Presentación

Ayudante: Camilo Andrés Lagos Vasquez

Problema 1

Dibuje el diagrama de fase correspondiente a

$$\boxed{\underline{Y}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underline{Y}}$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1 \end{aligned}}$$

$$e^{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

$$\underline{Y = \gamma e^{At}}$$

Paso 1: Exponenciar

$$\exp\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n$$

Notemos que $A^2 = I$, por lo que obtenemos

$$\exp\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} & \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \\ \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} & \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$$

$$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$A^0 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} t^0$$

$$A^1 = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} t^1$$

$$A^2 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} t^2$$

$$A^3 = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \qquad t^5$$

$$1 + \frac{t^2}{2!} + \frac{t^4}{4!}$$

$$\left|\begin{array}{l} \sinh' = \cosh \\ \cosh' = \sinh \end{array}\right|$$

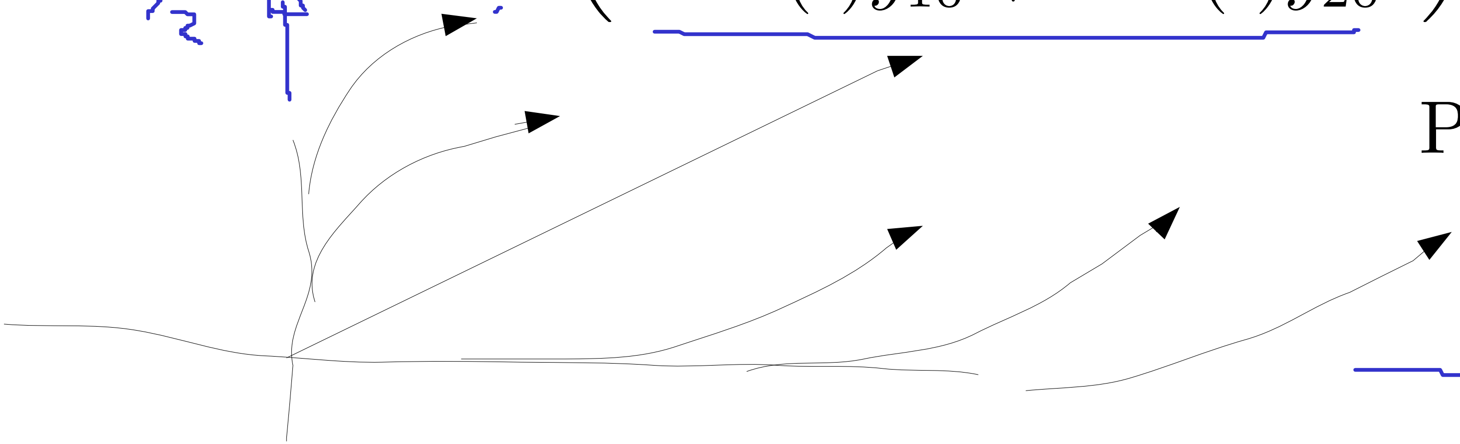
Paso 2: Reescribir

$$Y = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} \exp \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$$

$$\longrightarrow Y = \begin{pmatrix} \cosh(t)y_{10} + \sinh(t)y_{20} \\ \sinh(t)y_{10} + \cosh(t)y_{20} \end{pmatrix}$$

$$\frac{\sinh(x)}{\cosh(x)} \xrightarrow{x \to \infty} 1$$

Y_2



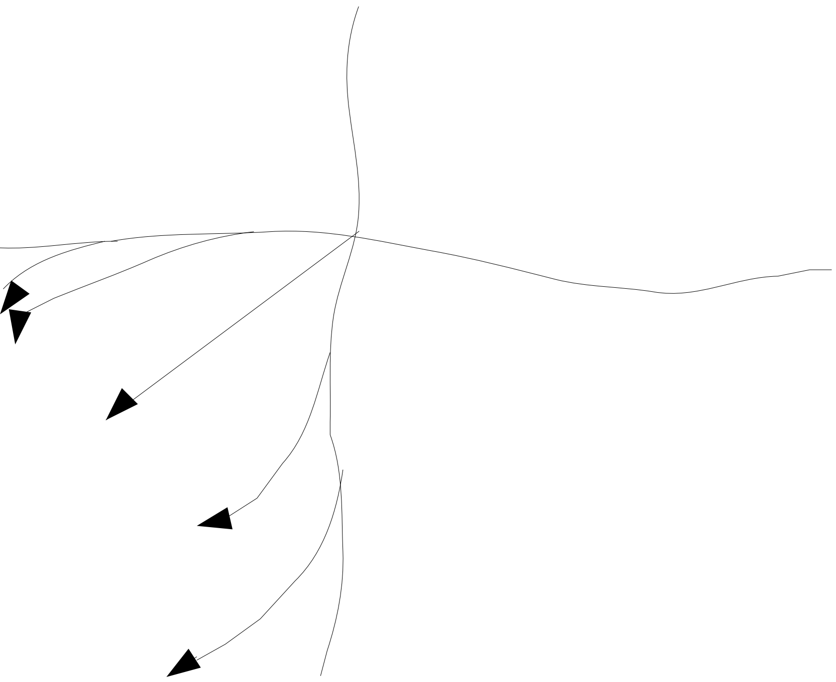
Primer cuadrante

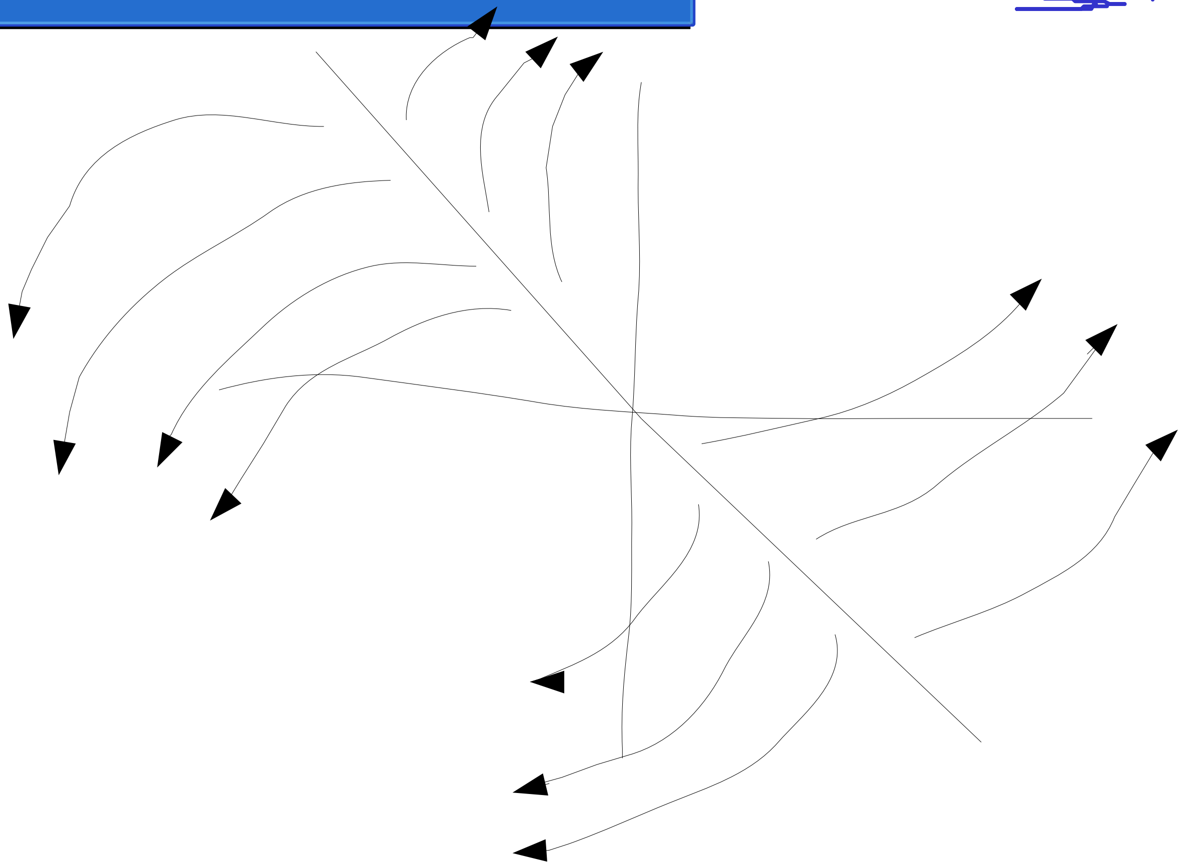
Y_1

$x \begin{pmatrix} A \\ A \end{pmatrix}$

Paso 3: Dibujar

$|\lambda_2| > |\lambda_1|$





Preguntas

Problema 2

Sea $A \in M(5)$ cuyo polinomio característico es $p(\lambda) = (\lambda - 1)^3(\lambda - 2 - i)(\lambda - 2 + i)$. Suponga que $\dim Ker(I - A) = 2$. Hallar la solución de $x' = Ax$ con condición inicial $x(0) = x_0$.

$$\lambda_1 = 1$$
$$\lambda_2 = 2 + i$$
$$\lambda_3 = 2 - i$$

$\mathrm{Ker}(I - A) = 2$

$J =$

Paso 1: Escribir forma de jordan

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Paso 2: Calcular exponenciales

$$A = QJQ^{-1}$$

$$\exp(At) = \exp(QJtQ^{-1}) = Q\exp(Jt)Q^{-1}$$

$$\exp\begin{pmatrix} 2t & t \\ -t & 2t \end{pmatrix} = e^{2t}\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

$$\exp\begin{pmatrix} t & t \\ 0 & t \end{pmatrix} = e^{t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$(aBa^{-1})^2 = aB\cancel{a^{-1}} \cdot \cancel{a}Ba^{-1}$$
$$= aB^2a^{-1}$$

$$\begin{pmatrix} 2t & t \\ -t & 2t \end{pmatrix} = \begin{pmatrix} 2t & 0 \\ 0 & 2t \end{pmatrix} + \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}$$

$$\exp\begin{pmatrix} 2t & t \\ -t & 2t \end{pmatrix} = \exp\begin{pmatrix} 2t & 0 \\ 0 & 2t \end{pmatrix} \cdot \exp\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} e^{2t}$$

$$B^0 = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \qquad B^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = B \qquad B^1 = \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}$$

$$B^2 = \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix}$$

$$\begin{pmatrix} t & t \\ 0 & t \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$$

$$\exp\begin{pmatrix} t & t \\ 0 & t \end{pmatrix} = \exp\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \exp\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$$

$$= e^{t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left[I + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} + 0 + 0 + \cdots \right]$$

$$= e^{t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$= e^{t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Paso 3: Escribir solución

$$A = e^t Q \begin{pmatrix} 1 & t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^t\cos(t) & e^t\sin(t) \\ 0 & 0 & 0 & -e^t\sin(t) & e^t\cos(t) \end{pmatrix} Q^{-1}$$

Extra: Sistema que representa

Si $Q = I$, podemos escribir el sistema de ecuaciones lineales

$$
\begin{aligned}
x_1' &= x_1 + x_2 \\
x_2' &= x_2 \\
x_3' &= x_3 \\
x_4' &= 2x_4 + x_5 \\
x_5' &= 2x_5 - x_4
\end{aligned}
$$

1 1

1

1

$$e^{x} = \sum_{0}^{\infty} \frac{x^{n}}{n!}$$

$$\boxed{e^{At} = \sum_{0}^{\infty} \frac{\textcircled{Q} A^{n} t^{n}}{n!} Q^{-1}} \qquad Q \boxed{\sum \frac{D^{n} t^{n}}{n!}} Q^{-1}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = - \underline{\underline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{0} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^{1} = \begin{pmatrix} \underline{0} & \underline{1} \\ -1 & 0 \end{pmatrix} \qquad \underline{0 + t + 0 - \frac{t^{3}}{3!} + \frac{t^{5}}{5!}}$$

$$A^{2} = \begin{pmatrix} \underline{-1} & \underline{0} \\ -0 & -1 \end{pmatrix}$$

$$A^{3} = \begin{pmatrix} 0 & \underline{-1} \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} C & S \\ -S & C \end{pmatrix}$$

$$A^{4} = \begin{pmatrix} 1 & \underline{0} \\ 0 & 1 \end{pmatrix} \qquad 1 + 0 - \frac{t^{2}}{2!} + 0 + \frac{t^{4}}{4!} + 0$$

$$A^{5} = \begin{pmatrix} 0 & \underline{1} \\ -1 & 0 \end{pmatrix}$$

Dudas

Problema 3

Resuelva la edo de segundo orden

$$x'' = ax' + \frac{3}{4}a^2x$$

$z' = az +$

$$\cancel{x' = z}$$

$$\cancel{z' = [illegible]z + 10x}$$

Paso 1: Reescribir

Sea $\boxed{x' = z} \rightarrow x'' = z'$ Reemplazandolo en la edo original

$$\left.\begin{aligned} x' &= z \\ z' &= az + \frac{3}{4}a^2 x \end{aligned}\right\}$$

Sistema de ecuaciones diferenciales

$$\begin{pmatrix} z \\ x \end{pmatrix}' = \boxed{\begin{pmatrix} a & \frac{3}{4}a^2 \\ 1 & 0 \end{pmatrix}} \begin{pmatrix} z \\ x \end{pmatrix} \leftrightarrow \text{matriz equivalente}$$

Paso 2: Calcular valores propios

$$\det\begin{pmatrix} a-\lambda & \frac{3}{4}a^2 \\ 1 & -\lambda \end{pmatrix} = 0 \longrightarrow \lambda^2 - a\lambda - \frac{3}{4}a^2 = 0$$

$$\lambda = \frac{a \pm 2a}{2}$$

$$\boxed{\lambda_1 = \frac{3}{2}a \qquad \lambda_2 = -\frac{1}{2}a}$$

Calculando vectores propios

λ_1

$$v_1 = k v_2$$

$$\begin{pmatrix} -\frac{1}{2}a & \frac{3}{4}a^2 \\ 1 & -\frac{3}{2}a \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

λ_2

$$\begin{pmatrix} \frac{3}{2}a & \frac{3}{4}a^2 \\ 1 & \frac{1}{2}a \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Paso 3: Escribir diagonalizacion

vector propio λ_1 vector propio λ_2

$$\begin{pmatrix} \frac{3}{2}a \\ 1 \end{pmatrix} \qquad \begin{pmatrix} -\frac{1}{2}a \\ 1 \end{pmatrix}$$

$$\boxed{Q = \begin{pmatrix} \frac{3}{2}a & -\frac{1}{2}a \\ 1 & 1 \end{pmatrix}} \longrightarrow A = Q \begin{pmatrix} \frac{3}{2}a & 0 \\ 0 & -\frac{1}{2}a \end{pmatrix} Q^{-1}$$

$$\begin{pmatrix} \vec{z} \\ x_0 \end{pmatrix} \xrightarrow{At} = \begin{pmatrix} \vec{z} \\ x_0 \end{pmatrix} \boxed{e^{QDtQ^{-1}}} = \begin{pmatrix} \vec{z} \\ x \end{pmatrix}$$

Paso 4: Escribir solución

$$\left(QDQ^{-1}\right)^{n} = QD^{n}Q^{-1}$$

$$\begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} z_0 \\ x_0 \end{pmatrix} \exp\begin{pmatrix} at & \frac{3}{4}a^2t \\ 1t & 0 \end{pmatrix}$$

$$= \underline{\begin{pmatrix} z_0 \\ x_0 \end{pmatrix}} Q \begin{pmatrix} \underline{\exp(\frac{3}{2}at)} & 0 \\ 0 & \underline{\exp(-\frac{1}{2}at)} \end{pmatrix} Q^{-1}$$

$$Y' = A\cancel{\#}$$

$$\boxed{A}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Preguntas

Ejercicios propuestos

$$A = B + C \qquad BC = CB$$

$$\exp(A) = \exp(B)\exp(C)$$