Chemically Reacting Flow: Theory & Practice
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Appendix B
Navier-Stokes Equations

The purpose of this appendix is to spell out explicitly the Navier-Stokes and mass-continuity equations in different coordinate systems. Although the equations can be expanded from the general vector forms, dealing with the stress tensor T usually makes the expansion tedious. Expansion of the scalar equations (e.g., species or energy) are much less trouble.

B.1 GENERAL VECTOR FORM

The equations in this section retain some compact notation, including the substantial derivative operator D/Dt, the divergence of the velocity vector $\nabla\cdot\mathbf{V}$, and the Laplacian operator ∇^2. The expansion of these operations into the various coordinate systems may be found in Appendix A.

B.1.1 Mass Continuity

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{V}) = 0 \qquad \frac{D\rho}{Dt} + \rho\, \nabla\cdot\mathbf{V} = 0 \tag{B.1}$$

B.1.2 Momentum, General Form

$$
\begin{aligned}
\rho\frac{D\mathbf{V}}{Dt} &= \rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right] = \rho\left[\frac{\partial \mathbf{V}}{\partial t} + \nabla\left(\frac{\mathbf{V}\cdot\mathbf{V}}{2}\right) + \mathbf{V}\times(\nabla\times\mathbf{V})\right] \\
&= \mathbf{f} + \nabla\cdot\mathsf{T} = \mathbf{f} - \nabla p + \nabla\cdot\mathsf{T}'
\end{aligned} \tag{B.2}
$$

B.1.3 Momentum, Constant Viscosity

$$
\begin{aligned}
\rho\frac{D\mathbf{V}}{Dt} &= \rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right] = \rho\left[\frac{\partial \mathbf{V}}{\partial t} + \nabla\left(\frac{\mathbf{V}\cdot\mathbf{V}}{2}\right) - \mathbf{V}\times(\nabla\times\mathbf{V})\right] \\
&= \mathbf{f} - \nabla p - \mu\nabla\times[(\nabla\times\mathbf{V})] + (\kappa + 2\mu)\nabla[\nabla\cdot\mathbf{V}] \\
&= \mathbf{f} - \nabla p + \mu\nabla^2\mathbf{V} + (\kappa + \mu)\nabla(\nabla\cdot\mathbf{V})
\end{aligned} \tag{B.3}
$$

B.1.4 Momentun, Incompressible and Constant Viscosity

$$
\rho\frac{D\mathbf{V}}{Dt} = \mathbf{f} - \nabla p + \mu\nabla^2\mathbf{V} \tag{B.4}
$$

B.2 STRESS COMPONENTS

The stress state is represented as a symmetric tensor T, whose components may be expanded into various coordinate systems. The specific-coordinate-system expansions of the divergence of the velocity vector $\nabla\cdot\mathbf{V}$ may be found in Section A.10.

B.2.1 Cartesian

The components of the velocity vector (u, v, w) align with the cartesian-coordinate directions (x, y, z).

$$
\begin{aligned}
\tau_{xx} &= -p + 2\mu\frac{\partial u}{\partial x} + \kappa\nabla\cdot\mathbf{V} \\
\tau_{yy} &= -p + 2\mu\frac{\partial v}{\partial y} + \kappa\nabla\cdot\mathbf{V} \\
\tau_{zz} &= -p + 2\mu\frac{\partial w}{\partial z} + \kappa\nabla\cdot\mathbf{V}
\end{aligned}
$$

$$
\begin{aligned}
\tau_{xy} = \tau_{yx} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
\tau_{yz} = \tau_{zy} &= \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\
\tau_{zx} = \tau_{xz} &= \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)
\end{aligned}
\tag{B.5}
$$

B.2.2 Cylindrical

The components of the velocity vector (u, v, w) align with the cylindrical-coordinate directions (z, r, θ).

$$
\begin{aligned}
\tau_{zz} &= -p + 2\mu\frac{\partial u}{\partial z} + \kappa \nabla\cdot\mathbf{V} \\
\tau_{rr} &= -p + 2\mu\frac{\partial v}{\partial r} + \kappa \nabla\cdot\mathbf{V} \\
\tau_{\theta\theta} &= -p + 2\mu\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{v}{r}\right)\kappa \nabla\cdot\mathbf{V} \\
\tau_{zr} = \tau_{rz} &= \mu\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right) \\
\tau_{r\theta} = \tau_{\theta r} &= \mu\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial r} - \frac{w}{r}\right) \\
\tau_{\theta z} = \tau_{z\theta} &= \mu\left(\frac{\partial w}{\partial z} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)
\end{aligned}
\tag{B.6}
$$

B.2.3 Spherical

The components of the velocity vector (v_r, v_θ, v_ϕ) align with the spherical-coordinate directions (r, θ, ϕ).

$$
\begin{aligned}
\tau_{rr} &= -p + 2\mu\frac{\partial v_r}{\partial r} + \kappa \nabla\cdot\mathbf{V} \\
\tau_{\theta\theta} &= -p + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \kappa \nabla\cdot\mathbf{V} \\
\tau_{\phi\phi} &= -p + 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right) + \kappa \nabla\cdot\mathbf{V} \\
\tau_{r\theta} = \tau_{\theta r} &= \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right] \\
\tau_{\theta\phi} = \tau_{\phi\theta} &= \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right] \\
\tau_{\phi r} = \tau_{r\phi} &= \mu\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]
\end{aligned}
\tag{B.7}
$$

B.2.4 Curvilinear

The components of the velocity vector (v_1, v_2, v_3) align with the curvilinear-coordinate directions (x_1, x_2, x_3).

$$\begin{aligned}
\tau_{11} &= -p + 2\mu\left(\frac{1}{h_1}\frac{\partial v_1}{\partial x_1} + \frac{v_2}{h_1 h_2}\frac{\partial h_1}{\partial x_2} + \frac{v_3}{h_3 h_1}\frac{\partial h_1}{\partial x_3}\right) + \kappa\nabla\cdot\mathbf{V} \\
\tau_{22} &= -p + 2\mu\left(\frac{1}{h_2}\frac{\partial v_2}{\partial x_2} + \frac{v_3}{h_2 h_3}\frac{\partial h_2}{\partial x_3} + \frac{v_1}{h_1 h_2}\frac{\partial h_2}{\partial x_1}\right) + \kappa\nabla\cdot\mathbf{V} \\
\tau_{33} &= -p + 2\mu\left(\frac{1}{h_3}\frac{\partial v_3}{\partial x_3} + \frac{v_1}{h_3 h_1}\frac{\partial h_3}{\partial x_1} + \frac{v_2}{h_2 h_3}\frac{\partial h_3}{\partial x_2}\right) + \kappa\nabla\cdot\mathbf{V} \\
\tau_{12} = \tau_{21} &= \mu\left[\frac{h_2}{h_1}\frac{\partial}{\partial x_1}\left(\frac{v_2}{h_2}\right) + \frac{h_1}{h_2}\frac{\partial}{\partial x_2}\left(\frac{v_1}{h_1}\right)\right] \\
\tau_{23} = \tau_{32} &= \mu\left[\frac{h_3}{h_2}\frac{\partial}{\partial x_2}\left(\frac{v_3}{h_3}\right) + \frac{h_2}{h_3}\frac{\partial}{\partial x_3}\left(\frac{v_2}{h_2}\right)\right] \\
\tau_{31} = \tau_{13} &= \mu\left[\frac{h_1}{h_3}\frac{\partial}{\partial x_3}\left(\frac{v_1}{h_1}\right) + \frac{h_3}{h_1}\frac{\partial}{\partial x_1}\left(\frac{v_3}{h_3}\right)\right]
\end{aligned} \tag{B.8}$$

B.3 CARTESIAN NAVIER-STOKES EQUATIONS

B.3.1 Mass Continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \tag{B.9}$$

B.3.2 x-Momentum

$$\begin{aligned}
\rho\frac{Du}{Dt} &= f_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} + \kappa\nabla\cdot\mathbf{V}\right] \\
&\quad + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]
\end{aligned} \tag{B.10}$$

B.3.3 y-Momentum

$$\begin{aligned}
\rho\frac{Dv}{Dt} &= f_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y} + \kappa\nabla\cdot\mathbf{V}\right] \\
&\quad + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]
\end{aligned} \tag{B.11}$$

B.3.4 z-Momentum

$$\rho \frac{Dw}{Dt} = f_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[2\mu \frac{\partial w}{\partial z} + \kappa \nabla\cdot\mathbf{V}\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \qquad \text{(B.12)}$$

B.4 CARTESIAN NAVIER-STOKES, CONSTANT VISCOSITY

B.4.1 x-Momentum, Constant Viscosity

$$\rho \frac{Du}{Dt} = f_x - \frac{\partial p}{\partial x} + \mu \nabla^2 u + (\mu + \kappa)\frac{\partial}{\partial x}\nabla\cdot\mathbf{V} \qquad \text{(B.13)}$$

B.4.2 y Momentum, Constant Viscosity

$$\rho \frac{Dv}{Dt} = f_y - \frac{\partial p}{\partial y} + \mu \nabla^2 v + (\mu + \kappa)\frac{\partial}{\partial y}\nabla\cdot\mathbf{V} \qquad \text{(B.14)}$$

B.4.3 z-Momentum, Constant Viscosity

$$\rho \frac{Dw}{Dt} = f_z - \frac{\partial p}{\partial z} + \mu \nabla^2 w + (\mu + \kappa)\frac{\partial}{\partial z}\nabla\cdot\mathbf{V} \qquad \text{(B.15)}$$

B.5 CYLINDRICAL NAVIER-STOKES EQUATIONS

B.5.1 Mass Continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial z} + \frac{1}{r}\frac{\partial r\rho v}{\partial r} + \frac{1}{r}\frac{\partial \rho w}{\partial \theta} = 0. \qquad \text{(B.16)}$$

B.5.2 z-Momentum

$$
\begin{aligned}
\rho\left(\frac{Du}{Dt}\right) &= \rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial z}+v\frac{\partial u}{\partial r}+\frac{w}{r}\frac{\partial u}{\partial \theta}\right) \\
&= f_z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial z}\left[2\mu\frac{\partial u}{\partial z}+\kappa\nabla\cdot\mathbf{V}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left[\mu r\left(\frac{\partial v}{\partial z}\frac{\partial u}{\partial r}\right)\right] \\
&\quad+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left(\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial w}{\partial z}\right)\right]
\end{aligned}
\tag{B.17}
$$

B.5.3 r-Momentum

$$
\begin{aligned}
\rho\left(\frac{Dv}{Dt}-\frac{w^2}{r}\right) &= \rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial z}+v\frac{\partial v}{\partial r}+\frac{w}{r}\frac{\partial v}{\partial \theta}-\frac{w^2}{r}\right) \\
&= f_r-\frac{\partial p}{\partial r}+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial u}{\partial r}\right)\right]+\frac{\partial}{\partial r}\left[2\mu\frac{\partial v}{\partial r}+\kappa\nabla\cdot\mathbf{V}\right] \\
&\quad+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left(\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial r}-\frac{w}{r}\right)\right]+\frac{2\mu}{r}\left[\frac{\partial v}{\partial r}-\frac{1}{r}\frac{\partial w}{\partial \theta}-\frac{v}{r}\right]
\end{aligned}
\tag{B.18}
$$

B.5.4 θ-Momentum

$$
\begin{aligned}
\rho\left(\frac{Dw}{Dt}+\frac{vw}{r}\right) &= \rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial z}+v\frac{\partial w}{\partial r}+\frac{w}{r}\frac{\partial w}{\partial \theta}+\frac{vw}{r}\right) \\
&= f_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{\partial}{\partial z}\left[\mu\left(\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial w}{\partial z}\right)\right] \\
&\quad+\frac{\partial}{\partial r}\left[\mu\left(\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial r}-\frac{w}{r}\right)\right] \\
&\quad+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{2\mu}{r}\frac{\partial w}{\partial \theta}+\kappa\nabla\cdot\mathbf{V}\right]+\frac{2\mu}{r}\left[\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial r}-\frac{w}{r}\right]
\end{aligned}
\tag{B.19}
$$

B.6 CYLINDRICAL NAVIER-STOKES, CONSTANT VISCOSITY

B.6.1 z-Momentum, Constant Viscosity

$$
\begin{aligned}
\rho\left(\frac{Du}{Dt}\right) &= \rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial z}+v\frac{\partial u}{\partial r}+\frac{w}{r}\frac{\partial u}{\partial \theta}\right) \\
&= f_z-\frac{\partial p}{\partial z}+\mu\nabla^2 u+(\kappa+\mu)\frac{\partial}{\partial z}(\nabla\cdot\mathbf{V})
\end{aligned}
\tag{B.20}
$$

B.6.2 r-Momentum, Constant Viscosity

$$\begin{aligned}\rho\left(\frac{Dv}{Dt}-\frac{w^2}{r}\right) &= \rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial z}+v\frac{\partial v}{\partial r}+\frac{w}{r}\frac{\partial v}{\partial \theta}-\frac{w^2}{r}\right) \\ &= f_r-\frac{\partial p}{\partial r}+\mu\left[\nabla^2 v-\frac{v}{r^2}-\frac{2}{r^2}\frac{\partial w}{\partial \theta}\right]+(\kappa+\mu)\frac{\partial}{\partial r}(\nabla\cdot\mathbf{V})\end{aligned} \tag{B.21}$$

B.6.3 θ-Momentum, Constant Viscosity

$$\begin{aligned}\rho\left(\frac{Dw}{Dt}+\frac{vw}{r}\right) &= \rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial z}+v\frac{\partial w}{\partial r}+\frac{w}{r}\frac{\partial w}{\partial \theta}+\frac{vw}{r}\right) \\ &= f_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\nabla^2 w-\frac{w}{r^2}+\frac{2}{r^2}\frac{\partial v}{\partial \theta}\right]+(\kappa+\mu)\frac{1}{r}\frac{\partial}{\partial \theta}(\nabla\cdot\mathbf{V})\end{aligned} \tag{B.22}$$

B.7 SPHERICAL NAVIER-STOKES EQUATIONS

B.7.1 Mass Continuity:

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho r^2 v_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho v_\phi)=0 \tag{B.23}$$

B.7.2 r-Momentum

$$\begin{aligned}\rho\left(\frac{Dv_r}{Dt}-\frac{v_\theta^2+v_\phi^2}{r}\right) &= f_r-\frac{\partial p}{\partial r}+\frac{\partial}{\partial r}\left[2\mu\frac{\partial v_r}{\partial r}+\kappa\nabla\cdot\mathbf{V}\right] \\ &\quad+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{\mu}{r}\frac{\partial v_r}{\partial \theta}\right] \\ &\quad+\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left[\frac{\mu}{r\sin\theta}\frac{\partial v_r}{\partial \phi}+\mu r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right] \\ &\quad+\frac{\mu}{r}\left[4\frac{\partial v_r}{\partial r}-\frac{2}{r}\frac{\partial v_\theta}{\partial \theta}-\frac{4v_r}{r}-\frac{2}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}-\frac{2v_\theta\cot\theta}{r}\right. \\ &\qquad\left.+r\cot\theta\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{\cot\theta}{r}\frac{\partial v_r}{\partial \theta}\right]\end{aligned} \tag{B.24}$$

B.7.3 θ-Momentum

$$
\begin{aligned}
\rho\left(\frac{Dv_\theta}{Dt}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right) &= f_\theta-\frac{1}{r}\frac{\partial p}{\partial\theta}+\frac{1}{r}\frac{\partial}{\partial\theta}\left[2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right)+\kappa\nabla\cdot\mathbf{V}\right]\\
&+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\mu\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right)+\frac{\mu}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right]\\
&+\frac{\partial}{\partial r}\left[\mu r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{\mu}{r}\frac{\partial v_r}{\partial\theta}\right] \qquad \text{(B.25)}\\
&+\frac{\mu}{r}\left[\frac{2\cot\theta}{r}\frac{\partial v_\theta}{\partial\theta}-\frac{2\cot\theta}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}-\frac{2v_\theta\cot^2\theta}{r}\right.\\
&\qquad\left.+3r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{3}{r}\frac{\partial v_r}{\partial\theta}\right]
\end{aligned}
$$

B.7.4 ϕ-Momentum

$$
\begin{aligned}
\rho\left(\frac{Dv_\phi}{Dt}+\frac{v_\phi v_r}{r}+\frac{v_\theta v_\phi\cot\theta}{r}\right) &= f_\phi-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[2\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right.\right.\\
&\qquad\left.\left.+\frac{v_r}{r}+\frac{v_\theta\cot\theta}{r}\right)+\kappa\nabla\cdot\mathbf{V}\right]\\
&+\frac{\partial}{\partial r}\left[\frac{\mu}{r\sin\theta}\frac{\partial v_r}{\partial\phi}+\mu r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right] \qquad \text{(B.26)}\\
&+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{\mu\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right)+\frac{\mu}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right]\\
&+\frac{\mu}{r}\left[\frac{3}{r\sin\theta}\frac{\partial v_r}{\partial\phi}+3r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right.\\
&\qquad\left.+\frac{2\cot\theta\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right)+\frac{2\cot\theta}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right]
\end{aligned}
$$

B.8 SPHERICAL NAVIER-STOKES, CONSTANT VISCOSITY

B.8.1 r-Momentum, Constant Viscosity

$$
\begin{aligned}
\rho\left(\frac{Dv_r}{Dt}-\frac{v_\theta^2+v_\phi^2}{r}\right) &= f_r-\frac{\partial p}{\partial r} \qquad \text{(B.27)}\\
&+\mu\left(\nabla^2 v_r-\frac{2v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{2v_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)\\
&+(\kappa+\mu)\frac{\partial}{\partial r}(\nabla\cdot\mathbf{V})
\end{aligned}
$$

B.8.2 θ-Momentum, Constant Viscosity

$$
\begin{aligned}
\rho\left(\frac{Dv_\theta}{Dt}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right) &= f_\theta-\frac{1}{r}\frac{\partial p}{\partial\theta} \\
&+\mu\left(\nabla^2 v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial\phi}\right) \\
&+(\kappa+\mu)\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla\cdot\mathbf{V})
\end{aligned}
\tag{B.28}
$$

B.8.3 ϕ-Momentum, Constant Viscosity

$$
\begin{aligned}
\rho\left(\frac{Dv_\phi}{Dt}+\frac{v_\phi v_r}{r}+\frac{v_\theta v_\phi\cot\theta}{r}\right) &= f_\phi-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} \\
&+\mu\left(\nabla^2 v_\phi-\frac{v_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi}\right. \\
&\qquad\left.+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\phi}\right) \\
&+(\kappa+\mu)\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\nabla\cdot\mathbf{V})
\end{aligned}
\tag{B.29}
$$

B.9 ORTHOGONAL CURVILINEAR NAVIER-STOKES

B.9.1 Mass Continuity

$$
\begin{aligned}
&\frac{\partial\rho}{\partial t}+\frac{V_1}{h_1}\frac{\partial\rho}{x_1}+\frac{V_2}{h_2}\frac{\partial\rho}{x_2}+\frac{V_3}{h_3}\frac{\partial\rho}{x_3} \\
&\quad+\frac{\rho}{h_1h_2h_3}\left[\frac{\partial(h_2h_3V_1)}{\partial x_1}+\frac{\partial(h_3h_1V_2)}{\partial x_2}+\frac{\partial(h_1h_2V_3)}{\partial x_3}\right]=0
\end{aligned}
\tag{B.30}
$$

B.9.2 x_1-Momentum

$$
\begin{aligned}
&\rho\left[\frac{\partial V_1}{\partial t}+\frac{V_1}{h_1}\frac{\partial V_1}{x_1}+\frac{V_2}{h_2}\frac{\partial V_1}{x_2}+\frac{V_3}{h_3}\frac{\partial V_1}{x_3}-V_2\left(\frac{V_2}{h_2h_1}\frac{\partial h_2}{\partial x_1}-\frac{V_1}{h_1h_2}\frac{\partial h_1}{\partial x_2}\right)\right. \\
&\qquad\left.+V_3\left(\frac{V_1}{h_1h_3}\frac{\partial h_1}{\partial x_3}-\frac{V_3}{h_3h_1}\frac{\partial h_3}{\partial x_1}\right)\right] \\
&= f_1-\frac{1}{h_1}\frac{\partial p}{\partial x_1}+\frac{1}{h_1}\frac{\partial}{\partial x_1}(\kappa\nabla\cdot\mathbf{V})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left\{2\mu h_2h_3\left(\frac{1}{h_1}\frac{\partial V_1}{\partial x_1}+\frac{V_2}{h_1h_2}\frac{\partial h_1}{\partial x_2}+\frac{V_3}{h_3h_1}\frac{\partial h_1}{\partial x_3}\right)\right\}\right.\\
&\qquad+\frac{\partial}{\partial x_2}\left\{\mu h_3h_1\left[\frac{h_2}{h_1}\frac{\partial}{\partial x_1}\left(\frac{V_2}{h_2}\right)+\frac{h_1}{h_2}\frac{\partial}{\partial x_2}\left(\frac{V_1}{h_1}\right)\right]\right\}\\
&\qquad\left.+\frac{\partial}{\partial x_3}\left\{\mu h_1h_2\left[\frac{h_1}{h_3}\frac{\partial}{\partial x_3}\left(\frac{V_1}{h_1}\right)+\frac{h_3}{h_1}\frac{\partial}{\partial x_1}\left(\frac{V_3}{h_3}\right)\right]\right\}\right]\\
&+\frac{\mu}{h_1h_2}\left\{\frac{h_2}{h_1}\frac{\partial}{\partial x_1}\left(\frac{V_2}{h_2}\right)+\frac{h_1}{h_2}\frac{\partial}{\partial x_2}\left(\frac{V_1}{h_1}\right)\right\}\frac{\partial h_1}{\partial x_2}\\
&+\frac{\mu}{h_1h_3}\left\{\frac{h_1}{h_3}\frac{\partial}{\partial x_3}\left(\frac{V_1}{h_1}\right)+\frac{h_3}{h_1}\frac{\partial}{\partial x_1}\left(\frac{V_3}{h_3}\right)\right\}\frac{\partial h_1}{\partial x_3}\\
&-\frac{2\mu}{h_1h_2}\left\{\frac{1}{h_2}\frac{\partial V_2}{\partial x_2}+\frac{V_3}{h_2h_3}\frac{\partial h_2}{\partial x_3}+\frac{V_1}{h_1h_2}\frac{\partial h_2}{\partial x_1}\right\}\frac{\partial h_2}{\partial x_1}\\
&-\frac{2\mu}{h_1h_3}\left\{\frac{1}{h_3}\frac{\partial V_3}{\partial x_3}+\frac{V_1}{h_3h_1}\frac{\partial h_3}{\partial x_1}+\frac{V_3}{h_2h_3}\frac{\partial h_3}{\partial x_2}\right\}\frac{\partial h_3}{\partial x_1}
\end{aligned}
\tag{B.31}
$$

B.9.3 x_2-Momentum

$$
\begin{aligned}
&\rho\left[\frac{\partial V_2}{\partial t}+\frac{V_1}{h_1}\frac{\partial V_2}{x_1}+\frac{V_2}{h_2}\frac{\partial V_2}{x_2}+\frac{V_3}{h_3}\frac{\partial V_2}{x_3}-V_3\left(\frac{V_3}{h_3h_2}\frac{\partial h_3}{\partial x_2}-\frac{V_2}{h_2h_3}\frac{\partial h_2}{\partial x_3}\right)\right.\\
&\qquad\qquad\left.+V_1\left(\frac{V_2}{h_2h_1}\frac{\partial h_2}{\partial x_1}-\frac{V_1}{h_1h_2}\frac{\partial h_1}{\partial x_2}\right)\right]\\
=\;& f_2-\frac{1}{h_2}\frac{\partial p}{\partial x_2}+\frac{1}{h_2}\frac{\partial}{\partial x_2}\left(\kappa\nabla\cdot\mathbf{V}\right)\\
&+\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left\{\mu h_2h_3\left[\frac{h_2}{h_1}\frac{\partial}{\partial x_1}\left(\frac{V_2}{h_2}\right)+\frac{h_1}{h_2}\frac{\partial}{\partial x_2}\left(\frac{V_1}{h_1}\right)\right]\right\}\right.\\
&\qquad+\frac{\partial}{\partial x_2}\left\{2\mu h_3h_1\left(\frac{1}{h_2}\frac{\partial V_2}{\partial x_2}+\frac{V_3}{h_2h_3}\frac{\partial h_2}{\partial x_3}+\frac{V_1}{h_1h_2}\frac{\partial h_2}{\partial x_1}\right)\right\}\\
&\qquad\left.+\frac{\partial}{\partial x_3}\left\{\mu h_1h_2\left[\frac{h_3}{h_2}\frac{\partial}{\partial x_2}\left(\frac{V_3}{h_3}\right)+\frac{h_2}{h_3}\frac{\partial}{\partial x_3}\left(\frac{V_2}{h_2}\right)\right]\right\}\right]\\
&+\frac{\mu}{h_2h_3}\left\{\frac{h_3}{h_2}\frac{\partial}{\partial x_2}\left(\frac{V_3}{h_3}\right)+\frac{h_2}{h_3}\frac{\partial}{\partial x_3}\left(\frac{V_2}{h_2}\right)\right\}\frac{\partial h_2}{\partial x_3}\\
&+\frac{\mu}{h_2h_1}\left\{\frac{h_2}{h_1}\frac{\partial}{\partial x_1}\left(\frac{V_2}{h_2}\right)+\frac{h_1}{h_2}\frac{\partial}{\partial x_2}\left(\frac{V_1}{h_1}\right)\right\}\frac{\partial h_2}{\partial x_1}\\
&-\frac{2\mu}{h_2h_3}\left\{\frac{1}{h_3}\frac{\partial V_3}{\partial x_3}+\frac{V_1}{h_3h_1}\frac{\partial h_3}{\partial x_1}+\frac{V_2}{h_2h_3}\frac{\partial h_3}{\partial x_2}\right\}\frac{\partial h_3}{\partial x_2}\\
&-\frac{2\mu}{h_2h_1}\left\{\frac{1}{h_1}\frac{\partial V_1}{\partial x_1}+\frac{V_2}{h_1h_2}\frac{\partial h_1}{\partial x_2}+\frac{V_3}{h_3h_1}\frac{\partial h_1}{\partial x_3}\right\}\frac{\partial h_1}{\partial x_2}
\end{aligned}
\tag{B.32}
$$

B.9.4 x_3-Momentum

$$
\begin{aligned}
&\rho\left[\frac{\partial V_3}{\partial t}+\frac{V_1}{h_1}\frac{\partial V_3}{x_1}+\frac{V_2}{h_2}\frac{\partial V_3}{x_2}+\frac{V_3}{h_3}\frac{\partial V_3}{x_3}-V_1\left(\frac{V_1}{h_1h_3}\frac{\partial h_1}{\partial x_3}-\frac{V_3}{h_3h_1}\frac{\partial h_3}{\partial x_1}\right)\right.\\
&\qquad\left.+V_2\left(\frac{V_3}{h_3h_2}\frac{\partial h_2}{\partial x_1}-\frac{V_2}{h_2h_3}\frac{\partial h_2}{\partial x_3}\right)\right]\\
&=f_3-\frac{1}{h_3}\frac{\partial p}{\partial x_3}+\frac{1}{h_3}\frac{\partial}{\partial x_3}\left(\kappa\boldsymbol{\nabla}\cdot\mathbf{V}\right)\\
&+\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left\{\mu h_2h_3\left[\frac{h_1}{h_3}\frac{\partial}{\partial x_3}\left(\frac{V_1}{h_1}\right)+\frac{h_3}{h_1}\frac{\partial}{\partial x_1}\left(\frac{V_3}{h_3}\right)\right]\right\}\right.\\
&\qquad+\frac{\partial}{\partial x_2}\left\{\mu h_3h_1\left[\frac{h_3}{h_2}\frac{\partial}{\partial x_2}\left(\frac{V_3}{h_3}\right)+\frac{h_2}{h_3}\frac{\partial}{\partial x_3}\left(\frac{V_2}{h_2}\right)\right]\right\}\\
&\quad\left.+\frac{\partial}{\partial x_3}\left\{2\mu h_1h_2\left(\frac{1}{h_3}\frac{\partial V_3}{\partial x_3}+\frac{V_1}{h_3h_1}\frac{\partial h_3}{\partial x_1}+\frac{V_2}{h_2h_3}\frac{\partial h_3}{\partial x_2}\right)\right\}\right]\\
&+\frac{\mu}{h_1h_3}\left\{\frac{h_1}{h_3}\frac{\partial}{\partial x_3}\left(\frac{V_1}{h_1}\right)+\frac{h_3}{h_1}\frac{\partial}{\partial x_1}\left(\frac{V_3}{h_3}\right)\right\}\frac{\partial h_3}{\partial x_1}\\
&+\frac{\mu}{h_2h_3}\left\{\frac{h_3}{h_2}\frac{\partial}{\partial x_2}\left(\frac{V_3}{h_3}\right)+\frac{h_2}{h_3}\frac{\partial}{\partial x_3}\left(\frac{V_2}{h_2}\right)\right\}\frac{\partial h_3}{\partial x_2}\\
&-\frac{2\mu}{h_3h_1}\left\{\frac{1}{h_1}\frac{\partial V_1}{\partial x_1}+\frac{V_2}{h_1h_2}\frac{\partial h_1}{\partial x_2}+\frac{V_3}{h_3h_1}\frac{\partial h_1}{\partial x_3}\right\}\frac{\partial h_1}{\partial x_3}\\
&-\frac{2\mu}{h_3h_2}\left\{\frac{1}{h_2}\frac{\partial V_2}{\partial x_2}+\frac{V_3}{h_2h_3}\frac{\partial h_2}{\partial x_3}+\frac{V_1}{h_1h_2}\frac{\partial h_2}{\partial x_1}\right\}\frac{\partial h_2}{\partial x_3}
\end{aligned}
\tag{B.33}
$$