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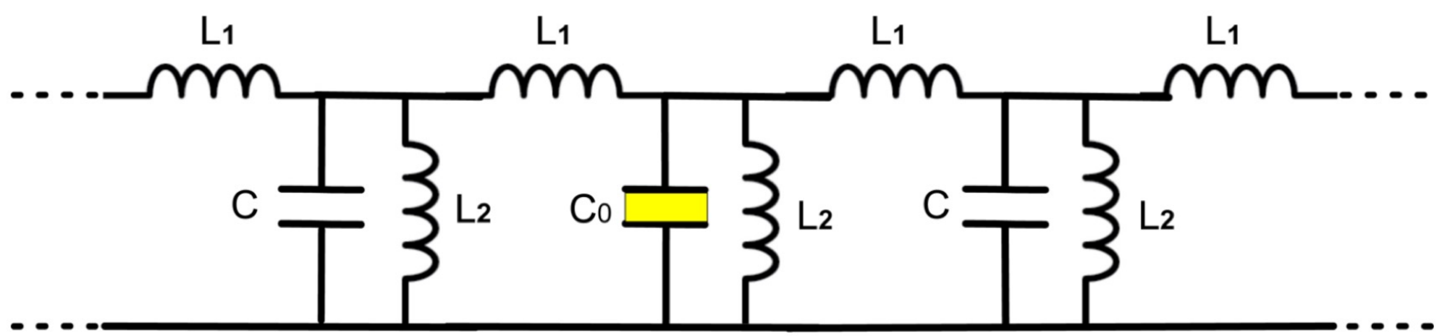
Linear impurity modes in an electrical lattice: Theory and experiment



$$\frac{d^2Q_n}{dt^2} = \frac{1}{L_1}(V_{n+1} + V_{n-1} - 2V_n) - \frac{1}{L_2}V_n.$$

The impurity at the center of our current considerations is assumed to be a capacitive one located at $n = 0$, with capacitance C_0. The rest of the capacitances are taken as identical and equal to C. In other words, $C_n = C + (C_0 - C)\delta_{n,0}$.

By taking $Q_n = C_n V_n$ and $V_n(t) \sim \cos(\Omega t + \phi_n)$, we arrive at the stationary equations, i.e., the eigenvalue problem:

$$\Omega^2 U_n = -\omega_1^2 (U_{n+1} + U_{n-1} - 2U_n) + \omega_2^2 U_n - (\Delta C/C)\, \Omega^2 \delta_{n,0}\, U_n, \tag{2}$$

$$z\, U_n = \gamma\, (U_{n+1} + U_{n-1}) + \epsilon\, \delta_{n,0}\, U_n \tag{3}$$

with

$$\begin{aligned} z &\equiv \Omega^2 - 2\omega_1^2 - \omega_2^2 \\ \gamma &\equiv -\omega_1^2 \\ \epsilon &= -(\Delta C/C)\Omega^2, \end{aligned}$$

where $\omega_1^2 = 1/(L_1 C)$, and $\omega_2^2 = 1/(L_2 C)$.

Equation (3) describes formally a single impurity tight-binding model, whose Hamiltonian is given by

$$H = H_0 + H_d. \tag{4}$$

Using Dirac's notation, we can express H_0 and H_d in terms of Wannier functions $\{|n\rangle\}$ as

$$H_0 = \gamma \sum_n (|n+1\rangle\langle n| + |n\rangle\langle n+1|), \tag{5}$$

the (undisturbed) lattice Hamiltonian, and

$$H_d = \epsilon\, |0\rangle\langle 0|, \tag{6}$$

$$\langle n|k\rangle = (1/\sqrt{N})\exp(ikn), \tag{7}$$

and eigenvalues,

$$z_k = 2\gamma\cos(k), \tag{8}$$

or, in terms of the electrical lattice parameters,

$$\Omega^2 = 4\,\omega_1^2\sin^2(k/2) + \omega_2^2. \tag{9}$$

Thus, the system is able to support the propagation of electrical waves that form a band extending (in terms of Ω^2) from ω_2^2 to $\omega_2^2 + 4\omega_1^2$. Notice that in the infinite lattice limit this band would be a continuous spectrum, while in the finite lattice case, the boundary conditions select a specific set of wave numbers k (e.g., $k_m = m\pi/N$ with m integer running up to N for periodic boundary conditions), and thus only the corresponding frequencies are observed.

Given a Hamiltonian H, the lattice Green's function is defined as [22–24]

$$G(z) = 1/(z - H). \tag{10}$$

In our case, $H = H_0 + H_d$. Treating H_d formally as a perturbation, we can express $G(z)$ as

$$G(z) = G^{(0)} + G^{(0)}\, H_d\, G^{(0)} + G^{(0)}\, H_d\, G^{(0)}\, H_d\, G^{(0)} + \cdots\,, \tag{11}$$

where $G^{(0)} = 1/(z - H_0)$ is the unperturbed Green function. Inserting $H_d = \epsilon|0\rangle\langle 0|$, we have

$$\begin{aligned} G(z) &= G^{(0)} + \epsilon\, G^{(0)}|0\rangle\langle 0|G^{(0)} \\ &\quad + \epsilon^2 G^{(0)}|0\rangle\langle 0|G^{(0)}|0\rangle\langle 0|G^{(0)} + \cdots \\ &= G^{(0)} + \epsilon\, G^{(0)}|0\rangle\left[\sum_{n=0}^{\infty}\epsilon^n G_{00}^{(0)n}\right]\langle 0|G^{(0)} \\ &= G^{(0)} + \frac{G^{(0)}|0\rangle\, \epsilon\, \langle 0|G^{(0)}}{1 - \epsilon\, G_{00}^{(0)}}. \end{aligned}$$

A. Electrical impurity in the bulk

Let us consider a capacitive defect that is located far from the boundaries of the system. In that case we have from Ref. [22] that:

$$G_{00}^{(0)}(z) = \frac{\mathrm{sgn}(z)}{\sqrt{z^2 - (2\gamma)^2}}. \tag{12}$$

After solving the energy equation $1/\epsilon = G_{00}^{(0)}(z_b)$, one obtains

$$z_b = \pm\sqrt{\epsilon^2 + (2\gamma)^2}. \tag{13}$$

In terms of our electrical parameters, this leads to

$$\Omega^2 = \frac{-\left(2\omega_1^2 + \omega_2^2\right) \pm \sqrt{4\omega_1^4 + (\delta - 1)^2\omega_2^2\left(4\omega_1^2 + \omega_2^2\right)}}{\delta(\delta - 2)}, \tag{14}$$

where $\delta = C_0/C$ is the capacitive mismatch. However, it by the residue of $G(z)$ at $z = z_b$,

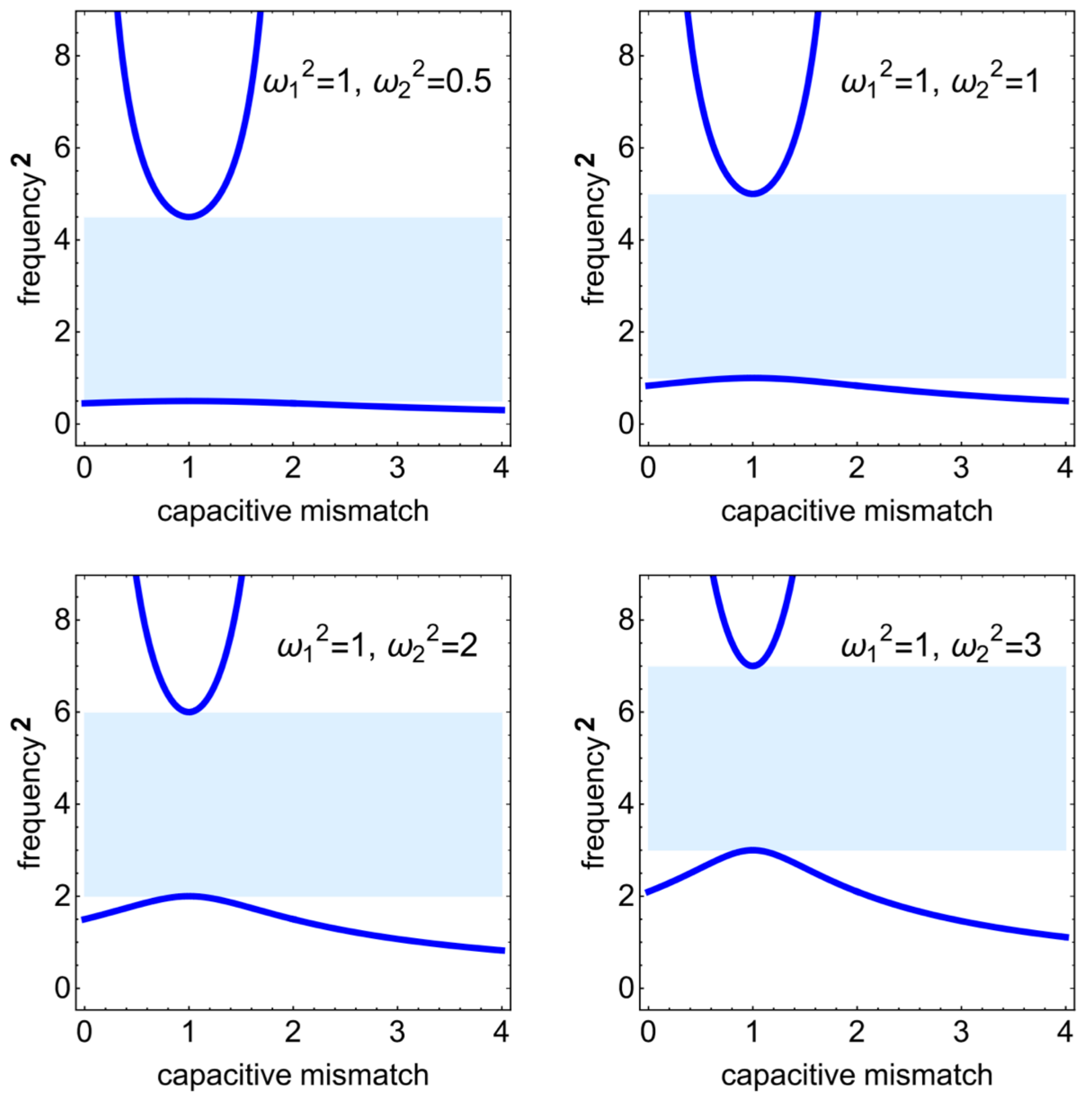
$$b_n = \frac{G_{n0}(z_b)}{\sqrt{-G'_{00}(z_b)}}, \tag{15}$$

where

$$G_{n0}(z) = \frac{\mathrm{sgn}(z)}{\sqrt{z^2 - (2\gamma)^2}} \left\{ -\left(\frac{z}{2|\gamma|}\right) + \mathrm{sgn}(z)\sqrt{\left(\frac{z}{2\gamma}\right)^2 - 1} \right\}^{|n|}. \tag{16}$$

We obtain:

$$b_n = \mathrm{sgn}(z_b)\frac{\left[z_b^2 - (2\gamma)^2\right]^{1/4}}{|z_b|^{1/2}} \left\{ -\left(\frac{z_b}{2|\gamma|}\right) + \mathrm{sgn}(z_b)\sqrt{\left(\frac{z_b}{2\gamma}\right)^2 - 1} \right\}^{|n|}, \tag{17}$$


$\omega_1{}^2$=1, $\omega_2{}^2$=0.5
$\omega_1{}^2$=1, $\omega_2{}^2$=1
$\omega_1{}^2$=1, $\omega_2{}^2$=2
$\omega_1{}^2$=1, $\omega_2{}^2$=3
frequency^2
capacitive mismatch
0
2
4
6
8
0
1
2
3
4

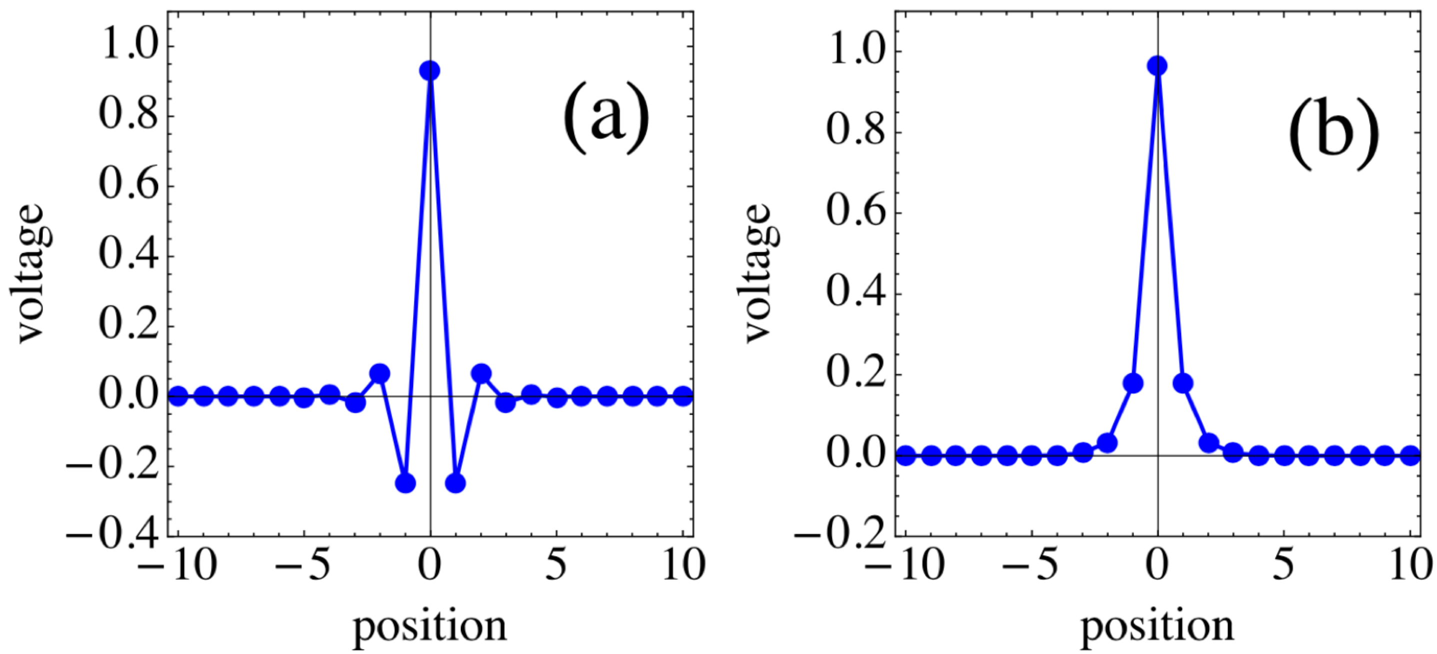
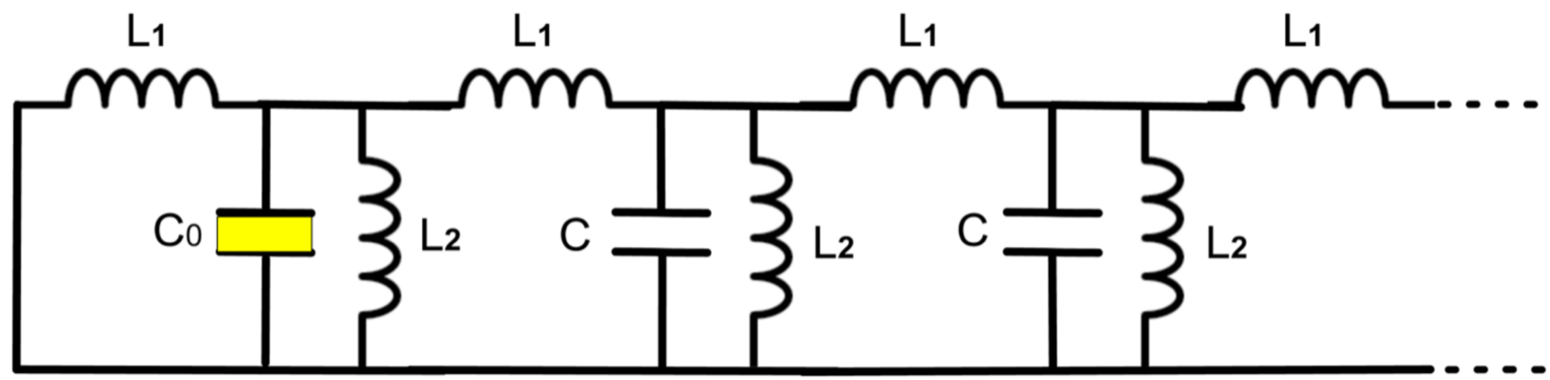
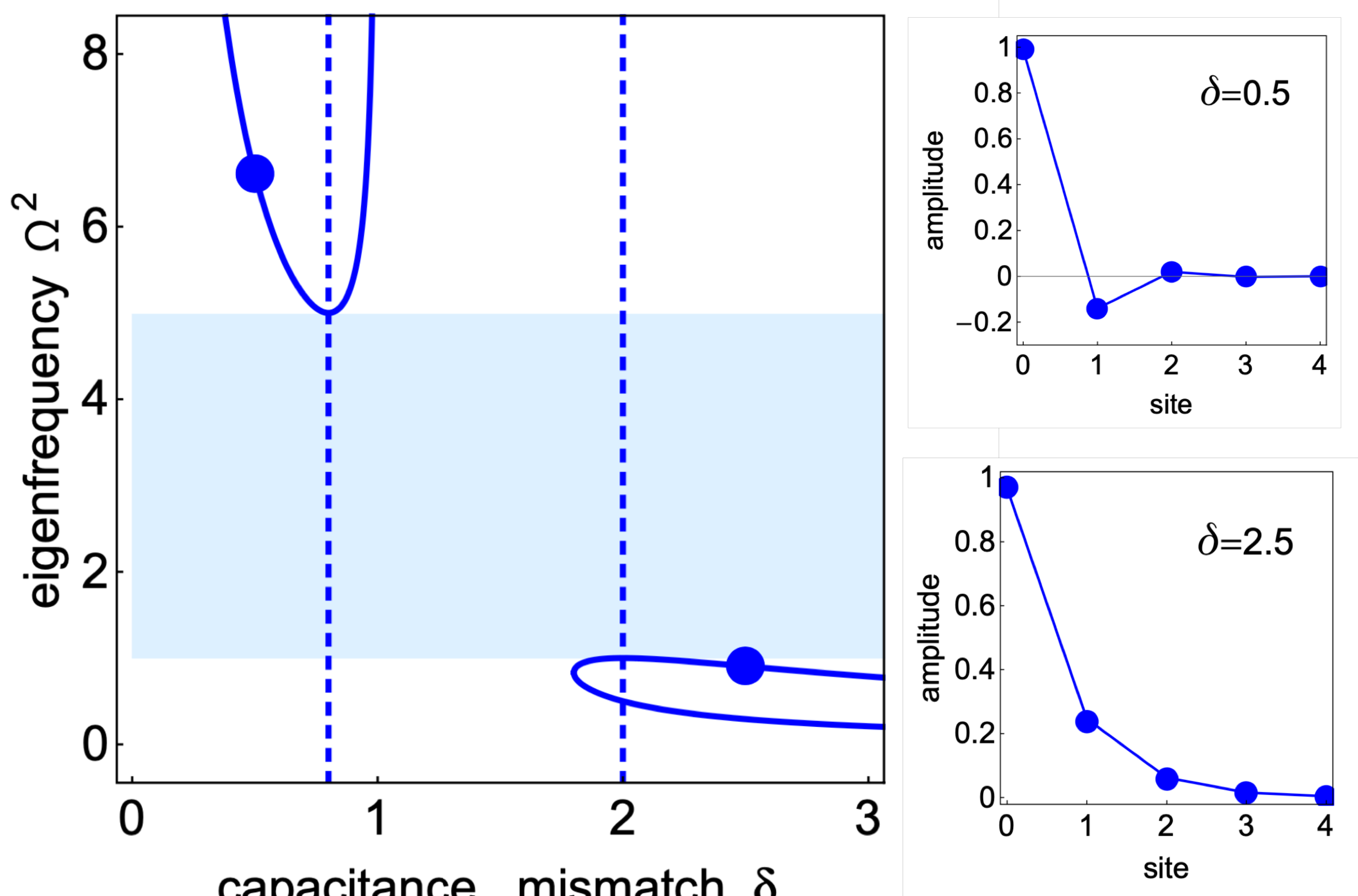


FIG. 3. Spatial profiles of the localized electrical bulk modes for $\omega_1 = 1 = \omega_2$; (a) $\delta = 0.5$ (upper branch) and (b) $\delta = 3$ (lower branch).

Electrical impurity at the boundary

Now we consider the case where the defect is placed at the very surface of a semi-infinite electrical array (Fig. 4). Computation of the proper Green's function for this case requires realizing that, even though the main formalism is still the same as in Sec. III A, the unperturbed Green's function $G^{(0)}_{mn}$ needs to be computed by the method of mirror images: since there is no lattice to the left of $n = 0$, $G^{(0)}_{mn}$ must vanish identically at $n = -1$. This means $G^{(0)}_{mn}(z) = G^{\infty}_{mn}(z) - G^{\infty}_{m,-n-2}(z)$, where $G^{\infty}_{mn}(z)$ is the unperturbed Green's function of the bulk case: $G^{\infty}_{mn}(z) = \mathrm{sgn}(z)(1/\sqrt{z^2 - 1})[z - \mathrm{sgn}(z)\sqrt{z^2 - 1}]^{|n-m|}$.

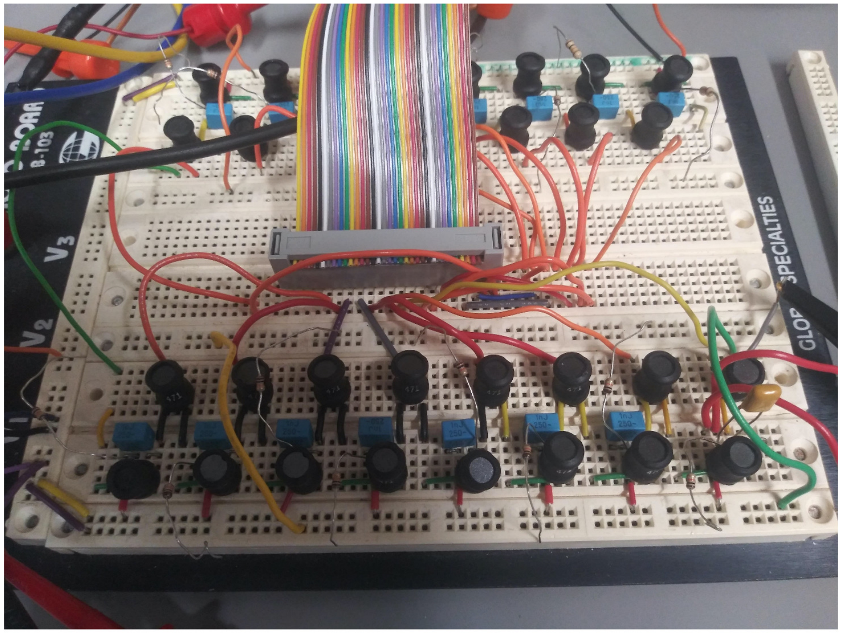

L1
L1
L1
L1
C0
L2
C
L2
C
L2

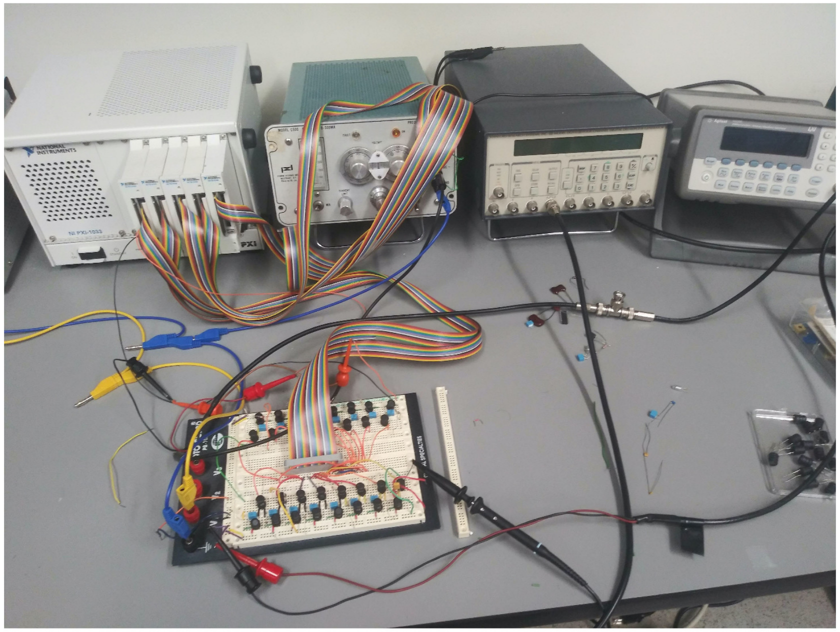


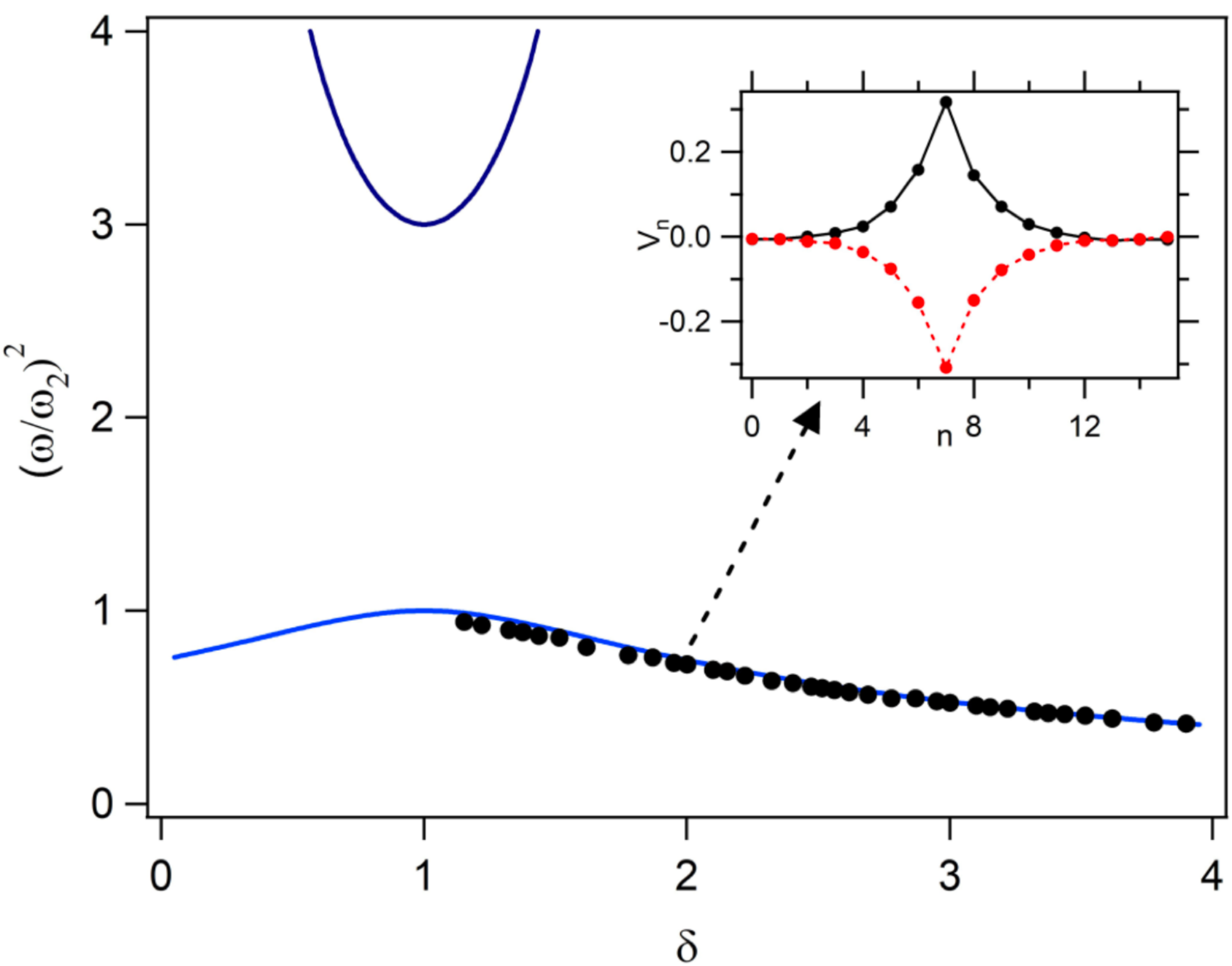
eigenfrequency Ω^2
capacitance mismatch δ
8
6
4
2
0
0
1
2
3
amplitude
site
δ=0.5
1
0.8
0.6
0.4
0.2
0
−0.2
0
1
2
3
4
amplitude
site
δ=2.5
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4

EL EXPERIMENTO

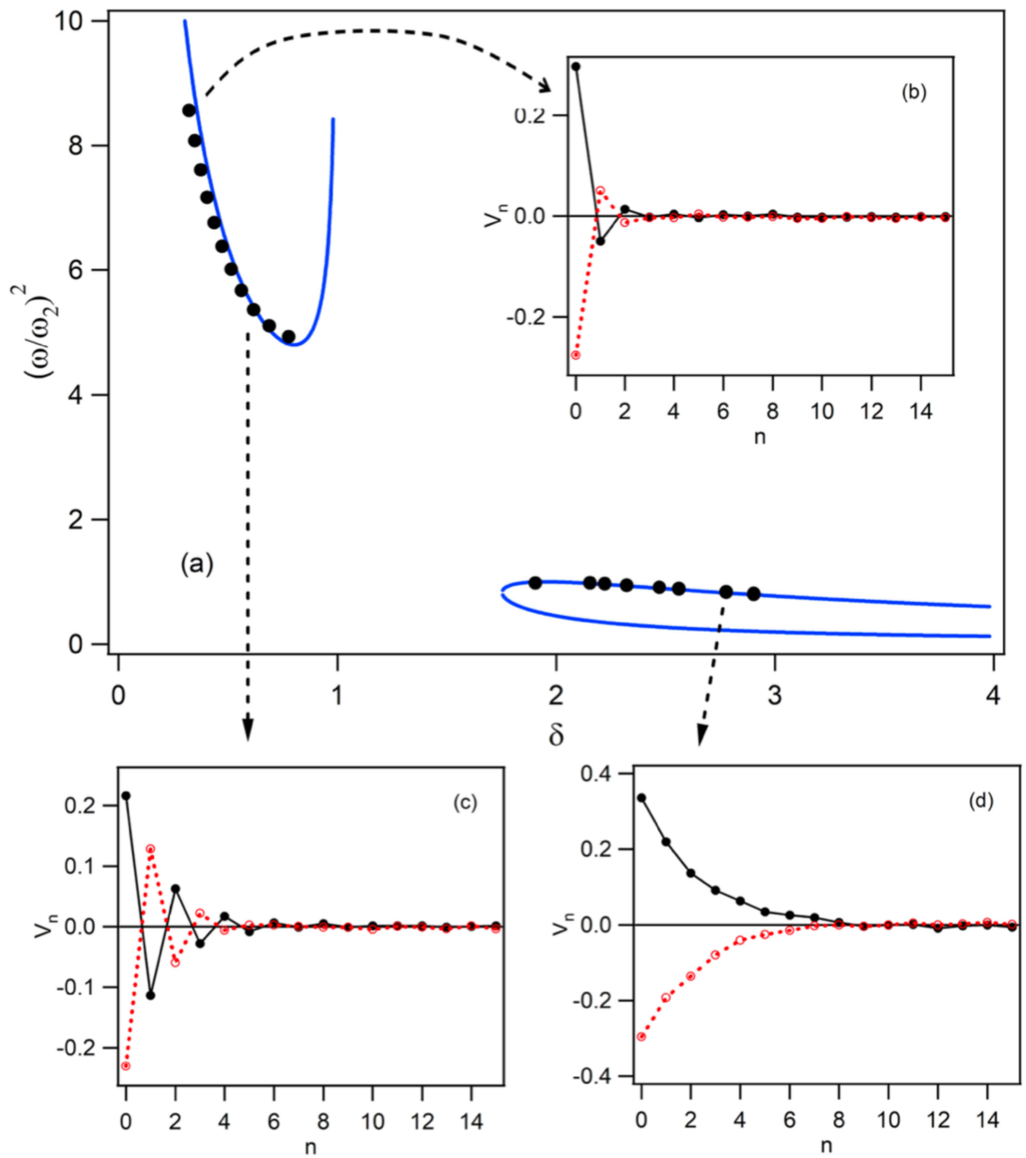
The experiment







$(\omega/\omega_2)^2$
δ
0
1
2
3
4
V_n
n
0.2
0.0
-0.2
0
4
8
12


(a)
(b)
(c)
(d)
$(\omega/\omega_2)^2$
δ
V_n
n