La Ecuación radial $\psi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)$

$$\breve{\Omega}\, Y = \ell(\ell+1)\, Y$$

$$-\frac{r}{R}\frac{d^2}{dr^2}(rR) + \ell(\ell+1) = \frac{2mr^2}{\hbar^2}\left(E - V(r)\right) \qquad (1)$$

$\boxed{\ell = 0}$

$$\frac{d^2}{dr^2}(rR) = -\frac{2m}{\hbar^2}\left[E - V(r)\right](rR) \qquad (2)$$

→ Ec. de S. 1D para $u(r) = rR$

si $V(r) = 0 \Rightarrow \frac{d^2}{dr^2}(rR) = -\frac{2mE}{\hbar^2}(rR)$

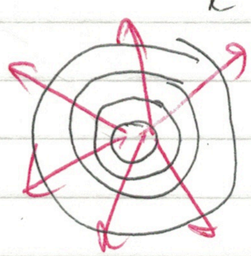
$$\Rightarrow rR = e^{\pm ikr}, \text{ donde } k = \sqrt{\frac{2mE}{\hbar^2}} \qquad (3)$$

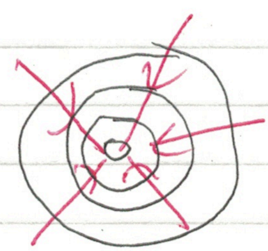
$$\Rightarrow \psi(r,\theta,\phi) \propto \frac{e^{\pm ikr}}{r}\, Y_{00}(\theta,\phi) \qquad (4)$$

→ cte.

incluyendo el tiempo:

$$\psi(r,t) \propto e^{i(kr-\omega t)},\ e^{-i(kr+\omega t)}$$





simetría esférica $\Rightarrow$ $l=0$
(no dep. de θ, ϕ)

$$V(r) = \frac{1}{2} m \omega^2 r^2 \qquad (5)$$

$$\frac{d^2}{dr^2}(rR) - \frac{2m}{\hbar^2}\left(\frac{1}{2} m\omega^2 r^2\right)(rR) = -\frac{2mE}{\hbar^2}(rR)$$

$$\frac{d^2}{dr^2}[rR] - \overset{a^2}{\frac{m^2\omega^2}{\hbar^2}} r^2 [rR] = -\overset{b}{\frac{2mE}{\hbar^2}}[rR] \qquad (6)$$

Menor eigenvalue: $\frac{\hbar\omega}{2}$?

con autofunción: $rR = e^{-ar^2/2}$? **NO!**

$$R(r) \xrightarrow[r\to 0]{} \text{finito} \Rightarrow \lim_{r\to 0} rR(r) = 0$$

$$u_1(x) = x e^{-ax^2/2} \longleftrightarrow rR(r) = r e^{-ar^2/2}$$

$a \equiv m^2\omega^2/\hbar^2$

$$\Rightarrow R(r) = e^{-ar^2/2}$$
con $E = \frac{3}{2}\hbar\omega$ ✓ } demostrar como ejercicio

Potencial centrífugo

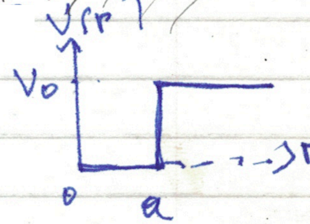
$$\frac{d^2}{dr^2}(rR) = -\frac{2m}{\hbar^2}\left(E - V(r) - \frac{\hbar^2 l(l+1)}{2mr^2}\right)(rR) \qquad (1)$$

$$V_{ef}(r) = V(r) + \frac{\hbar^2 l(l+1)}{2mr^2} \qquad (2)$$

← potencial centrífugo

Aplicación: pozo FINITO en 3D:

$$V(r) = \begin{cases} V_0 & r > a \\ 0 & r < a \end{cases}$$



caso $l = 0$

$r < a$: $\frac{d^2}{dr^2}(rR) = -\frac{2m}{\hbar^2} E \, (rR) = -k^2 (rR)$

$\Rightarrow rR = A\cos(kr) + B\sin(kr)$

$\Rightarrow R(r) = \frac{A}{r}\cos(kr) + \frac{B}{r}\sin(kr)$

$$\psi(r,\theta,\phi) = \frac{A}{r}\cos(kr) + \frac{B}{r}\sin(kr) \qquad (3)$$

como $\psi(0,\theta,\phi)$ debe ser finito $\Rightarrow A = 0$

$$\boxed{\psi(r,\theta,\phi) = \frac{B}{r}\sin(kr)} \quad (r < a) \qquad (4)$$

$\underline{r>a}$: $\frac{d^2}{dr^2}(rR) = -\frac{2m(E-V_0)}{\hbar^2}(rR) \equiv \alpha^2 (rR)$

Para un estado ligado $E<V_0$

$$\Rightarrow \boxed{rR = C e^{-\alpha r}} \quad (r>a) \qquad (5)$$

$$\boxed{\psi(r,\theta,\phi) = \frac{C}{r} e^{-\alpha r}} \quad (r>a) \qquad (6)$$

$\underline{r=a}$ $\quad B\sin(ka) = C e^{-\alpha a} \qquad (7)$

$(rR)_{a^-} = (rR)_{a^+}$

$(rR)'_{a^-} = (rR)'_{a^+}$: $\quad kB\cos(ka) = -\alpha C e^{-\alpha a} \qquad (8)$

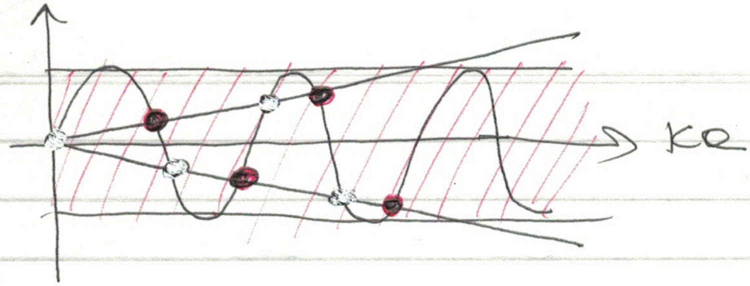
$$\Rightarrow \boxed{-\alpha = k\cot(ka)} \qquad (9)$$

y como $k = \sqrt{2mE/\hbar^2}$ y $\alpha = \sqrt{2m(V_0-E)/\hbar^2}$

$$\Rightarrow k^2 + \alpha^2 = \frac{2mV_0}{\hbar^2} \equiv \beta^2 \qquad (10)$$

(9) : $\rightarrow$ $\quad k^2 + k^2\cot^2(ka) = \beta^2$

$$\frac{k^2}{\sin^2(ka)} = \beta^2 \Rightarrow \boxed{\sin(ka) = \pm\frac{ka}{\beta a}} \qquad (11)$$



Junto con $\cot(Ka) < 0$ (de

pero que $\exists$ al menos 1 solución, $\frac{\pi}{2} < Ka < \pi$

pero $Ka < \beta a \Rightarrow$ si $\beta a < \pi/2$ NO $\exists$ B.S.

o sea $\frac{2mV_0a^2}{\hbar^2} < \left(\frac{\pi}{2}\right)^2$ NO $\exists$ Estado ligado

si $\frac{\pi}{2} < \beta a < \frac{3\pi}{2} \rightarrow$ 1 Estados ligado

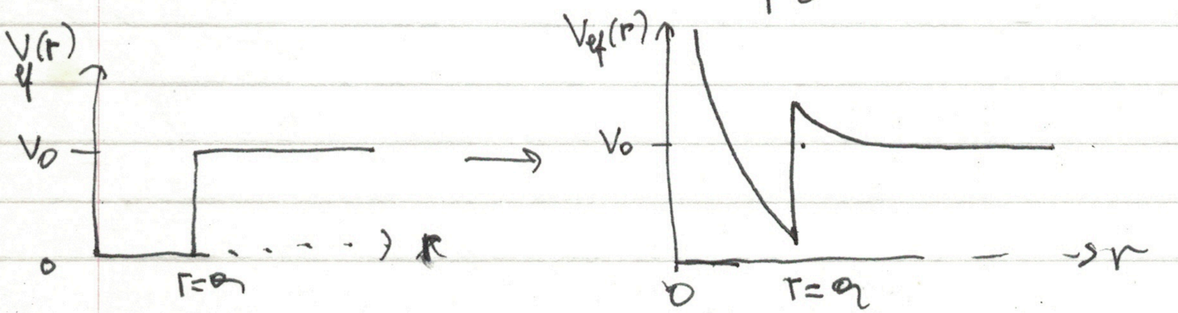
$\frac{3\pi}{2} < \beta a < \frac{5\pi}{2} \rightarrow$ 2 Estados ligados

⋮ etc.

Caso $l \neq 0$: la ec. radial se complica un poco

$$\frac{d^2}{dr^2}[rR] = -K^2[rR] + \frac{l(l+1)}{r^2}[rR] \qquad (r<a)$$

$$\frac{d^2}{dr^2}[rR] = \alpha^2[rR] + \frac{l(l+1)}{r^2}[rR] \qquad (r>a)$$



pot. se hace más angosto y menos profundo

Los soluciones para $r<a$, $r>a$ son las "fn. esféricas de Bessel" $\{j_\ell(kr), n_\ell(kr)\}$

$\to$ fn. esférica de Neumann

$$R(r) = j_0(kr) = \sin(kr)/kr$$

$$R(r) = j_1(kr) = \frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr}$$

$$j_2(kr) = \left[\frac{3}{(kr)^3} - \frac{1}{kr}\right]\sin(kr) - \frac{3\cos(kr)}{(kr)^2}$$

$$n_0(kr) = -\cos(kr)/kr$$

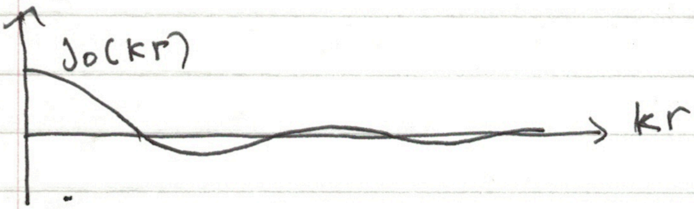
$$n_1(kr) = \frac{-\cos(kr)}{(kr)^2} - \frac{\sin(kr)}{kr}$$

$$n_2(kr) = \left[\frac{-3}{(kr)^3} + \frac{1}{kr}\right]\cos(kr) - \frac{3\sin(kr)}{(kr)^2}$$

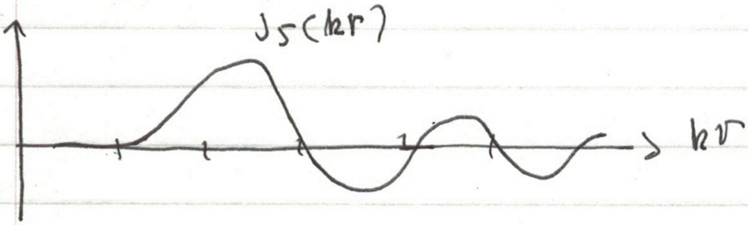
Ej: verificar que $j_0(kr)$ y $n_0(kr)$ son soluciones de la ec. radial para $r<a$ y $\ell=0$.

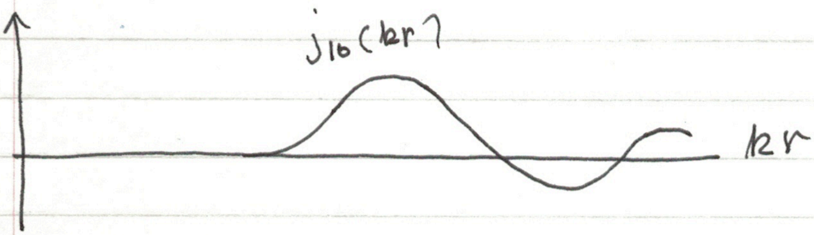
Ejercicio: Verifiquen que $j_2(kr)$ y $n_2(kr)$ son soluciones de

$$\frac{d^2}{dr^2}(rR) = -k^2(rR) + \frac{6\hbar^2}{2mr^2}(rR).$$



A medida que crece l, las j_l son "empujados" hacia kr mayores.





$$\underbrace{|\vec{L}|}_{\sqrt{\hbar^2 l(l+1)}\,\simeq\,\hbar l} = |\vec{r}\times\vec{p}| = |\vec{r}||\vec{p}|\sin\phi \le |\vec{r}||\vec{p}| = r\hbar k$$

$$\therefore \quad \hbar l \le \hbar r k \Rightarrow \boxed{kr \geq l}$$

$r<a$: La soluc. debe ser del tipo $j_l(kr)$, porque $n_l(kr)$ diverge en el origen.

construimos:

$$h_\ell^{(1)}(i\alpha r) = j_\ell(i\alpha r) + i\, n_\ell(i\alpha r)$$

"Función esférica de Henkel del 1er tipo"

$$h_1^{(1)}(i\alpha r) = i\left(\frac{1}{\alpha r} + \frac{1}{(\alpha r)^2}\right) e^{-\alpha r}$$

$$h_2^{(1)}(i\alpha r) = \left(\frac{1}{\alpha r} + \frac{3}{(\alpha r)^2} + \frac{3}{(\alpha r)^3}\right) e^{-\alpha r}$$

Por lo tanto, para $\ell = 1$:

$$R(r) = A\, j_1(kr) \qquad (r<a)$$
$$R(r) = B\, h_1^{(1)}(i\alpha r) \qquad (r>a)$$

cont. de R'/R en $r=a$ $\Rightarrow$ $\left(\dfrac{(d/dr)[j_1(kr)]}{j_1(kr)}\right)_{r=a} = \left(\dfrac{(d/dr)[h_1^{(1)}(i\alpha r)]}{h_1^{(1)}(i\alpha r)}\right)_{r=a}$

igual que $(rR)/(rR)'$

$$\Rightarrow \frac{\dfrac{d}{dr}\left[\dfrac{\sin(kr)}{(kr)^2} - \dfrac{\cos(kr)}{kr}\right]_{r=a}}{\dfrac{\sin(ka)}{(ka)^2} - \dfrac{\cos(ka)}{ka}} = \frac{\dfrac{d}{dr}\left[i e^{-\alpha r}\left(\dfrac{1}{\alpha r} + \dfrac{1}{(\alpha r)^2}\right)\right]_{r=a}}{i e^{-\alpha a}\left(\dfrac{1}{\alpha a} + \dfrac{1}{(\alpha a)^2}\right)}$$

$$\Rightarrow \frac{\frac{-2\sin(ka)}{k^2a^3} + \frac{2\cos(ka)}{ka^2} + \frac{\sin(ka)}{a}}{\frac{\sin(ka)}{(ka)^2} - \frac{\cos(ka)}{ka}} = \frac{-\frac{1}{a} - \frac{2}{\alpha a^2} - \frac{2}{\alpha^2 a^3}}{\frac{1}{\alpha a} + \frac{1}{(\alpha a)^2}}$$

lo cual puede simplificarse a (ejercicio):

$$\frac{\cot(ka)}{ka} - \left(\frac{1}{ka}\right)^2 = \frac{1}{\alpha a} + \left(\frac{1}{\alpha a}\right)^2 \quad (1)$$

De las defs. de k y α, tenemos

$$(ka)^2 + (\alpha a)^2 = \frac{2mV_0a^2}{\hbar^2} \quad (2)$$

(1) y (2) deben ser resueltas gráfica/. para hallar los niveles de energía para $\underline{l=1}$.

$ka \equiv x$; $\alpha a \equiv y$

$$\frac{\cot(x)}{x} - \frac{1}{x^2} = \frac{1}{y} + \frac{1}{y^2} \quad / x^2y^2 \quad y \quad x^2 + y^2 = R_0^2$$

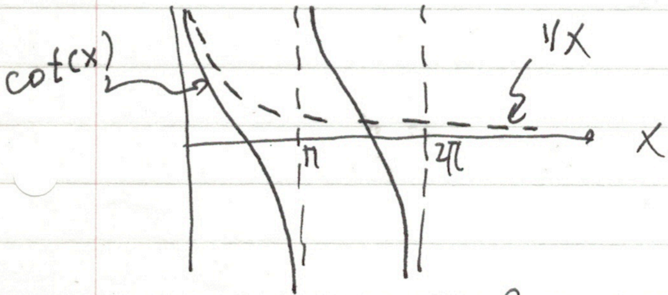
$$\to y^2x\cot(x) - y^2 = x^2y + x^2$$

$$y^2x\cot(x) - x^2y = x^2 + y^2 = R_0^2$$

$$(R_0^2 - x^2)x\cot(x) - x^2\sqrt{R_0^2 - x^2} = R_0^2$$

Ej: Dem. que no $\exists$ estado ligado para $l=1$ cuando $V_0 < \frac{\pi^2\hbar^2}{2ma^2}$.

Sol: para $ka < \pi$, $\cot(ka) < \frac{1}{ka}$ $\Rightarrow$ LHS de (1) es Negativo



y como el RHS de (1) es positivo $\Rightarrow$ No $\exists$ sol. si $ka < \pi$.

De (2) $\Rightarrow$ $\frac{2mV_0a^2}{\hbar^2} \geq (ka)^2$

por lo tanto, cuando $ka \geq \pi$ $\Rightarrow$ $\frac{2mV_0a^2}{\hbar^2} \geq \pi^2$

y o sea $\exists$ al menos 1 sol.

$\Rightarrow V_0 \geq \frac{\pi^2\hbar^2}{2ma^2}$ //

Nota: Es posible tener una solución en $ka = \pi$

$\Rightarrow \cot(ka) \to \infty \Rightarrow \alpha a \to 0$

$\Rightarrow \frac{2mV_0a^2}{\hbar^2} = \pi^2 \Rightarrow V_0 = \frac{\pi^2\hbar^2}{2ma^2}$

otra solución aparece cuando $ka = 2\pi$

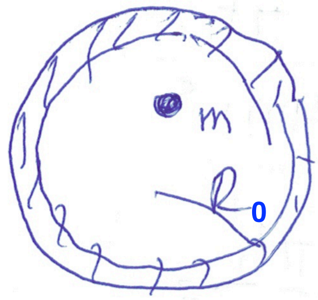
$\Rightarrow V_0 = \frac{4\pi^2\hbar^2}{2ma^2}$.

pozo
infinito
esferico

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{2mr^2}{\hbar^2}E=0$$

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\underbrace{\left(\frac{2mE}{\hbar^2}\right)}_{k^2}R=0$$

$$\boxed{R''+\frac{2}{r}R'+k^2R=0}$$



$$R=\frac{U}{r} \Rightarrow R'=\frac{U'}{r}-\frac{U}{r^2}$$

$$\Rightarrow R''=\frac{U''}{r}-\frac{U'}{r^2}-\frac{U'}{r^2}+\frac{2U}{r^3}$$

$$\frac{U''}{r}-\frac{2U'}{r^2}+\frac{2}{r^3}U+\frac{2}{r}\left[\frac{U'}{r}-\frac{U}{r^2}\right]+k^2R=0$$

$$\frac{U''}{r}-\cancel{\frac{2U'}{r^2}}+\cancel{\frac{2U}{r^3}}+\cancel{\frac{2U'}{r^2}}-\cancel{\frac{2U}{r^3}}+k^2R=0$$

$$\frac{U''}{r}+k^2\frac{U}{r} \Rightarrow \boxed{U''+k^2U=0}$$

$$U=A\sin(kr)+B\cos(kr)$$

$$k^2=\frac{n^2\pi^2}{R_0^2}=\frac{2mE}{\hbar^2}$$

$$R(r)=\frac{A}{r}\sin(kr)+\frac{B}{r}\cos(kr)$$

$$\Rightarrow \boxed{E=\frac{\pi^2\hbar^2}{2mR_0^2}\cdot n^2}$$

$R(0)=$ finito $\Rightarrow B=0$

$R(R_0)=0 \Rightarrow \sin(kR_0)=0 \Rightarrow kR_0=n\pi$

$k=\frac{n\pi}{R_0}$

$$\Rightarrow \boxed{R(r)=\frac{A}{r}\sin\left(\frac{n\pi}{R_0}r\right)}$$

$$P = \frac{\partial E}{\partial V} = \frac{\partial E}{\partial r} \cdot \frac{\partial r}{\partial V} = \frac{\partial E/\partial r}{\partial V/\partial r}$$

presion interna

$$V = \frac{4\pi}{3} R^3 \Rightarrow \partial V/\partial r = 4\pi R^2$$

$$\Rightarrow P = \frac{1}{4\pi r2} \frac{\partial E}{\partial r} = \frac{1}{4\pi r2} \cdot \frac{\partial}{\partial r}\left[\frac{\pi^2 \hbar^2}{2mr^2}\right] = \frac{1}{4\pi r^2} \cdot \frac{\pi^2 \hbar^2}{2m} \cdot \frac{(-2)}{r^3}$$

$$= \frac{\pi \hbar^2}{4mr^5}$$

$$\boxed{P = \frac{\pi \hbar^2}{4mr^5}}$$