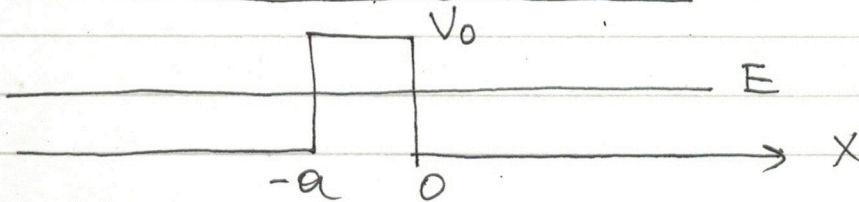
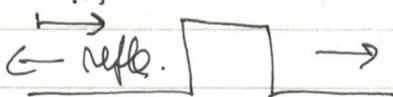


Penetración de barrera

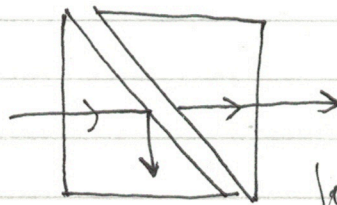
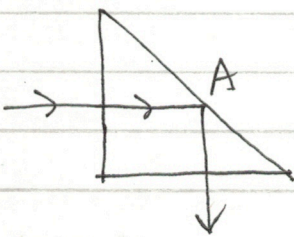


clásico: la partícula NO puede pasar si $E < V_0$.

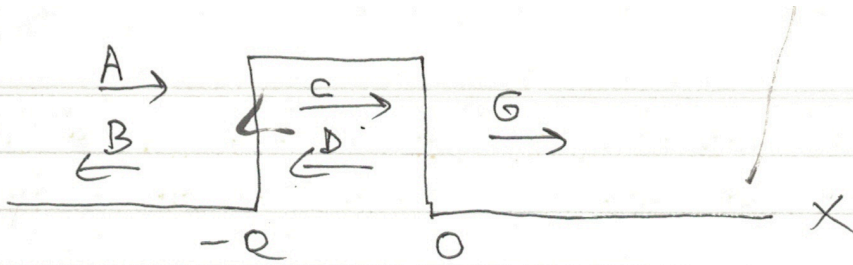
$$E = \frac{p^2}{2m} + V \Rightarrow \frac{p^2}{2m} = E - V < 0 \text{ dentro de la barrera} \Rightarrow \text{momento imaginario!} \rightarrow t$$

cuánticamente:  $\leftarrow \text{refl.} \quad \text{transmitido} \rightarrow$

óptico: reflexión total interna.



los prismas se acoplan por los campos evanescentes



$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$u = A e^{ikx} + B e^{-ikx}$$

$$x < -a$$

$$u = C e^{\alpha x} + D e^{-\alpha x}$$

$$-a < x < 0$$

$$u = G e^{ikx}$$

$$x > 0$$

$$\alpha = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

Cont. en $x=0$

cont. en $x=-a$

$$u' : C + D = G$$

$$u' : \alpha(C - D) = i k G$$

$$A e^{-ika} + B e^{ika} = C e^{-\alpha a} + D e^{\alpha a}$$

$$ik(A e^{-ika} - B e^{ika}) = \alpha(C e^{-\alpha a} - D e^{\alpha a})$$

$$\begin{cases} \alpha C + \alpha D = \alpha G \\ \alpha C - \alpha D = i k G \end{cases}$$

$$2\alpha D = (\alpha - i k) G$$

$$\Rightarrow 2\alpha C = (\alpha + i k) G$$

$$C = \frac{(\alpha + i k) G}{2\alpha}$$

$$D = \frac{(\alpha - i k) G}{2\alpha}$$

$$\Rightarrow C e^{-\alpha a} + D e^{\alpha a} = \left[\left(\frac{\alpha + i k}{2\alpha} \right) e^{-\alpha a} + \left(\frac{\alpha - i k}{2\alpha} \right) e^{\alpha a} \right] G = \left[\cosh(\alpha a) - \frac{i k}{\alpha} \sinh(\alpha a) \right] G$$

$$\Rightarrow C e^{-\alpha a} - D e^{\alpha a} = \left[\left(\frac{\alpha + i k}{2\alpha} \right) e^{-\alpha a} - \left(\frac{\alpha - i k}{2\alpha} \right) e^{\alpha a} \right] G = \left[\frac{i k}{\alpha} \cosh(\alpha a) - \sinh(\alpha a) \right] G$$

$$A e^{-ika} + B e^{ika} = G \left[\cosh(\alpha a) - \frac{i k}{\alpha} \sinh(\alpha a) \right]$$

$$A e^{-ika} - B e^{ika} = G \frac{\alpha}{i k} \left[\frac{i k}{\alpha} \cosh(\alpha a) - \sinh(\alpha a) \right]$$

$$= G \left[\cosh(\alpha a) - \frac{\alpha}{i k} \sinh(\alpha a) \right]$$

next: Despejar B/A (relac con el coefic de reflexion)

$$\frac{A e^{-i k a} + B e^{i k a}}{A e^{-i k a} - B e^{i k a}} = \frac{\cosh(\alpha a) - \frac{i k}{\alpha} \sinh(\alpha a)}{\cosh(\alpha a) - \frac{\alpha}{i k} \sinh(\alpha a)}$$

$$\frac{(1 + (B/A) e^{2 i k a}) (\cosh(\alpha a) - \frac{\alpha}{i k} \sinh(\alpha a))}{(1 - (B/A) e^{2 i k a}) (\cosh(\alpha a) - \frac{i k}{\alpha} \sinh(\alpha a))} = 1$$

$$\cosh(\alpha a) - \frac{\alpha}{i k} \sinh(\alpha a) + \left(\frac{B}{A}\right) e^{2 i k a} (\cosh(\alpha a) - \frac{\alpha}{i k} \sinh(\alpha a)) =$$

$$(\cosh(\alpha a) - \frac{i k}{\alpha} \sinh(\alpha a)) - (B/A) e^{2 i k a} (\cosh(\alpha a) - \frac{i k}{\alpha} \sinh(\alpha a))$$

$$\left(\frac{B}{A}\right) e^{2 i k a} \left[\cosh(\alpha a) - \frac{\alpha}{i k} \sinh(\alpha a) + \cosh(\alpha a) - \frac{i k}{\alpha} \sinh(\alpha a) \right] =$$

$$\cancel{\cosh(\alpha a)} - \frac{i k}{\alpha} \sinh(\alpha a) - \cancel{\cosh(\alpha a)} + \frac{\alpha}{i k} \sinh(\alpha a)$$

$$e^{2 i k a} (B/A) \left[2 \cosh(\alpha a) - \left(\frac{\alpha}{i k} + \frac{i k}{\alpha} \right) \sinh(\alpha a) \right] = \left(\frac{\alpha}{i k} - \frac{i k}{\alpha} \right) \sinh(\alpha a)$$

$$\frac{B}{A} = e^{-2 i k a} \frac{\left(\frac{\alpha}{i k} - \frac{i k}{\alpha} \right) \sinh(\alpha a)}{\left[2 \cosh(\alpha a) - \left(\frac{\alpha}{i k} + \frac{i k}{\alpha} \right) \sinh(\alpha a) \right]}$$

$$= e^{-2 i k a} \frac{(\alpha^2 + k^2) \sinh(\alpha a)}{\left[2 i k \alpha \cosh(\alpha a) + (k^2 - \alpha^2) \sinh(\alpha a) \right]}$$

$$\Rightarrow R = \frac{|B|^2}{|A|^2} = \left[1 + \frac{4(k\alpha)^2}{(k^2 + \alpha^2)^2 \sinh^2(\alpha a)} \right]^{-1}$$

para $k^2 = 2mE/\hbar^2$, $\alpha^2 = 2m(V_0 - E)/\hbar^2$

Resonancias

$$R = \left[1 + \frac{4 \left(\frac{2m}{\hbar^2} \right) E \cdot \frac{2m}{\hbar^2} (V_0 - E)}{\left(\frac{2m}{\hbar^2} \right) V_0^2 \sinh^2(\alpha a)} \right] = \left(1 + \frac{4E(V_0 - E)}{V_0^2 \sinh^2(\alpha a)} \right)$$

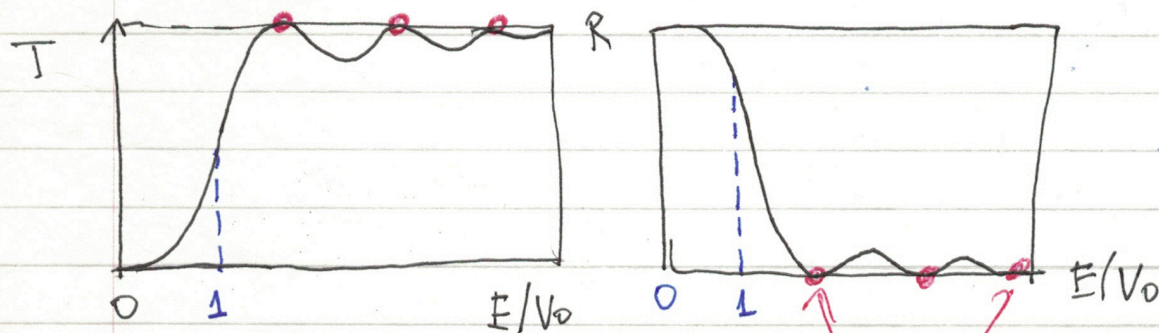
$$R = \left[1 + \frac{4(E/V_0)(1 - (E/V_0))}{\sinh^2 \left[\sqrt{\frac{2mV_0 a^2}{\hbar^2} \left(1 - \frac{E}{V_0} \right)} \right]} \right]^{-1}$$

Definir al para $V_0 = 8\hbar^2/m a^2 \Rightarrow R = \left[1 + \frac{4(E/V_0)(1 - E/V_0)}{\sinh^2 \left[4\sqrt{1 - \frac{E}{V_0}} \right]} \right]^{-1}$

Por su parte, $T = 1 - R$ (verdad?)

$$= \left[1 + \frac{\sinh^2 \left[\sqrt{\frac{2mV_0 a^2}{\hbar^2} \left(1 - \frac{E}{V_0} \right)} \right]}{4(E/V_0)(1 - (E/V_0))} \right]^{-1}$$

También sirve para $E > V_0$: $(\sinh i\alpha) = i \sin(\alpha)$ o $\sinh(i\alpha) = -i \sin(\alpha)$



Resonancias

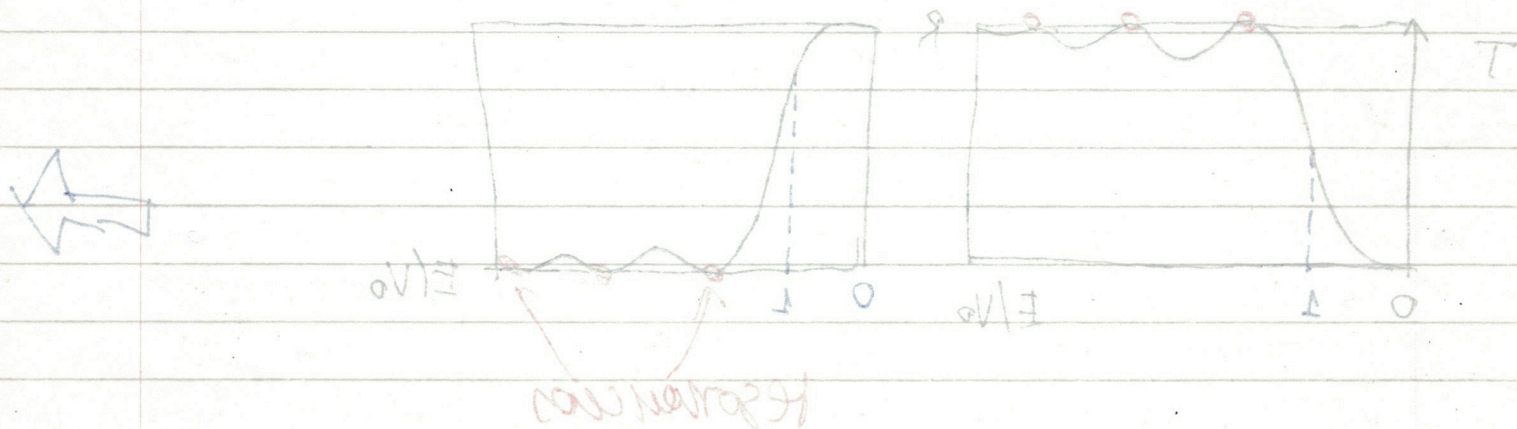
Resonancias cuando $T=1$
 $R=0$

$$E > V_0 \quad T = \left[1 + \frac{\sin^2 \left(\sqrt{\frac{2m\alpha^2 V_0}{\hbar^2} \left(\frac{E}{V_0} - 1 \right)} \right)}{4 \left(\frac{E}{V_0} \right) \left(\frac{E}{V_0} - 1 \right)} \right]^{-1}$$

$$T=1 \Rightarrow \sqrt{\frac{2m\alpha^2 V_0}{\hbar^2} \left(\frac{E}{V_0} - 1 \right)} = n\pi$$

$$\Rightarrow E_n = V_0 + \frac{\pi^2 \hbar^2 n^2}{2ma^2} \quad n=0, 1, 2, \dots$$

ojo: No es cuantización de energías, son energías resonantes.



En general, $\langle \sigma \rangle = \int_{-\infty}^{\infty} \psi^*(x) \sigma \psi(x) dx$

Values dispersion e incertez

$$\Delta x \equiv x - \langle x \rangle$$

$$\delta x \equiv \sqrt{\langle (\Delta x)^2 \rangle} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \quad (3)$$

igual! $\delta p \equiv \sqrt{\langle p_x^2 \rangle - \langle p_x \rangle^2}$ $\int_{-\infty}^{\infty} |\psi|^2 = 1 \quad (4)$

se $\psi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \Rightarrow |\psi|^2 = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$

$$\Rightarrow \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2\sigma^2}} dx - \left[\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2\sigma^2}} dx \right]^2 = \sigma^2$$

$$\Rightarrow \boxed{\delta x = \sqrt{\sigma^2 - 0} = \sigma} \quad (5)$$

$$\hat{p}\psi = -i\hbar \frac{\partial \psi}{\partial x} = (-i\hbar) \left(\frac{-2x}{4\sigma^2} \right) \psi = \frac{i\hbar x}{2\sigma^2} \psi$$

$$\begin{aligned} \hat{p}^2 \psi &= (-i\hbar)^2 \frac{\partial^2 \psi}{\partial x^2} = (-i\hbar) \left(\frac{i\hbar}{2\sigma^2} \right) \frac{\partial}{\partial x} (x\psi) \\ &= \frac{\hbar^2}{2\sigma^2} \left[\psi + x \frac{\partial \psi}{\partial x} \right] = \frac{\hbar^2}{2\sigma^2} \left(\psi + x \left(\frac{-x}{2\sigma^2} \right) \psi \right) \\ &= \frac{\hbar^2}{2\sigma^2} \left[1 - \frac{x^2}{2\sigma^2} \right] \psi \end{aligned}$$

$$\begin{aligned} \langle p^2 \rangle &= \frac{\hbar^2}{2\sigma^2} \underbrace{\int_{-\infty}^{\infty} |\psi|^2 dx}_1 - \frac{\hbar^2}{(2\sigma^2)} \underbrace{\int_{-\infty}^{\infty} x^2 |\psi|^2 dx}_{\sigma^2} \\ &= \frac{\hbar^2}{2\sigma^2} - \frac{\hbar^2}{4\sigma^2} = \frac{\hbar^2}{4\sigma^2} \end{aligned}$$

$$\Rightarrow \boxed{\delta p_x = \frac{\hbar}{2\sigma}} \quad (6)$$

$$\Rightarrow \delta x \delta p_x = \sigma \cdot \frac{\hbar}{2\sigma} = \frac{\hbar}{2} \quad (7)$$

$[\delta x \delta p_x > \hbar/2 \text{ per wave } \psi \text{ no generalization}]$

Teo. de Ehrenfest

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi \quad (1)$$

$$\text{con } \hat{H} = \frac{\hat{p}^2}{2m} + V \quad (2)$$

sea $\hat{A}$ operador arbitrario

$$\frac{d}{dt} \langle A \rangle = \frac{d}{dt} \int \psi^* A \psi = \int \frac{\partial \psi^*}{\partial t} A \psi + \int \psi^* \frac{\partial A}{\partial t} \psi + \int \psi^* A \frac{\partial \psi}{\partial t}$$

$$(1) \rightarrow \frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} \hat{H} \psi, \quad \text{y} \quad \frac{\partial \psi^*}{\partial t} = \frac{i}{\hbar} \hat{H} \psi^* \quad (3)$$

$$\Rightarrow \frac{d}{dt} \langle A \rangle = \frac{i}{\hbar} \langle \hat{H} \psi | A \psi \rangle + \langle \psi | \frac{\partial A}{\partial t} \psi \rangle - \frac{i}{\hbar} \langle \psi | A \hat{H} \psi \rangle$$

$$= \frac{i}{\hbar} \langle \psi | [\hat{H}, \hat{A}] \psi \rangle + \langle \frac{\partial A}{\partial t} \rangle \quad (4)$$

$$\frac{d \langle \hat{A} \rangle}{dt} = \frac{i}{\hbar} \langle [\hat{H}, \hat{A}] \rangle + \langle \frac{\partial A}{\partial t} \rangle \quad (5)$$

$$([A, B] = AB - BA)$$

Aplicaciones: $\hat{A} = \hat{x}$

$$[\hat{H}, \hat{x}] = \left[\frac{\hat{p}^2}{2m} + V, \hat{x} \right] = \frac{1}{2m} [\hat{p}^2, \hat{x}] = \frac{1}{2m} (\hat{p}^2 \hat{x} - \hat{x} \hat{p}^2)$$

$$(\hat{p}^2 \hat{x} - \hat{x} \hat{p}^2) \psi = -\hbar^2 \frac{d^2}{dx^2} (x \psi) + \hbar^2 x \frac{d^2}{dx^2} \psi$$

$$= -\hbar^2 (2\psi' + x\psi'' - x\psi'') = -2\hbar^2 \frac{\partial \psi}{\partial x}$$

$$\therefore [H, x] = \frac{1}{2m} \cdot (-2)\hbar^2 \frac{\partial \psi}{\partial x} = -\frac{\hbar^2}{m} \frac{\partial \psi}{\partial x}$$

$$\Rightarrow \frac{d\langle x \rangle}{dt} = \left\langle \frac{i}{\hbar} \left(-\frac{\hbar^2}{m} \right) \frac{\partial \psi}{\partial x} \right\rangle = \frac{\langle p \rangle}{m}$$

$$\boxed{\frac{d\langle x \rangle}{dt} = \frac{\langle p \rangle}{m}} \quad (6)$$

$$(2) \quad \check{A} = \check{P}$$

$$[H, p] = \left[\frac{p^2}{2m} + V, p \right] = [V(x), p]$$

$$\begin{aligned} [V, p] \psi &= (Vp)\psi - (pV)\psi = V(-i\hbar \frac{\partial}{\partial x})\psi + i\hbar \frac{\partial}{\partial x} (V\psi) \\ &= -i\hbar \underbrace{(\frac{\partial \psi}{\partial x}) V} + i\hbar \underbrace{(\frac{\partial V}{\partial x}) \psi} + i\hbar V \underbrace{(\frac{\partial \psi}{\partial x})} \end{aligned}$$

$$\Rightarrow [V, p] = i\hbar \frac{\partial V}{\partial x}$$

$$\Rightarrow \frac{d\langle p \rangle}{dt} = \frac{i}{\hbar} \cdot i\hbar \frac{\partial V}{\partial x} = -\frac{\partial V}{\partial x} = F \quad (7)$$

$$(6), (7) \Rightarrow \boxed{F = m \frac{d^2 \langle x \rangle}{dt^2}} \quad (8)$$