Chapter 5

Plasma Kinetic Theory

5.1 Klimontovich Equation

5.1.1 Introduction

Start from first principles => exact plasma description

Single particle:

- location $\mathbf{X}_1(t)$, velocity $\mathbf{V}_1(t)$
  => 6 degree of freedom
  => six-dimensional space
- Density of the particle in this space: $N(\mathbf{x}, \mathbf{v}, t) = \delta(\mathbf{x} - \mathbf{X}_1(t))\, \delta(\mathbf{v} - \mathbf{V}_1(t))$

with: $\delta(\mathbf{x} - \mathbf{X}_1(t)) = \delta(x - X_1(t))\, \delta(y - Y_1(t))\, \delta(z - Z_1(t))$

$\delta -$ Dirac delta function

Consider N_{0s} **particles** of species s:

Density of this distribution in phase space:

$$N_s(\mathbf{x}, \mathbf{v}, t) = \sum_{i=1}^{N_{0s}} \delta(\mathbf{x} - \mathbf{X}_i(t))\, \delta(\mathbf{v} - \mathbf{V}_i(t))$$

and for all species:

$$N = \sum_s N_s(\mathbf{x}, \mathbf{v}, t)$$

Particle motion:

$$\begin{aligned}
\dot{\mathbf{X}}_i &= \mathbf{V}_i(t) \\
\dot{\mathbf{V}}_i &= \frac{q_s}{m_s}\mathbf{E}^m(\mathbf{X}_i(t), t) + \frac{q_s}{m_s}\mathbf{V}_i(t) \times \mathbf{B}^m(\mathbf{X}_i(t), t)
\end{aligned}$$

To solve the equations of motion we need Maxwell's equations

$$\begin{aligned}
\nabla \cdot \mathbf{E}^m(\mathbf{x},t) &= \frac{1}{\epsilon_0}\rho_c(\mathbf{x},t) \\
\nabla \cdot \mathbf{B}^m(\mathbf{x},t) &= 0 \\
\nabla \times \mathbf{E}^m(\mathbf{x},t) + \frac{\partial \mathbf{B}^m(\mathbf{x},t)}{\partial t} &= 0 \\
\nabla \times \mathbf{B}^m(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial \mathbf{E}^m(\mathbf{x},t)}{\partial t} &= \mu_0 \mathbf{j}^m(\mathbf{x},t)
\end{aligned}$$

(m stands for microscopic fields) with the charge and current densities (sources)

$$\begin{aligned}
\rho_c^m(\mathbf{x},t) &= \sum_s q_s \int_{-\infty}^{\infty} d^3v N_s(\mathbf{x},\mathbf{v},t) \\
\mathbf{j}^m(\mathbf{x},t) &= \sum q_s \int_{-\infty}^{\infty} d^3v \mathbf{v} N_s(\mathbf{x},\mathbf{v},t)
\end{aligned}$$

The above equations fully determine the entire sytem of particles.

Initial value problem:

$$N_s(\mathbf{x},\mathbf{v},t=0) \quad => \quad \mathbf{E}^m(\mathbf{x},t=0)\,,\ \mathbf{B}^m(\mathbf{x},t=0)$$

=> Integrate equations in time.

5.1.2 Klimontovich Equation

Time evolution of the distribution function $N_s(\mathbf{x},\mathbf{v},t)$:

$$\begin{aligned}
\frac{\partial N_s(\mathbf{x},\mathbf{v},t)}{\partial t} &= -\sum_{i=1}^{N_{0s}} \dot{\mathbf{X}}_i \cdot \nabla_{\mathbf{x}} \delta(\mathbf{x}-\mathbf{X}_i(t))\,\delta(\mathbf{v}-\mathbf{V}_i(t)) \\
&\quad -\sum_{i=1}^{N_{0s}} \dot{\mathbf{V}}_i \cdot \nabla_{\mathbf{v}} \delta(\mathbf{x}-\mathbf{X}_i(t))\,\delta(\mathbf{v}-\mathbf{V}_i(t))
\end{aligned}$$

Note:

$$\begin{aligned}
\frac{\partial f(a-b)}{\partial a} &= -\frac{\partial f(a-b)}{\partial b} \\
\frac{df(g(t))}{dt} &= \frac{df}{dg}\frac{dg}{dt}
\end{aligned}$$

Substitute: $\dot{\mathbf{X}}_i$ and $\dot{\mathbf{V}}_i$

$$\begin{aligned}\frac{\partial N_s(\mathbf{x},\mathbf{v},t)}{\partial t} &= -\sum_{i=1}^{N_{0s}} \mathbf{v}\cdot\nabla_{\mathbf{x}}\delta(\mathbf{x}-\mathbf{X}_i(t))\,\delta(\mathbf{v}-\mathbf{V}_i(t)) \\ &\quad -\sum_{i=1}^{N_{0s}} \left\{\frac{q_s}{m_s}\mathbf{E}^m(\mathbf{x},t)+\frac{q_s}{m_s}\mathbf{v}\times\mathbf{B}^m(\mathbf{x},t)\right\}\cdot\nabla_{\mathbf{v}}\delta(\mathbf{x}-\mathbf{X}_i(t))\,\delta(\mathbf{v}-\mathbf{V}_i(t))\end{aligned}$$

where we used $f(a)\delta(a-b) = f(b)\delta(a-b)$.

Exercise: Prove that the last equation for $N_s(\mathbf{x},\mathbf{v},t)$ is correct and in particular that one can replace $\mathbf{V}_i(t)\times\mathbf{B}^m(\mathbf{X}_i(t),t)$ with $\mathbf{v}\times\mathbf{B}^m(\mathbf{x},t)$ in this equation.

As the final step we can now take the $\mathbf{v}\cdot\nabla_{\mathbf{x}}$ and $\left\{\frac{q_s}{m_s}\mathbf{E}^m(\mathbf{x},t)+\frac{q_s}{m_s}\mathbf{v}\times\mathbf{B}^m(\mathbf{x},t)\right\}$ in front of the summation which yields the **Klimontovich equation**

$$\frac{\partial N_s}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}}N_s + \frac{q_s}{m_s}(\mathbf{E}^m+\mathbf{v}\times\mathbf{B}^m)\cdot\nabla_{\mathbf{v}}N_s = 0$$

Together with the Maxwell's equation and the definitions for charge and current densities this provides a full description of the plasma dynamics!

However, since the distribution is a distribution of delta functions it still requires basically to follow all individual particles which in typical application is not feasible even on modern supercomputers.

Properties of the Klimontovich equation

- Incompressibility in phase space: Hypothetical point particle at $\mathbf{x}$, $\mathbf{v}$
  total time derivative

  $$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{d\mathbf{x}}{dt}\cdot\nabla_{\mathbf{x}} + \frac{d\mathbf{v}}{dt}\cdot\nabla_{\mathbf{v}}$$

  => Klimontovich equation

  $$\frac{DN_s(\mathbf{x},\mathbf{v},t)}{Dt} = 0$$

  => along each (hypothetical) path $N_s is$ constant!

- Conservation of particles (continuity): $\partial f/\partial t + \nabla_{\mathbf{r}}\cdot(\mathbf{v}f) = 0$
  In 6-dimensional phase space we can define $\nabla_{\mathbf{R}} = (\nabla_{\mathbf{x}}, \nabla_{\mathbf{v}})$ and $\mathbf{V} = (d\mathbf{x}/dt,\, d\mathbf{v}/dt)$ =>

  $$\frac{\partial N_s}{\partial t} + \nabla_{\mathbf{R}}\cdot(\mathbf{V}N_s) = 0$$

  Klimontovich eq. must satisfy continuity!

5.1.3 Plasma Kinetic Equation

The Klimontovich distribution is a distribution of δ functions => need to reduce amount of information (we know that a plasma behave collectively so it is not necessary to follow each individual particle.).

=> generate smooth distribution using an appropriate average

Rigorous way:

- Ensemble average over infinite number of realizations, e.g., with a temperature contact => statistical mechanics

Alternatively:

- Define boxes size Δx, Δv with $\Delta x \ll \lambda_{de}$ and count particles in range $[\mathbf{x}, \mathbf{v}]$ to $[\mathbf{x}+\Delta\mathbf{x}, \mathbf{v}+\Delta\mathbf{v}]$ => $f_s = \frac{n_s}{\Delta x^3 \Delta v^3}$

Define fluctuations

$$\begin{aligned} N_s(\mathbf{x},\mathbf{v},t) &= f_s(\mathbf{x},\mathbf{v},t) + \delta N_s(\mathbf{x},\mathbf{v},t) \\ \mathbf{E}^m(\mathbf{x},t) &= \mathbf{E}(\mathbf{x},t) + \delta\mathbf{E}(\mathbf{x},t) \\ \mathbf{B}^m(\mathbf{x},t) &= \mathbf{B}(\mathbf{x},t) + \delta\mathbf{B}(\mathbf{x},t) \end{aligned}$$

such that: $\langle \delta N_s \rangle, \langle \delta \mathbf{E} \rangle, \langle \delta \mathbf{B} \rangle = 0$

=>

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_s + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f_s = \frac{q_s}{m_s}\langle(\delta\mathbf{E}+\mathbf{v}\times\delta\mathbf{B})\cdot\nabla_{\mathbf{v}}\delta N_s\rangle$$

- Left side - collective effects
- right side -collisional effects

Continuum limit: $N_0 \to \infty$

- right side: fluctuations $\delta N_s \sim N_0^{1/2}$ (statistical mechanics)
  $\delta\mathbf{E} \sim e\delta N_s \sim \frac{1}{N_0} N_0^{1/2} \sim N_0^{-1/2}$

=> right side $\to$ const

=> left side $\sim N_0 \to \infty$

Which yields the collisionless **Boltzmann** equations:

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_s + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f_s = 0$$

Complemented with Maxwells equations and with the definitions for charge and current density

$$\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{x}, t) &= \frac{1}{\epsilon_0}\rho_c(\mathbf{x}, t) \\
\nabla \cdot \mathbf{B}(\mathbf{x}, t) &= 0 \\
\nabla \times \mathbf{E}(\mathbf{x}, t) &= -\frac{\partial \mathbf{B}(\mathbf{x}, t)}{\partial t} \\
\nabla \times \mathbf{B}(\mathbf{x}, t) &= \mu_0 \mathbf{j}(\mathbf{x}, t) + \frac{1}{c^2}\frac{\partial \mathbf{E}(\mathbf{x}, t)}{\partial t} \\
\rho_c(\mathbf{x}, t) &= \sum_s q_s \int_{-\infty}^{\infty} d^3v f_s(\mathbf{x}, \mathbf{v}, t) \\
\mathbf{j}(\mathbf{x}, t) &= \sum q_s \int_{-\infty}^{\infty} d^3v \mathbf{v} f_s(\mathbf{x}, \mathbf{v}, t)
\end{aligned}$$

yield the **Vlasov** equations.

5.2 Liouville Equation

5.2.1 Concept of a system

Motivation:

Use Liouville equation => derivation of a kinetic equation (right hande side of the Boltzmann equation)

Note: Klimontovich equation - Behaviour of individual particles

One particle:

- spatial coordinate of the system $\mathbf{x}_1 = (x_1, y_1, z_1)$
- velocity coordinate of the system $\mathbf{v}_1 = (v_{x1}, v_{y1}, v_{z1})$
- Particle orbit (as before) by $\mathbf{X}_1(t)$ and $\mathbf{V}_1(t)$
- System coordinates: $(\mathbf{x}_1, \mathbf{v}_1) = (x_1, y_1, z_1, v_{x1}, v_{y1}, v_{z1})$ (6 coord)
- Density of systems:
  $N(\mathbf{x}, \mathbf{v}, t) = \delta(\mathbf{x}_1 - \mathbf{X}_1(t))\,\delta(\mathbf{v}_1 - \mathbf{V}_1(t))$
  One system consisting of one particle

2 particles:

- 12 coordinates for our system
- Phase space: $(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2) = (x_1, y_1, z_1, v_{x1}, v_{y1}, v_{z1}, x_2, y_2, z_2, v_{x2}, v_{y2}, v_{z2})$

- Density:
  $N(\mathbf{x}, \mathbf{v}, t) = \delta(\mathbf{x}_1 - \mathbf{X}_1(t))\,\delta(\mathbf{v}_1 - \mathbf{V}_1(t))\,\delta(\mathbf{x}_2 - \mathbf{X}_2(t))\,\delta(\mathbf{v}_2 - \mathbf{V}_2(t))$
  1 system consiting of 2 particles

Generalisation to N_0 particles => Phase space has 6N coordinates

Density:

$$N(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, ...\mathbf{x}_{N_0}, \mathbf{v}_{N_0}, t) = \prod_{i=0}^{N_0} \delta(\mathbf{x}_i - \mathbf{X}_i(t))\,\delta(\mathbf{v}_i - \mathbf{V}_i(t))$$

5.2.2 Liouville equation

Interested in the time evolution of N and with

$$\frac{\partial \delta(\mathbf{x}_i - \mathbf{X}_i(t))}{\partial t} = -\frac{d\mathbf{X}_i(t)}{dt} \cdot \nabla_{\mathbf{x}_i} \delta(\mathbf{x}_i - \mathbf{X}_i(t))$$

=>

$$\begin{aligned} \frac{\partial N(\mathbf{x}, \mathbf{v}, t)}{\partial t} &= -\sum_{i=1}^{N_0} \dot{\mathbf{X}}_i \cdot \nabla_{\mathbf{x}_i} \prod_{i=0}^{N_0} \delta(\mathbf{x}_i - \mathbf{X}_i(t))\,\delta(\mathbf{v}_i - \mathbf{V}_i(t)) \\ &\quad -\sum_{i=1}^{N_0} \dot{\mathbf{V}}_i \cdot \nabla_{\mathbf{v}_i} \prod_{i=0}^{N_0} \delta(\mathbf{x}_i - \mathbf{X}_i(t))\,\delta(\mathbf{v}_i - \mathbf{V}_i(t)) \end{aligned}$$

As before we can substitute (note $\mathbf{x}_i$ instead of $\mathbf{X}_i(t)$ and $\mathbf{v}_i$ instead of $\mathbf{V}_i(t)$ in the Lorentz force!):

$$\begin{aligned} \dot{\mathbf{X}}_i &= \mathbf{V}_i(t) \\ \dot{\mathbf{V}}_i &= \frac{q_s}{m_s}\left[\mathbf{E}^m(\mathbf{x}_i, t) + \mathbf{v}_i \times \mathbf{E}^m(\mathbf{x}_i, t)\right] \end{aligned}$$

Such that

$$\frac{\partial N}{\partial t} + \sum_{i=1}^{N_0} \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} N + \sum_{i=1}^{N_0} \dot{\mathbf{V}}_i \cdot \nabla_{\mathbf{v}_i} N = 0$$

which is **Liouville's equation.**

Properties:

(a)

$$\frac{D}{Dt} N(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, ...\mathbf{x}_{N_0}, \mathbf{v}_{N_0}, t) = 0$$

with

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{N_0} \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} + \sum_{i=1}^{N_0} \dot{\mathbf{V}}_i \cdot \nabla_{\mathbf{v}_i}$$

=> incompressibility

(b) Continuity

$$
\begin{aligned}
\mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} N &= \nabla_{\mathbf{x}_i} \cdot (\mathbf{v}_i N) \\
\dot{\mathbf{V}}_i \cdot \nabla_{\mathbf{v}_i} &= \nabla_{\mathbf{v}_i} \cdot \left(\dot{\mathbf{V}}_i N\right)
\end{aligned}
$$

because

$$\nabla_{\mathbf{v}_i} \cdot \dot{\mathbf{V}}_i = \nabla \cdot \left\{ \frac{q_s}{m_s} \left[\mathbf{E}^m (\mathbf{x}_i, t) + \mathbf{v}_i \times \mathbf{E}^m (\mathbf{x}_i, t)\right] \right\} = 0$$

=>

$$\frac{\partial N}{\partial t} + \sum_{i=1}^{N_0} \nabla_{\mathbf{x}_i} \cdot (\mathbf{v}_i N) + \sum_{i=1}^{N_0} \nabla_{\mathbf{v}_i} \cdot \left(\dot{\mathbf{V}}_i N\right) = 0$$

Probability density:

Ensemble of systems N:

Def.:

$$f_{N_0} (\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, ...\mathbf{x}_{N_0}, \mathbf{v}_{N_0}, t)\, d\mathbf{x}_1 d\mathbf{v}_1 d\mathbf{x}_2 d\mathbf{v}_2 ... d\mathbf{x}_{N_0} d\mathbf{v}_{N_0}$$

is the proability that

$\mathbf{X}_1 (t)$ is in the interval $[\mathbf{x}_1, \mathbf{x}_1 + d\mathbf{x}_1]$, $\mathbf{X}_2 (t)$ is in the interval $[\mathbf{x}_2, \mathbf{x}_2 + d\mathbf{x}_2]$, ..

$\mathbf{V}_1 (t)$ is in the interval $[\mathbf{v}_1, \mathbf{v}_1 + d\mathbf{v}_1]$, $\mathbf{V}_2 (t)$ is in the interval $[\mathbf{v}_2, \mathbf{v}_2 + d\mathbf{v}_2]$, ..

Probability is conserved along trajectory:

each fluid element moves along the trajectory as a probability

With $\nabla_{\mathbf{x}_i} \cdot \mathbf{v}_i = 0$ and $\nabla_{\mathbf{v}_i} \cdot \dot{\mathbf{V}}_i = 0$ =>

$$\frac{\partial f_{N_0}}{\partial t} + \sum_{i=1}^{N_0} \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} f_{N_0} + \sum_{i=1}^{N_0} \dot{\mathbf{V}}_i \cdot \nabla_{\mathbf{v}_i} f_{N_0} = 0$$

or

$$\frac{D}{Dt} f_{N_0} (\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, ...\mathbf{x}_{N_0}, \mathbf{v}_{N_0}, t) = 0$$

We have now a probability distribution function which determines the kinetic evolution exactly, however, this distribution is in a $6N_0$ dimensional space. Thus there is now reduction in complexity compared to Klimotovich equation!