Caos

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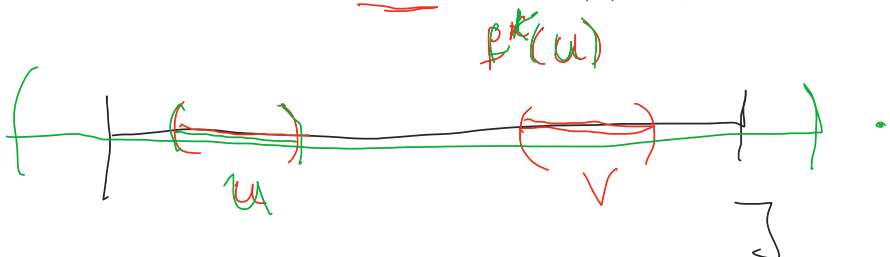
$S^1, \mathbb{R} \supset J$

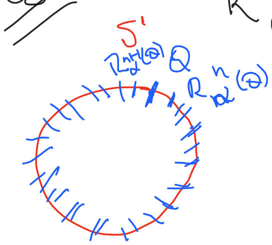
Definition 8.1. $f: J \to J$ is said to be topologically transitive if for any pair of open sets $U, V \subset J$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.



obs: f es topológicamente transitivo si $\exists\, x \in J$ tq $\overline{O_f(x)} = J$

Ej: $R_\alpha : S^1 \to S^1$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$

$\theta \to R_\alpha(\theta) = \theta + \alpha 2\pi$



Aquí $\forall x \in S^1$ $\overline{O_{R_\alpha}(x)} = S^1$

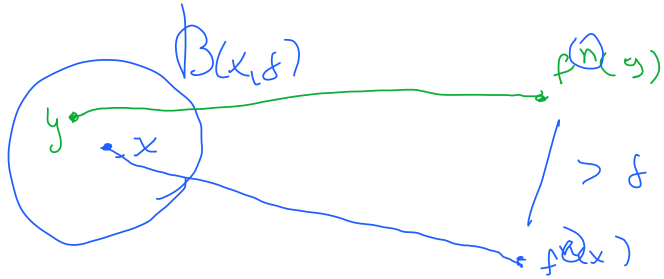
$\therefore$ R_α es topológicamente transitiva.

Definition 8.2. $f: J \to J$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in J$ and any neighborhood N of x, there exists $y \in N$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.

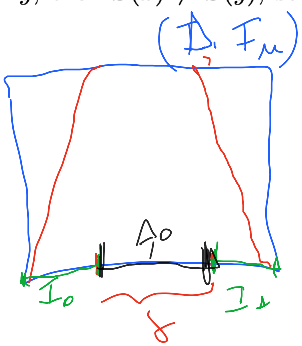
f es s.c/R c.i $\exists\, \delta$ $\forall\, x \in J$

$y\ \forall \varepsilon > 0\ \exists\ y \in B(x, \varepsilon)$

$y\ n \geq 0$ tq $|f^n(x) - f^n(y)| > \delta$



Example 8.3. The quadratic map $\mu x(1-x)$ with $\mu > 2+\sqrt{5}$ possesses sensitive dependence on initial conditions on Λ. To see this, choose δ less than the diameter of A_0, where A_0 is the gap between I_0 and I_1. Let $x, y \in \Lambda$. If $x \neq y$, then $S(x) \neq S(y)$, so the itineraries of x and y must differ in at



$\xrightarrow{S} (\Sigma_2, \sigma)$

Conjugación topológica entre (Λ, F_μ) y (Σ_2, σ)

$\Rightarrow$ Probaremos que (Σ_2, σ) es sensible c/r a c.i.

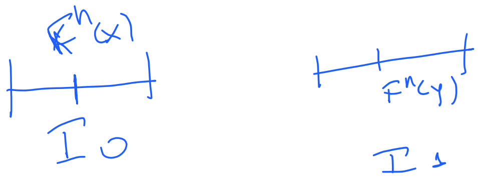
$\exists\ \delta = \mathrm{diam}(A_0) = d_{\#}(I_0, I_1)$

$\forall\ x, y \in \Lambda \Rightarrow \delta(x) \neq \delta(y) = (x_i)_{i\geq 0}, (y_i)_{i\geq 0}$
$x \neq y$

$\Rightarrow \exists\ n$ t.q.

$x_n \neq y_n$

$\therefore F_\mu,\ \mu > 2+\sqrt{5}$
es sensible C/R
a C.I.

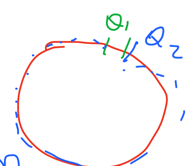


$|F^n(x) - F^n(y)| > \delta$

Example 8.4. An irrational rotation of the circle is topologically transitive but not sensitive to initial conditions, since all points remain the same distance apart after iteration.

$R_\alpha : S^1 \to S^1$
$\theta \to R_\alpha(\theta) = \theta + \alpha 2\pi\ ,\ \alpha \in \mathbb{R}\setminus\mathbb{Q}$

R_α Top transitiva



$\forall \theta_1, \theta_2$ $\quad R_\alpha^n(\theta_1) = \theta_1 + \alpha 2\pi \cdot n$
$R_\alpha(\theta_2) = \theta_2 + \alpha 2\pi \cdot n$

Pero $\quad d(\theta_1, \theta_2) = d(R_\alpha^n \theta_1, R_\alpha^n(\theta_2))$

$\Rightarrow R_\alpha$ No es sensible C/R
a C.I.

Definition 8.5. Let V be a set. $f: V \to V$ is said to be chaotic on V if

1. f has sensitive dependence on initial conditions.
2. f is topologically transitive.
3. periodic points are dense in V. $\overline{Per(f)} = V$

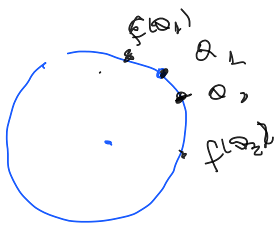
Obs: Para que una aplicación sea caótica, debemos tener 3 cosas

1°) debe ser impredecible
2°) debe ser indescomponible
3°) debe tener regularidad.

Example 8.6. $f: S^1 \to S^1$ given by $f(\theta) = 2\theta$ is chaotic. As we have seen, the angular distance between two points is doubled upon iteration of f. Hence f is sensitive to initial conditions. Topological transitivity also follows from this observation since any small arc in S^1 is eventually expanded by some f^k to cover all of S^1 and, in particular, any other arc in S^1. The density of periodic points was established in §1.3. We remark that this map possesses a strong form of sensitive dependence called expansiveness.

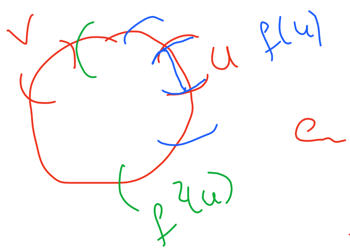
$$f: S^1 \longrightarrow S^1$$
$$\theta \longrightarrow f(\theta) = 2\theta$$

1°) es sensible c/r c.i



$$\{f^n(\theta)\} = \{2^n\theta\}$$

2°) Top transitiva ...



$f^k(U) \cap V \neq \phi$.

en $f(\theta) = 2\theta$ $\exists$ k tq

$f^k(U) = S^1$

$\Rightarrow$ $f^k(U) \cap V \neq \phi$

$\Rightarrow$ f top. transitiva

3) $f(\theta) = 2\theta$ $\quad f^n(\theta) = 2^n\theta$

pts periódicos

$2^n\theta = \theta$

$\theta = \frac{k}{2^n - 1}$ $\quad \forall k \in \{0, 1, \dots, 2^n - 1\}$ $\Rightarrow$ n pts periódicos de periodo n.

$\Rightarrow \overline{Per(f)} = S^1$

Así, f es caótica.

obs: f es más que caótica, pues es sensible c/R a todas las c.i., esto significa que f es expansiva.

Definition 8.7. $f: J \to J$ is expansive if there exists $\nu > 0$ such that, for any $x, y \in J$, $x \neq y$, there exists n such that $|f^n(x) - f^n(y)| > \nu$.

Obs: f es expansiva si $\exists\, \nu > 0$ tq $\forall\, x, y$ en J satisfaciendo $|f^n(x) - f^n(y)| < \nu$ $\Rightarrow$ $\boxed{x = y}$

Example 8.8. The quadratic maps $F_\mu(x) = \mu x(1-x)$ are chaotic on Λ when $\mu > 2 + \sqrt{5}$.

- Sensible c/R C.I ✓
- Top. transitiva ✓ $((\Sigma_2, \sigma)$ ✓ $s = 0\,1\,00\,1\,000\,1\,0000\,1$ una densa en Σ_2
- $Per(F_\mu) = \Lambda$ ✓ $((\Sigma_2, \sigma)$ ✓ $)$

Obs: El ejemplo anterior, muestra que para F_μ, $\mu > 2 + \sqrt{5}$, tenemos caos en $\Lambda \subset I = [0,1]$. Sin embargo Λ es pequeño en I. Pues es un subconjunto propio con medida cero.
El sgte ejemplo muestra q' hay caos para $\mu = 4$ en I.

Example 8.9. $F_4(x) = 4x(1-x)$ is chaotic on the interval $I = [0, 1]$.

Para notar que $([0,1], F_4)$ es caótica. Construiremos una conjugación topológica con $(S^1, 2\theta)$

$(I, F_4\ \ )$

Pero hay que

$$([-1,+1],\ 2x^2-1)$$

$$(S^1,\ 2\theta)$$

para p otra conjugación topológica intermedia $([-1,+1],\ 2x^2-1)$

Sea h_1 conjugación top. Esto es

$$\begin{array}{ccc} (S^1, 2\theta) & \xrightarrow{h_1} & ([-1,1],\ \underbrace{2x^2-1}_{p}) \\ \theta & \longrightarrow & h_1(\theta) = \cos\theta \end{array}$$

$$\begin{tikzcd} S^1 \arrow[r, "2\theta = F"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\circlearrowleft"] & S^1 \arrow[d, "h_1"] \\ {[-1,1]} \arrow[r, "2x^2-1=p"'] & {[-1,1]} \end{tikzcd}$$

$p \circ h_1 = h_1 \circ F$

$$p(\cos\theta) = 2\cos^2\theta - 1$$

$$h_1(2\theta) = \cos(2\theta)$$

Conjugar top. h_2, $(p(t) = 2t^2-1)$

$$([-1,1],\ p) \xrightarrow{h_2} ([0,1],\ F_4)$$

$$t \longrightarrow h_2(t) = \frac{1}{2}(1-t)$$

$$\boxed{F_4 = 4x(1-x)}$$

tq $F_4 \circ h_2 = h_2 \circ p$

$$F_4\Big(\underbrace{\tfrac{1}{2}(1-t)}_{x}\Big) = 4\underbrace{\tfrac{1}{2}(1-t)}_{x}\Big[1-\underbrace{\tfrac{1}{2}(1-t)}_{x}\Big]$$

$$= 2(1-t) - (1-t)^2 = 1-t^2 \checkmark$$

$$h_2(2t^2-1) = \frac{1}{2}(1-2t^2+1) = 1-t^2$$

$$\begin{tikzcd}
S^1 \arrow[r, "2\theta = f"] \arrow[d, "h_1"'] & S^1 \arrow[d, "h_1"] \\
{[-1,1]} \arrow[r, "2t^2-1"] \arrow[d, "h_2"'] & {[-1,1]} \arrow[d, "h_2"] \\
{[0,1]} \arrow[r, "F_4"] & {[0,1]}
\end{tikzcd}$$

$(S^1, 2\theta)$ es top conjugado a $([0,1], F_4)$

Pues f ...

$$\boxed{F_4 \circ \underbrace{h_2 \circ h_1}_{H} = \underbrace{h_2 \circ h_1}_{H} \circ f}$$

$\Rightarrow$ como $2\theta = f$ es continua en S^1)

$\Rightarrow$ $F_4 = 4x(1-x)$ es continua en $[0,1]$)