{O}_y = \mathfrak{O}_z \sqcup \{x\} \sqcup \mathfrak{O}_y ,$$

where $x := \dots 00000 \dots$ and $y := \dots 00\dot{1}00 \dots$.

Exercise 8.31. Let $S = \{0, 1, 2\}$ and $z \in \{0, 1, 2\}^{\mathbb{Z}}$ containing 1 and 2 just once.

Compute the closure of the orbit of z .

Remark 8.32. The dictionary of a subset $A \subset S^{\mathbb{Z}}$ is $\mathcal{D}(A) := \bigcup_{x \in A} \mathcal{D}_x$ (all the finite words contained in some infinite word belonging to A) .

Properties of such dictionarie?

Connectios between dictionaries and closed invariant sets?

Corollary 8.33. $(\mathbb{Z}, \theta, \Sigma)$ *is pointwise transitive .*

Proof. We must construct a point x with dense orbit. Since $S^\star$ is countable, let $u^{(1)}, u^{(2)}, \ldots$ an enumeration of the elements of $S^\star$. The sequence

$$x := \ldots u^{(3)} u^{(2)} \dot{u}^{(1)} u^{(2)} u^{(3)} \ldots$$

has a dense orbit, by Theorem 8.27. Todas las palabras finitas aparecen en x ; el dicctionario de x coincide con $S^{\mathbb{Z}}$. So any element $y \in \Sigma$ belongs to $\overline{\mathfrak{D}}_x = \Sigma$. □

Exercise 8.34. Show that there is an infinity of dense orbits.

Lemma 8.35. *Let $x, y \in S^{\mathbb{Z}}$, $j, k \in \mathbb{Z}$ and $u = \{u_0, \ldots, u_p\}, v = \{v_0, \ldots, v_q\} \in S^\star$. We have*

$$\mathbb{Z}_x^y = \big\{ n \in \mathbb{Z} \,\big|\, x_{i+n} = y_i \,,\, \forall i \big\} \,, \tag{8.1}$$

$$\begin{aligned} \mathbb{Z}_x^{C_k[v]} &= \big\{ n \in \mathbb{Z} \,\big|\, x_{k+n+i} = v_i \,,\, \forall\, i = 0, \ldots, q-1 \big\} \\ &= \big\{ n \in \mathbb{Z} \,\big|\, x_{[k+n, k+n+q-1]} = v \big\} \end{aligned} \tag{8.2}$$

$$\mathbb{Z}_{C_j[u]}^{C_k[v]} = \big\{ n \in \mathbb{Z} \,\big|\, \exists\, x \in S^{\mathbb{Z}},\, x_{[j;j+p-1]} = u \,,\, x_{[k+n, k+n+q-1]} = v \big\} \,. \tag{8.3}$$

Proof. The first identity follows from the transcription of the condition $\theta_n(x) = y$.

For the second identity we write

$$\begin{aligned} n \in \mathbb{Z}_x^{C_k[v]} &\Leftrightarrow \theta_n(x) \in C_k[v] \\ &\Leftrightarrow x \in \theta_{-n}\big(C_k[v]\big) = C_{k+n}[v] \\ &\Leftrightarrow x_{k+n+i} = v_i \,,\, \forall\, i = 0, \ldots, q-1 \\ &\Leftrightarrow x_{[k+n;k+n+q-1]} = v \,. \end{aligned}$$

And for the last one:

$$\begin{aligned} n \in \mathbb{Z}_{C_j[u]}^{C_k[v]} &\Leftrightarrow \exists\, x \in C_j[u] \,,\, n \in \mathbb{Z}_x^{C_k[v]} \\ &\Leftrightarrow \exists\, x \in S^{\mathbb{Z}},\, x_{j+i} = u_i \,,\, \forall\, i = 0, \ldots, p-1 \,,\, x_{k+n+i} = v_i \,,\, \forall\, i = 0, \ldots, q-1 \\ &\Leftrightarrow \exists\, x \in S^{\mathbb{Z}},\, x_{[j;j+p-1]} = u \,,\, x_{[k+n,k+n+q-1]} = v \,. \end{aligned}$$

□

Example 8.36. In particular, if a, b are letters, we have

$$\mathbb{Z}_{C_j[a]}^{C_k[b]} = \big\{ n \in \mathbb{Z} \,\big|\, \exists\, x \in S^{\mathbb{Z}},\, x_j = a \,,\, x_{k+n} = b \big\} \,. \tag{8.4}$$

But don't forget that $C_j[a], C_b[b]$ are "large" cylinders. We are mainly interested in small cylinders (neighborhoods), coming from large words u, v .

Definition 8.37. A point $\sigma \in \Sigma$ is called *recurrent* if it satisfies one of the following three equivalent conditions:

(c) $\sigma \in \mathcal{L}_\sigma$,

(c) there is a divergent net $(a_i)_{i\in I} \subset \mathsf{G}$ such that $a_i \bullet \sigma \to \sigma$,

(d) G_σ^U is not relatively compact for any open neighborhood U of σ .

Theorem 8.38. *The point $x \in S^{\mathbb{Z}}$ is recurrent if and only if every word which occurs in x does so at places j with arbitrarily large j .*

Proof. The point x is recurrent

if and only if $\mathbb{Z}_x^{C_k[v]}$ is not finite as soon as the cylinder $C_k[v]$ is a neighborhood of x ,

meaning that if v appears in x at some position k then $\{n \in \mathbb{Z} \,|\, x_{[k+n,k+n+q-1]} = v\}$ is infinite

meaning that if v appears in x at some position k then it appears in x at an infinity of positions. □

Exercise 8.39. The point $z \in \{0,1\}^{\mathbb{Z}}$ from Exercise 8.30 is not recurrent. It is not enough to check letters (one-letter words); have a look at (moderately) larger centered subwords!

Exercise 8.40. Formulate definitions and statements for "positive recurrence" and "negative recurrence".

Theorem 8.41. *The point $x \in \Sigma$ is almost periodic if and only if*

(Cond. A) every finite word $v = v_0 \dots v_{q-1}$ that appears in x appears with bounded gaps, i. e.

$$\{n \in \mathbb{Z} \,|\, v \text{ appears in } x \text{ at place } n\}$$

is syndetic in $\mathbb{Z}$.

Remark first that (Cond. A) is equivalent to

(Cond. A_0) every finite word v that appears in x at position 0 appears with bounded gaps, i. e.

$$\{n \in \mathbb{Z} \,|\, v \text{ appears in } x \text{ at place } n\}$$

is syndetic in $\mathbb{Z}$.

The equivalence between (Cond. A) and (Cond. A_0) follows from the fact that almost periodicity of points is invariant under the action.

Proof. The point x is almost periodic if and only if

$$\mathbb{Z}_x^{C_k[v]} = \mathbb{Z}_x^{\theta_{-k}(C_0[v])} = -k + \mathbb{Z}_x^{C_0[v]} = -k + \{n \in \mathbb{Z} \,|\, x_{[n,n+q-1]} = v\}$$

is syndetic as soon as the cylinder $C_k[v]$ is a neighborhood of x , i. e. as soon as $x_{[k,k+q-1]} = v$.

But this is precisely (Cond A). □

Exercise 8.42. Show using the two characterizations Theorem 8.25 and Theorem 8.45 that periodicity implies almost periodicity.

Corollary 8.43. *There exist points in Σ which are almost periodic but not periodic.*

Proof. Both the Fibonacci and the Morse sequences are such examples. □

Definition 8.44. The action $(\mathsf{G}, \theta, \Sigma)$ is *strongly mixing* if the complement of G_U^V is relatively compact for every $U, V \in \mathrm{Top}^*(\Sigma)$.

Theorem 8.45. *The dynamical system* $(\mathbb{Z}, \theta, \Sigma)$ *is strongly mixing, so it is also weakly mixing and non-wandering.*

Proof. We already know that strongly mixing implies both weakly mixing and non-wandering.

One only needs to show that the complement of $\mathbb{Z}_{C_j[u]}^{C_k[v]}$ is finite for any j, k, u, v . Note that

$$\mathbb{Z}_{C_j[u]}^{C_k[v]} = \mathbb{Z}_{\theta_{-j}(C_0[u])}^{\theta_{-k}(C_0[v])} = j - k + \mathbb{Z}_{C_0[u]}^{C_0[v]} ,$$

so it is enough to take $j = 0 = k$. Replacing u and v by another finite word w containing both of them, it is enough to show that

the complement of $\mathbb{Z}_{C_0[w]}^{C_0[w]}$ *is finite for any* $w = w_1 \dots w_{p-1} \in S^\star$,

or, equivalently, that

there exists $N \in \mathbb{N}$ *such that*

$$\{n \in \mathbb{Z} \,|\, |n| \geq N\} \subset \mathbb{Z}_{C_0[w]}^{C_0[w]} = \big\{ n \in \mathbb{Z} \,\big|\, \exists\, y \in S^{\mathbb{Z}},\, y_{[0;p-1]} = w \,,\, y_{[n,n+p-1]} = w \big\} ,$$

This can be rephrased as the existence of some $N \in \mathbb{N}$ such that

$$|n| \geq N \;\Rightarrow\; \exists\, y \in S^{\mathbb{Z}},\, y_{[0;p-1]} = w \,,\, y_{[n,n+p-1]} = w \,.$$

GUESS WHYYYYYYYY ! $\square$

9 Subhifts

Definition 9.1. A *subshift* is a closed invariant subset Λ of $\left(\mathbb{Z}, \theta, \Sigma = S^{\mathbb{Z}}\right)$, seen as a sub-dynamical system.

Example 9.2. Orbit closures (but not orbits, in general) are subshifts.

Example 9.3. If $\emptyset \neq T \subset S$, then

$$\{x \in \Sigma \,|\, x_j \in T\,,\, \forall\, j \in \mathbb{Z}\}$$

is a subshift that can be identified to $T^{\mathbb{Z}}$.

Definition 9.4.

- *The dictionary* $\mathrm{Di}(x)$ *of* the sequence $x \in S^{\mathbb{Z}}$ is the family of all the finite words appearing in x.
- *The dictionary* $\mathrm{Di}(B)$ *of* the subset $B \subset S^{\mathbb{Z}}$ is the family of all the finite words appearing in some sequence $x \in B$:
$$\mathrm{Di}(B) = \bigcup_{x \in B} \mathrm{Di}(x) \subset S^{\star} := \bigsqcup_{m \in \mathbb{N}} S^m.$$
- The elements of $\mathrm{Di}(B)$ are called B*-admissible words*.

Remark 9.5. The dictionary is very far from determining the family.

It is not at all amazing to have $A \neq B$ but $\mathrm{Di}(A) = \mathrm{Di}(B)$.

Exercise 9.6. Let $\Lambda \subset \Sigma$ be a subshift.

- $x \in \Lambda$ has a dense orbit in Λ if and only if each Λ-admissible word appears in x.
- Λ is minimal if and only if every Λ-admissible word appears in all the sequences of Λ.
- Λ is pointwise transitive (equivalent to topologically transitive) if and only if there is a sequence $x \in \Lambda$ containing all the dictionary of Λ.
- Λ is non-wandering if for every $w \in \mathrm{Di}(\Lambda)$ and for every $n \in \mathbb{N}$ there exists $y \in \Lambda$ in which w appears at two different places j and $j + k$, with $k \geq n$. What happens for $\Lambda = \Sigma$?

Definition 9.7.
- *The dictionary* $\mathrm{Di}(x)$ of the sequence $x \in S^{\mathbb{Z}}$ is the family of all the finite words appearing in x .
- *The dictionary* $\mathrm{Di}(B)$ *of* the subset $B \subset S^{\mathbb{Z}}$ is the family of all the finite words appearing in some sequence $x \in B$:
$$\mathrm{Di}(B) = \bigcup_{x \in B} \mathrm{Di}(x) \subset S^\star := \bigsqcup_{m \in \mathbb{N}} S^m .$$
- The elements of $\mathrm{Di}(B)$ are called B*-admissible words*.

Lemma 9.8. *If* $B \subset \Sigma$ *, then* $\mathrm{Di}(B) = \mathrm{Di}(\overline{B})$.

Proof. Since $B \subset \overline{B}$, one surely has $\mathrm{Di}(B) \subset \mathrm{Di}(\overline{B})$.

Let now $\overline{B} \ni x = \lim_m x^m$, with $x^m \in B$, for every $m \in \mathbb{N}$.

Let $w \in \mathrm{Di}(x)$, meaning that $x_{[k,k+|w|]} = w$ for some $k \in \mathbb{Z}$.

There exists $M \in \mathbb{N}$ such that for every $m > M$ one has
$$x^m_{[k,k+|w|]} = x_{[k,k+|w|]} = w \,,$$
so that for such a m we get $w \in \mathrm{Di}(x^m) \subset \mathrm{Di}(B)$. Therefore
$$\mathrm{Di}(\overline{B}) = \bigcup_{x \in \overline{A}} \mathrm{Di}(x) \subset \mathrm{Di}(B) \,.$$
□

Exercise 9.9. Another proof, using cylinders.

Corollary 9.10. *If* $A \subset \Sigma$*, then* $\mathrm{Di}(A) = \mathrm{Di}\big[\mathrm{Sat}(A)\big] = \mathrm{Di}\big[\,\overline{\mathrm{Sat}(A)}\,\big]$.

Remark 9.11. We have defined a map
$$2^\Sigma \ni A \to \mathrm{Di}(A) \in 2^{S^\star} .$$
We have seen that it is not at all injective.
Is it surjective? Given $P \subset S^*$, is there some $A \subset \Sigma$ such that $\mathrm{Di}(A) = P$?
Note that for every subword u of $v \in \mathrm{Di}(A)$ one has $u \in \mathrm{Di}(A)$. This does not happen for all the subsets P of $S^\star$.

Definition 9.12. If $P \subset S^*$, we set
$$\hat\Lambda(P) := \big\{ y \in \Sigma \,\big|\, y_{[k,k+n]} \in P \,,\, \forall\, k \in \mathbb{Z} \,,\, n \in \mathbb{N} \big\}$$
and (if it is non-void) we call it *the subshift defined by* P.

Proposition 9.13. *If non-void,* $\hat\Lambda(P)$ *is a subshift and* $\mathrm{Di}\big[\hat\Lambda(P)\big] \subset P$ *. The inclusion may be strict.*

Proof. By its very definition, $\hat\Lambda(P)$ is invariant.

All the conditions for $y \in \hat\Lambda(P)$ are "finite type". From this you infer that $\hat\Lambda(P)$ is closed.

The inclusion $\mathrm{Di}\big[\hat\Lambda(P)\big] \subset P$ follows from the definition.

While P is arbitrary, $\hat\Lambda(P)$ is restricted. □

Exercise 9.14. Let $R \subset S^*$. Then $\hat{\Lambda}(R) \neq \emptyset$ if and only if the next two conditions hold:

(i) For every subword u of $v \in R$ one has $u \in R$,

(ii) If $w \in R$, there exist $u_1, u_2 \in R$ non-void such that $u_1 w u_2 \in R$.

Proposition 9.15. *Every subshift can be obtained by Definition 9.12:*
If $\Lambda \subset \Sigma$ is a subshift (closed and invariant), setting

$$\mathrm{Di}(\Lambda) := \big\{ u \in S^* \,\big|\, u \,\text{appears in some}\; y \in \Lambda \big\}$$

we will have $\hat{\Lambda}\big[\mathrm{Di}(\Lambda)\big] = \Lambda$.

Proof. The inclusion $\hat{\Lambda}\big[\mathrm{Di}(\Lambda)\big] \supset \Lambda$ follows from the definition.

For the opposite inclusion, we pick $y \in \hat{\Lambda}\big[\mathrm{Di}(\Lambda)\big]$ and show that it belongs to $\overline{\Lambda} = \Lambda$.

Let $C_j[u]$ be a cylindrical neighborhood of y. Then $u \in \mathrm{Di}(\Lambda)$, thus it appears in some sequence $x \in \Lambda$. By invariance, we may assume that it occurs at place j, which means that $x \in C_j[u] \cap \Lambda \neq \emptyset$.

Game over ! $\square$

Corollary 9.16. *This is the way you get orbit closures:*

$$\overline{\mathfrak{O}}_x = \hat{\Lambda}[\mathrm{Di}(x)]\,,$$

where

$$\mathrm{Di}(x) := \big\{ u \in S^* \,\big|\, u \,\text{appears in}\; x \big\}\,.$$

Proof. By Corollary 9.10 and Propositions 9.15 one has

$$\overline{\mathfrak{O}}_x = \hat{\Lambda}\big[\mathrm{Di}(\overline{\mathfrak{O}}_x)\big] = \hat{\Lambda}\big[\mathrm{Di}(\overline{\mathrm{Sat}(x)}\,\big] = \hat{\Lambda}[\mathrm{Di}(x)]\,.$$

$\square$

Remark 9.17. Let us set

$$\mathrm{Subshift}(\Sigma) \equiv \mathrm{Cl}_{\mathrm{inv}}(\Sigma) := \big\{ \Lambda \subset \;\big|\, \Lambda \;\text{closed and invariant} \big\}$$

for the family of all the subshifts of $(\mathbb{Z}, \theta, \Sigma)$. Then $\mathrm{Di} : \mathrm{Subshift}(\Sigma) \to 2^{S^\star}$ is an injective map and $\hat{\Lambda}$ is a left inverse.

Definition 9.18. For every $P \subset S^\star$ we set

$$\tilde{\Lambda}(P) := \big\{ x \in \Sigma \,\big|\, \text{no}\; u \in P \;\text{occurs in}\; x \big\}$$

Example 9.19. P is formed of all the words containing a or o. Por exemplo "chiques" y "chiquxs" are allowed to appear in elements of $\tilde{\Lambda}(P)$, but .....

Example 9.20. P is formed of all the Dirty Words. Give some examples !

Proposition 9.21. *If non-void, $\tilde{\Lambda}(P)$ is a subshift.*

Exercise 9.22. One has $\tilde{\Lambda}(P) = \hat{\Lambda}\big(S^\star \backslash P\big)$. So the two approaches are equivalent.

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