

la Ecuación radial

$$\Psi(r, \theta, \phi) = R(r) Y(\theta, \phi)$$

$$\hat{L}^2 Y = l(l+1) Y$$

$$-\frac{r}{R} \frac{d^2}{dr^2} (rR) + l(l+1) = \frac{2m r^2}{\hbar^2} (E - V(r)) \quad (1)$$

$$\underline{l=0} \quad \frac{d^2}{dr^2} (rR) = -\frac{2m}{\hbar^2} (E - V(r)) (rR) \quad (2)$$

→ Ec. de S. 1D para $u(r) = rR$

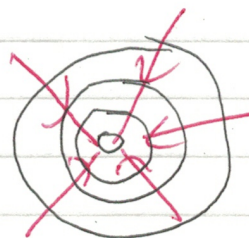
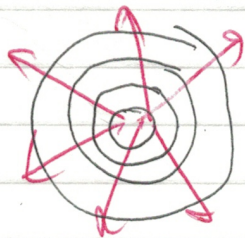
$$\text{Si } V(r) = 0 \Rightarrow \frac{d^2}{dr^2} (rR) = -\frac{2mE}{\hbar^2} (rR)$$

$$\Rightarrow rR = e^{\pm ikr}, \text{ donde } k = \sqrt{\frac{2mE}{\hbar^2}} \quad (3)$$

$$\Rightarrow \Psi(r, \theta, \phi) \propto \frac{e^{\pm ikr}}{r} Y_{00}(\theta, \phi) \rightarrow \underline{\text{cte.}} \quad (4)$$

Incluyendo el tiempo:

$$\Psi(r, t) \propto e^{i(kr + \omega t)}, \quad e^{-i(kr + \omega t)}$$



simetría esférica $\Rightarrow l=0$
(no dep. de θ, ϕ)

$$V(r) = \frac{1}{2} m \omega^2 r^2 \quad (5)$$

$$\frac{d^2}{dr^2} (rR) = \frac{2m}{\hbar^2} \left(\frac{1}{2} m \omega^2 r^2 \right) (rR) = -\frac{2mE}{\hbar^2} (rR)$$

$$\frac{d^2}{dr^2} [rR] - \frac{m\omega^2}{\hbar^2} r^2 [rR] = -\frac{2mE}{\hbar^2} [rR] \quad (6)$$

Mein eigenvalue: $\frac{\hbar\omega}{2}$?

con autofunción! $rR = e^{-er^2/2}$? NO!

~~pero~~ $R(r) \xrightarrow{r \rightarrow 0} \text{finito} \Rightarrow \lim_{r \rightarrow 0} rR(r) = 0$

$$u(x) = x e^{-ax^2/2} \quad \leftrightarrow \quad rR(r) = r e^{-er^2/2}$$

$$a \equiv \frac{m\omega^2}{\hbar}$$

$$\Rightarrow R(r) = e^{-er^2/2}$$

con $E = \frac{3}{2} \hbar \omega$ ✓

} demostros
como
ejercicio

Potencial centrifugo

$$\frac{d^2}{dr^2} (rR) = -\frac{2m}{\hbar^2} \left(E - V(r) - \frac{\hbar^2 l(l+1)}{2mr^2} \right) (rR) \quad (1)$$

$$V_{\text{ef}}(r) = V(r) + \frac{\hbar^2 l(l+1)}{2mr^2} \quad \leftarrow \text{potencial centrifugo} \quad (2)$$

Aplicación: pozo ^{FINITO} en 3D:

$$V(r) = \begin{cases} V_0 & r > a \\ 0 & r < a \end{cases}$$

como $l=0$

$$r < a: \quad \frac{d^2}{dr^2} (rR) = -\frac{2m}{\hbar^2} E (rR) = -k^2 (rR)$$

$$\Rightarrow rR = A \cos(kr) + B \sin(kr)$$

$$\Rightarrow R(r) = \frac{A}{r} \cos(kr) + \frac{B}{r} \sin(kr)$$

$$\psi(r, \theta, \phi) = \frac{A}{r} \cos(kr) + \frac{B}{r} \sin(kr) \quad (3)$$

como $\psi(0, \theta, \phi)$ debe ser finito $\Rightarrow A=0$

$$\boxed{\psi(r, \theta, \phi) = \frac{B}{r} \sin(kr)} \quad (r < a) \quad (4)$$



$r > a$: $\frac{d^2}{dr^2} (rR) = -\frac{2m(E - V_0)}{\hbar^2} (rR) \equiv \alpha^2 (rR)$

Para um estado ligado $E < V_0$

$$\Rightarrow \boxed{rR = d e^{-\alpha r}} \quad (r > a) \quad (5)$$

$$\boxed{\psi(r, \theta, \phi) = \frac{d}{r} e^{-\alpha r}} \quad (r > a) \quad (6)$$

$r = a$ $B \sin(ka) = C e^{-\alpha a}$ (7)

$(rR)'_a = (rR)'_{a+}$

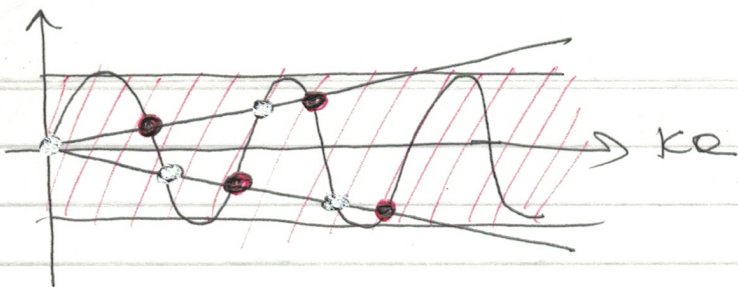
$(rR)'_{a-} = (rR)'_{a+}$: $kB \cos(ka) = -\alpha C e^{-\alpha a}$ (8)

$$\Rightarrow \boxed{-\alpha = k \cot(ka)} \quad (9)$$

y como $k = \sqrt{2mE/\hbar^2}$ y $\alpha = \sqrt{2m(V_0 - E)/\hbar^2}$

$$\Rightarrow k^2 + \alpha^2 = \frac{2mV_0}{\hbar^2} \equiv \beta^2 \quad (10)$$

(9) $\Rightarrow k^2 + k^2 \cot^2(ka) = \beta^2$
 $\frac{k^2}{\sin^2(ka)} = \beta^2 \Rightarrow \boxed{\sin(ka) = \pm \frac{ka}{\beta a}} \quad (11)$



Junto con
 $\cot(ka) < 0$ (de)

pero que $\exists$ al menos 1 solución, $\frac{\pi}{2} < ka < \pi$

pero $ka < \beta a \Rightarrow$ si $\beta a < \pi/2$ No $\exists$ B.S.

o sea $\frac{2mV_0a^2}{\hbar^2} < \left(\frac{\pi}{2}\right)^2$ NO $\exists$ Estados ligados.

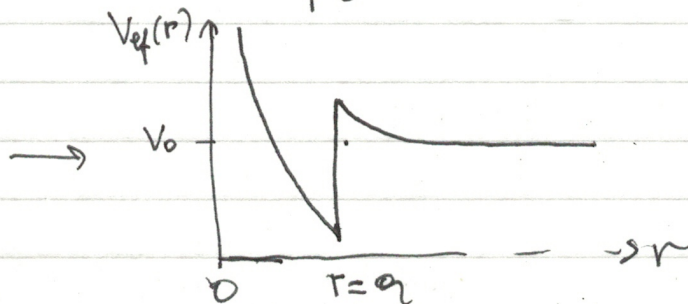
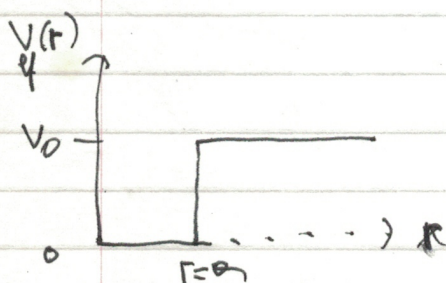
si $\frac{\pi}{2} < \beta a < \frac{3\pi}{2} \rightarrow 1$ Estados ligados

$\frac{3\pi}{2} < \beta a < \frac{5\pi}{2} \rightarrow 2$ Estados ligados
 ... etc.

Cuero $l \neq 0$: la ec. radial se complica un poco

$$\frac{d^2}{dr^2} [rR] = -k^2 [rR] + \frac{l(l+1)}{r^2} [rR] \quad (r < a)$$

$$\frac{d^2}{dr^2} [rR] = \alpha^2 [rR] + \frac{l(l+1)}{r^2} [rR] \quad (r > a)$$



pot. se hace
 más angosto
 y más
 profundo

Los **soluciones** para $r < a$, $r > a$ son las "fm. esféricas de Bessel" $\{j_l(kr), n_l(kr)\}$
 $\rightarrow$ fn. esférica de Neumann

$$R(r) = j_0(kr) = \sin(kr)/kr$$

$$R(r) = j_1(kr) = \frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr}$$

$$j_2(kr) = \left[\frac{3}{(kr)^3} - \frac{1}{kr} \right] \sin(kr) - \frac{3 \cos(kr)}{(kr)^2}$$

$$n_0(kr) = -\cos(kr)/kr$$

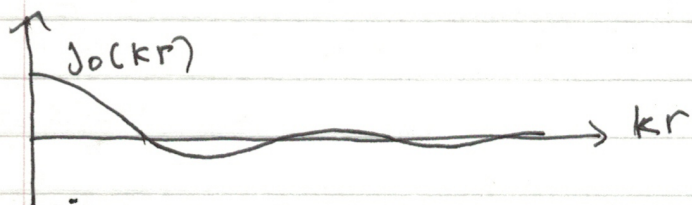
$$n_1(kr) = \frac{-\cos(kr)}{(kr)^2} - \frac{\sin(kr)}{kr}$$

$$n_2(kr) = \left[\frac{-3}{(kr)^3} + \frac{1}{kr} \right] \cos(kr) - \frac{3 \sin(kr)}{(kr)^2}$$

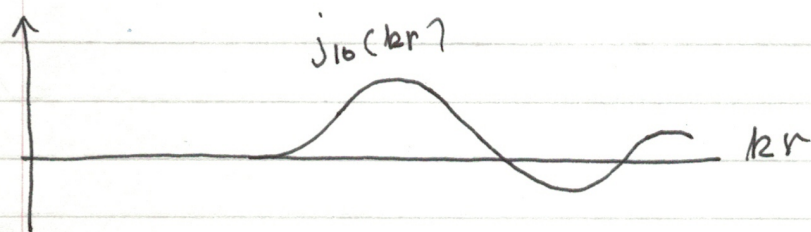
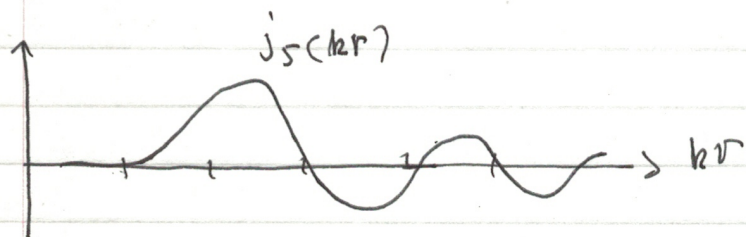
Ej: verifiquen que $j_0(kr)$ y $n_0(kr)$ son soluciones de la ec. radial para $r < a$ y $l=0$.

Ejercicio: Verificar que $j_2(kr)$ y $n_2(kr)$ son soluciones de

$$\frac{d^2}{dr^2}(rR) = -k^2(rR) + \frac{6\hbar^2}{2mr^2}(rR)$$



A medida que crece el
los se son "empujados"
hacia kr mayores.



$$|\vec{L}| = |\vec{r} \times \vec{p}| = |\vec{r}| |\vec{p}| \sin \alpha \leq |\vec{r}| |\vec{p}| = r \hbar k$$

$$\sqrt{\hbar^2 l(l+1)} \approx \hbar l$$

$$\therefore \hbar l \leq \hbar r k \Rightarrow$$

$$kr \geq l$$

RCQ: La soluc. debe ser del tipo $j_l(kr)$, porque $n_l(kr)$ diverge en el origen.

constantes.

$$h_e^{(1)}(iar) = j_e(iar) + i n_e(iar)$$

"Función esférica de Henkel del 1er tipo"

$$h_1^{(1)}(iar) = i \left(\frac{1}{dr} + \frac{1}{(dr)^2} \right) e^{-dr}$$

$$h_2^{(1)}(iar) = \left(\frac{1}{dr} + \frac{3}{(dr)^2} + \frac{3}{(dr)^3} \right) e^{-dr}$$

Por lo tanto, para $l=1$:

$$R(r) = A j_1(kr) \quad (r < a)$$

$$R(r) = B h_1^{(1)}(iar) \quad (r > a)$$

cond. de R'/R en $r=a \Rightarrow \left(\frac{(d/dr)[j_1(kr)]}{j_1(kr)} \right)_{r=a} = \left(\frac{(d/dr)[h_1^{(1)}(iar)]}{h_1^{(1)}(iar)} \right)_{r=a}$
igual que $(R')/(R)$

$$\Rightarrow \frac{\frac{d}{dr} \left[\frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr} \right]_{r=a}}{\frac{\sin(ka)}{(ka)^2} - \frac{\cos(ka)}{ka}} = \frac{\frac{d}{dr} \left[i e^{-dr} \left(\frac{1}{dr} + \frac{1}{(dr)^2} \right) \right]_{r=a}}{i e^{-da} \left(\frac{1}{da} + \frac{1}{(da)^2} \right)}$$

$$\Rightarrow \frac{\frac{-2\sin(ka)}{k^2 a^3} + \frac{2\cos(ka)}{ka^2} + \frac{\sin(ka)}{a}}{\frac{\sin(ka)}{(ka)^2} - \frac{\cos(ka)}{ka}} = \frac{-\frac{1}{a} - \frac{2}{a^2} - \frac{2}{a^3}}{\frac{1}{a} + \left(\frac{1}{a}\right)^2}$$

lo cual puede simplificarse e (ejercicio):

$$\frac{\cot(ka)}{ka} - \left(\frac{1}{ka}\right)^2 = \frac{1}{a} + \left(\frac{1}{a}\right)^2 \quad (1) \quad \checkmark$$

De las def. de k y α , tenemos

$$(ka)^2 + (\alpha a)^2 = \frac{2mV_0 a^2}{\hbar^2} \quad (2)$$

(1) y (2) deben ser resueltas algebraicamente para hallar los niveles de energía para $l=1$.

$$ka \equiv x ; \alpha a \equiv y$$

$$\frac{\cot(x)}{x} - \frac{1}{x^2} = \frac{1}{y} + \frac{1}{y^2} \quad \text{y} \quad x^2 + y^2 = R_0^2$$

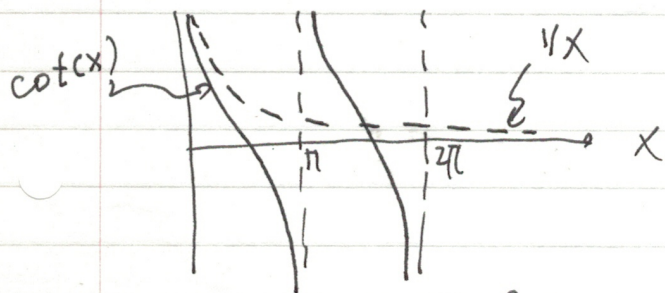
$$\rightarrow y^2 x \cot(x) - y^2 = x^2 y + x^2$$

$$y^2 x \cot(x) - x^2 y = x^2 + y^2 = R_0^2$$

$$(R_0^2 - x^2) x \cot(x) - x^2 \sqrt{R_0^2 - x^2} = R_0^2$$

Ej: Dem. que no $\exists$ estado ligado para $l=1$
cuando $V_0 < \frac{\pi^2 \hbar^2}{2ma^2}$.

Sol: para $ka < \pi$, $\cot(ka) < \frac{1}{ka} \Rightarrow$ LHS de (1)
es Negativo



y como el RHS de (1) es
positivo $\Rightarrow$ No $\exists$ sol. si
 $ka < \pi$.

$$\text{De (2)} \Rightarrow \frac{2mV_0a^2}{\hbar^2} \geq (ka)^2$$

por lo tanto, cuando $ka \geq \pi \Rightarrow \frac{2mV_0a^2}{\hbar^2} \geq \pi^2$

$$\Rightarrow V_0 \geq \frac{\pi^2 \hbar^2}{2ma^2} //$$

$\Rightarrow$ No sea $\exists$
al menos 1 sol.

Note: Es posible tener una solución en $ka = \pi$

$$\Rightarrow \cot(ka) \rightarrow \infty \Rightarrow a\alpha \rightarrow 0$$

$$\Rightarrow \frac{2mV_0a^2}{\hbar^2} = \pi^2 \Rightarrow V_0 = \frac{\pi^2 \hbar^2}{2ma^2}$$

otra solución aparece cuando $ka = 2\pi$

$$\Rightarrow V_0 = \frac{4\pi^2 \hbar^2}{2ma^2}$$

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{2mr^2 E}{\hbar^2} = 0$$

pozo
infinito
esferico

$$\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} + \underbrace{\left(\frac{2mE}{\hbar^2} \right)}_{K^2} R = 0$$

$$R'' + \frac{2}{r} R' + K^2 R = 0$$



$$R = \frac{U}{r} \Rightarrow R' = \frac{U'}{r} - \frac{U}{r^2}$$

$$\Rightarrow R'' = \frac{U''}{r} - \frac{U'}{r^2} - \frac{U'}{r^2} + \frac{2U}{r^3}$$

$$\frac{U''}{r} - \frac{2U'}{r^2} + \frac{2U}{r^3} + \frac{2}{r} \left[\frac{U'}{r} - \frac{U}{r^2} \right] + K^2 R = 0$$

$$\frac{U''}{r} - \frac{2U'}{r^2} + \frac{2U}{r^3} + \frac{2U'}{r^2} - \frac{2U}{r^3} + K^2 R = 0$$

$$\frac{U''}{r} + K^2 \frac{U}{r} \Rightarrow \boxed{U'' + K^2 U = 0}$$

$$U = A \sin(Kr) + B \cos(Kr)$$

$$R(r) = \frac{A}{r} \sin(Kr) + \frac{B}{r} \cos(Kr)$$

$$R(0) = \text{finite} \Rightarrow B = 0$$

$$R(R_0) = 0 \Rightarrow \sin(KR_0) = 0 \Rightarrow \begin{aligned} KR_0 &= n\pi \\ K &= \frac{n\pi}{R_0} \end{aligned}$$

$$\Rightarrow \boxed{R(r) = \frac{A}{r} \sin\left(\frac{n\pi r}{R_0}\right)}$$

$$K = \frac{n\pi}{R_0} = \frac{2mE}{\hbar^2}$$

$$\Rightarrow \boxed{E = \frac{\pi^2 \hbar^2}{2mR_0^2} \cdot n^2}$$

$$P = \frac{\partial E}{\partial V} = \frac{\partial E}{\partial r} \cdot \frac{\partial r}{\partial V} = \frac{\partial E / \partial r}{\partial V / \partial r}$$

pression interna

$$V = \frac{4\pi R^3}{3} \Rightarrow \partial V / \partial r = 4\pi R^2$$

$$\Rightarrow P = \frac{1}{4\pi r^2} \frac{\partial E}{\partial r} = \frac{1}{4\pi r^2} \frac{\partial}{\partial r} \left[\frac{\pi \hbar^2}{2m r^2} \right] = \frac{1}{4\pi r^2} \cdot \frac{\pi \hbar^2}{2m} \cdot \frac{(-2)}{r^3}$$

$$= \frac{\pi \hbar^2}{4m r^5}$$

$$P = \frac{\pi \hbar^2}{4m r^5}$$