THE BINARY TREE AND ITS AVATARS: FROM XIANTIAN TO THE ETERNAL SYMMETREE…

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In this paper we focus on Shao Yong's visualisations and interpretations of Yi Jing hexagrams, and trace back their connections with key notions in mathematics and science, emerging much later in Western culture. We move from Shao Yong's insightful Xiantian diagram illustrating the genesis of the hexagrams (very likely the first avatar of the binary tree in human) up to the contemporary "eternal symmetree" of Stanford physicists, a model of a multiverse in the context of eternal inflation in cosmology. We present illustrative examples of how we exploit Shao Yong's constructions as a trigger in mathematics education, at various educational levels, with humanistic as well as scientific students, including pre-service and in-service teachers. We finally discuss, from an enactivist perspective, some epistemological caveats regarding the approach to the history of mathematics suggested by our title.

INTRODUCTION

Yi Jing, the ancient Chinese oracle, became known in Europe only in 1700 thanks to the French Jesuit missionary Joachim Bouvet, who sent from China to his friend, the German mathematician and polymath Gottfried Wilhelm Leibnitz, Shao Yong's 8 x 8 square of the 64 hexagrams (Leibnitz, Cook & Rosemont, 1994; Aiton & Shimao, 1981; Ryan, 1966). See Fig. 1 (Marshall, 2015). Bouvet, also a mathematician, had noticed the isomorphism between the hexagram sequence as given by Shao Yong (1011-1077) and the numbers 0 to 63 written in the binary way discovered by Leibnitz: a remarkable epistemological convergence of an ancient Chinese insight and a much more recent European one on the binary dynamics of the Universe, which was a common concern of Shao Yong and Leibnitz (Aiton & Shimao, 1981; Birdwhistell, 1989).

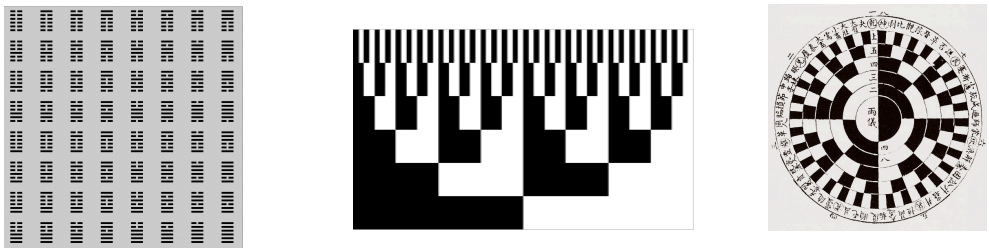

Figure 1: Shao Yong's 8 x 8 square, rectangular and circular Xiantian.

Shao Yong's Xian Tian (Fig. 1) is very likely the first explicit avatar of the binary tree in human civilisation; in either rectangular or circular form, it "explains" the generation, through binary branching, of the hexagrams (where black=Yin, white=Yang). Arithmetically, the first branching may be *interpreted* as the splitting of the integers into even and odd, and so on (congruence mod powers of 2). So in Xiantian numbers are ascending *random walks* (because hexagrams are randomly chosen), a

much later notion in the West (Pearson, 1905). Expanding the tree up to infinity, we are lead to 2-adic numbers (as the boundary at infinity, "including" the integers), which only in 1908 were fathomed by Hensel (Dickson, 1910) in a purely algebraic way, motivated by the study of Diophantine equations. Interestingly Xian Tian surfaces much later again, as the "eternal symmetree", a discrete combinatorial model for a multiverse in eternal inflation in cosmology (Harlow et al., 2012). See Fig. 2.

Figure 2: Three-step causal tree and causal future of *a*, for *p=2* (from Harlow et al., 2012)

Moreover, notice that Shao Yong and Darwin share the insight that the current state of a system may be the outcome of a branching process unfolding in time: Shao Yong considered the 64 hexagrams coming out of Xiantian and Darwin, the sundry living species on earth coming out of the Tree of Life…

ILLUSTRATIVE EXAMPLES OF CLASSROOM ACTIVITIES

We present below some illustrative examples of "didactical walks" in the classroom, suggested by the insights and connections above, involving the following learners at the University of Chile: **A**. First year humanistic students, who intend to major in Law or Social Sciences or Philosophy, taking a one semester general mathematics course. **B**. Pre-service secondary mathematics-physics teachers taking elementary number theory. **C**. Prospective mathematicians, scientists and mathematics-physics teachers taking a history of mathematics course. **D**. In-service primary school teachers enrolled in a professional development program at the University of Chile. Our usual methodology is to have the students work autonomously on given or constructed problems in random small groups of two to four.

Discovery approach to Shao Yong's diagrams:

Learners were prompted to stand up and try to register how many they were in the classroom, in a primitive way, *without counting themselves*. We hinted: what is the most basic and natural action a group of people can engage into? The idea emerged rather easily: to match up in couples, to pair… So they tried to match up in pairs, and realised that there *may be* an "odd man out". Then the pairs did the same, and so on. When the pairing game was over, they noticed eventually an unmatched person, an unmatched pair, and so on, and codified this as a chain of YES and NO. in a symbolic, less verbal way. They came up with dichotomic symbolic notations like crosses and circles, or ones and zeros, so re-discovering the binary system. When told about the ancient Yi Jing coding for Yang (continuous line) and Yin (broken line), they associated them with *presence* and *absence* and could write their number as an hexagram! After completing this activity most learners reported having understood for the first time the binary codification of a number. We posed then the challenge of drawing a *synthetic* picture of the whole binary 64 hexagram sequence, so that just a glimpse of it should be enough to reconstruct it correctly. Here learners D did better than learners A: In a class of 30 teachers (25 female, 5 male), after 30 m. working in groups, cutting out and putting together the hexagrams, 4 of them came up with diagrams like rectangular Xiantian and one with one *circular* Xiantian of its own!

Archeologic exploration of Yi Jing.

In this activity, learners received a copy of Shao Yong's square and tried to make sense of it. Some recognized there the Yi Jing hexagrams. They wondered about the "logic" of their sequence, from the all yin hexagram to the all yang one, say. They saw the square as a 2x2 matrix whose row and column indices were trigrams! They were intrigued because after the all-yin hexagram, yang appears in the first line (from the top), then moves to the second, but then, instead of moving to the third, it pops up in the first *and* second line. They found no logic in this phenomenon, which arose over and over again. One teacher student however said: "Look, it is like the electrons in atoms: before going to the next higher energy orbital, you have to fill in the lower energy orbitals! Other students, as well as T'ai Chi or martial arts practitioners we interviewed, found a somatic metaphoric explanation: Putting a Yang line first, you take a step forward. Moving Yang to the second position you take a second step. A Westerner would then take a third step. But in martial arts or T'ai Chi before moving forward again you would transfer your weight backward to take impulse to move forward again. And you keep doing this everywhere, like ebb and flow. More arithmetically minded students recognized the binary expansion of numbers in the square, interpreting Yang or Yin as *presence* or *absence* of 2^n.

Grasping Yi Jing in a glimpse

We shew a glimpse of Xiantian to learners A, C and D (ca. 200 msec.) and asked them to try to reconstruct the whole diagram. We got a wide spectrum of drawings. See Fig. 3 (Soto-Andrade, 2008), for a sample. Case studies with professional mathematicians and mathematics educators gave also disparate performances. Some mathematics educators did better than mathematicians (especially algebraists), because they "saw" the binary tree in Xian Tian, as roughly 10% of our students did.

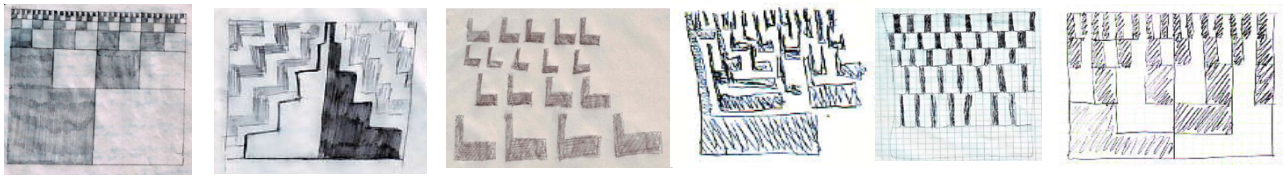

Figure 3. Students' reconstructions of Xiantian

Numbers are ascending paths in a binary tree.

Learners C, working in random groups, after realising that Xiantian suggests that whole numbers can be metaphorized as ascending walks in a binary tree, extended it to infinitely many generations and tried to fathom its boundary at infinity. They spotted there negative integers and some fractions and converged eventually to the Hensel's algebraic notation of 2-adic numbers as series of powers of 2 (Dickson, 1910). They also related this process to solving an infinite chain of congruences mod the powers of 2. They also wondered whether the boundary at infinity carried with some natural metric…

DISCUSSION, CONCLUSSIONS AND CAVEATS

In our classroom activities, particularly with students C, who are on a separate historical track (Fried, 2001), we have used Shao Yong's diagrammatic mathematical and philosophical insights as *triggers* of autonomous mathematical and epistemological reflexions. Sundry unexpected connections with notions constructed much later elsewhere also emerged. The question arises why Shao Yong felt the need of such a synthetic and process oriented view, as expressed in Xiantian. Very likely because of his Taoist and Buddhist background (Birdwhistell, 1989). Notice *en passant* that some of our didactical

walks are mutually incompatible: either we show the students a glimpse of Xiantian with the challenge of reconstruct it or we try to stimulate them to feel unhappy with the analytical 8x8 square and long for a synthetic insight of the hexagrams. Regarding the spin-offs of activities, we have noticed that the work the students did in a separate history of mathematics track fed back, in a circular way, to their understanding of mathematical contents they were supposed to master. They often reported having understood for the first time something they had learned in a formal, abstract and disconnected way.

An enactivist caveat is required though, regarding our title, which may suggest that we see the binary tree as a well-defined mathematical object "standing out there", which is represented by different cultures in different guises (avatars). The binary tree (a particular kind of graph) is in fact a typical Western mathematical construction, which we automatically project onto Xiantian. This happened when we shew Xiantian to Bourbaki himself (i.e. one of his foremost avatars): he saw instantly the binary tree. Our posture is however "non Whiggish" (Clark et al., 2018; Fried, 2001): we try instead to *listen* to Shao Yong (Arcavi & Isoda, 2007), and accept that this may enhance our understanding of "something" that we have constructed as a binary tree nowadays in our culture. Indeed trees are often seen as hierarchical *structures,* but Xiantian suggests a (dichotomic branching) *process*, pertaining to a changing and flowing cosmos, a Taoist view indeed (Birdwhistell, 1989). A bit Whiggish claim here would be that Shao Yong was a forerunner of Darwin… Epistemological disorientation (Clark et al., 2018) may help us to accept that other cultures have had insights ours is blind to. This applies to the hexagram sequence in Shao Yong's square: it may emerge or be fathomed from binary arithmetic, from physics, from martial arts, from dance, from theatre, among other avenues.

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