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MATHEMATICS TEACHER EDUCATION AROUND THE WORLD

A SPIRAL TASK AS A MODEL FOR IN-SERVICE TEACHER EDUCATION

ABSTRACT. The spiral approach has long been used by curriculum designers to deepen students' knowledge of scientific and mathematical concepts and to bring students to higher levels of abstraction. The benefits of a spiral approach, however, can also be extended to teacher education. This paper describes a spiral activity employed by the *Kidumatica* program not only to raise the level of teachers' content knowledge, but also to promote discussion and collaboration among teachers teaching at different grade levels. The activity was designed to take a single problem situation and develop it in ways appropriate to the different grade levels. At each stage, the teachers are encouraged to discuss teaching approaches required by the students at each grade level and the relationships between the different stages of the development.

KEY WORDS: community of practice, in-service, spiral-curriculum, spiral-task

The spiral approach in curriculum design has been well known and well used almost since Bruner (1960) first introduced it in the early 1960's. Indeed, although it can degenerate into mere repetition (Schmidt, McKnight, & Raizen 1996), its potential to bring students to higher levels of abstraction and deeper levels of understanding, particularly in science education (e.g., Aldridge, 1992; DeBoer, 1991), still remains in force. The benefits of a spiral approach, both those just mentioned and others, can also be extended to teacher education. This paper presents an instance of that possibility – a spiral task employed in an in-service program, called *Kidumatica*, for mathematics teachers.

The task, which will be discussed in detail below, had three goals. One of these goals was to provide an example of a spiral activity involving the development of a single significant mathematical problem that teachers could emulate within the context of their own school curricula. A second goal was to deepen the teachers' knowledge of the interconnections between graphs, algebraic equations, functions, and derivatives. Both of these goals, of course, can be traced back to the spiral approach as it is usually applied for the direct benefit of students. However, a third goal had specifically to do with teachers. It was to encourage communication among novice, expert, middle school and high school teachers, as well as among teachers teaching advanced, average, and challenged students.

This last goal was very much in line with the general objective of the *Kidumatica* program. For *Kidumatica* was designed not only to address the professional development of specific groups of teachers, but, at the same time, to mould a teacher population that sees itself as participating in a common endeavor. It did not aim to erase the differences within the teaching population, which is quite diverse (Amit, 2000), but to encourage communication within it so that our mathematics teachers could begin to see themselves as a single community. Thus, the position the *Kidumatica* program adopted with regards to its teachers was that which Stigler and Hiebert (1999) said Japanese teachers adopted with regards to students, namely, that "They view differences in the mathematics class as a resource for both students and teachers. Individual differences are beneficial for the class because they produce a range of ideas and solution methods that provide the material for students' discussion and reflection" (p. 94). Formulated differently, what we confronted in the *Kidumatica* program is what Gimenez (Gimenez, et al. (2004)) has called the challenge of "reconciling commonalities and differences."

Because of the close connection between the goals of the spiral task, which is the focus of this paper, and those of *Kidumatica*, we shall begin with a description of the *Kidumatica* program. Next, the rationale and theoretical basis of the spiral task will be discussed. Finally, the spiral task itself will be presented in detail.

THE *KIDUMATICA* IN-SERVICE PROGRAM

The *Kidumatica* Program ran for 7 years. Its establishment in 1995 came in a time of reform in science and technology education in Israel. Within that reform movement, mathematics education in particular was seen to have a crucial role. In response, the *Kidumatica* program was set up with the mission to raise the level of school mathematics teaching and to crystallize an active community of mathematics teachers (Amit & Fried, 2002; Amit & Hillman, 1999).

As an in-service program, three characteristics of *Kidumatica* stood out. (1) It was quite extensive, each round of the program being 3 years of full-day weekly meetings where each meeting comprised three to four different workshops; (2) it integrated all aspects of mathematics teaching in the middle and secondary schools, including specific mathematical topics, pedagogy, history of mathematics, technology, and research issues; (3) although the teacher-educators were each responsible for specific subjects, as a rule all were present and participated in every workshop.

About the last, a few words ought to be said. The constant presence of the teacher-educators and the character their interaction with the participating teachers was significant for several reasons (a) it allowed the teachers and the teacher-educators to develop a particularly close relationship; (b) the teacher-educators were in an ideal position to function as role models for the teachers (here it ought to be emphasized that all the teacher-educators were at the time or had been once themselves classroom teachers); (c) it created a strong feeling of cooperation between the teachers and the teacher-educators.

Although some teachers actively teach in grades 7–12, for the most part it is possible to divide the teacher population into middle school teachers (grades 7–9), high school teachers (grades 10–12), and department chairpersons. Teaching experience varies greatly among the teachers; the range up until now has been from 5 to 36 years. The teacher distribution for the 2001 school year, for example, is given in Table I.

Part of the time, teachers worked in groups, according to whether they were middle school teachers, high school teachers, or department chairpersons. But part of the time they worked together in common sessions. These common sessions attempted to give the business of the particular groups a bridging context centered on some issue or subject of general interest. While it was true that in the discussions ensuing from the common sessions, teachers did typically relate their own needs to the subject at hand, the need was felt to create activities that sewed the different needs of the various teachers into one fabric in a

TABLE I
Kidumatica: Teacher Population for 2001

Middle school only (Grades 7–9)	High school only (Grades 10–12)	All grades (Grades 7–12)	Department chairpersons
34 41.5%	28 34.1%	20 24.4%	15 18.2%

Total number of schools = 31.
Total number of teachers = 82.
Note: Department chairpersons are included among the teachers.

more directed and active way. This is how the 'spiral' activity, which will now be described, was born.

GENERAL RATIONALE AND THEORETICAL BASIS

The basic strategy of the 'spiral' activity was to take a single problem situation and show how it can be taken up over and over again in the 7^{th}, 8^{th} , 9^{th}, 10^{th}, and 11^{th} grades; teachers teaching, say, 8^{th} algebra are meant to find common ground with teachers teaching, say, maximum/minimum problems in the 11^{th} grade. The theoretical motivation for the activity plainly has its provenance in Bruner's 'spiral curriculum' whereby basic ideas are continually revisited so that the curriculum continually "turns on itself at higher levels..." (Bruner, 1960, p. 13).

Bruner, it may be recalled, saw this kind of curriculum following upon his well-known thesis that "any subject can be taught effectively in some intellectually honest form to any child at any stage of development" (Bruner, 1960, p. 33). Although that thesis may be problematic, it does contain an important message, namely, that, *at every level of teaching*, the teaching task is serious and intellectually respectable. This basic tenet, no less than the overall structure borrowed from the 'spiral curriculum', is central to the *Kidumatica* activity. Indeed, one of the problems we needed to overcome was the lack of appreciation sometimes felt by teachers of more advanced levels towards teachers of more elementary levels.

In this connection, the fact that the 'spiral' activity was primarily directed, on the face of it, towards mathematical content was important: far from embarrassing the teachers at the more elementary levels, the concentration on content serves to challenge the common assumption that because one understands mathematics as it is taught in the upper levels of high school one knows what is required to explain material at the lower middle school level. Ball and Bass (2000) have emphasized that turning away from one's 'compressed' mathematical knowledge and 'decompressing' it, to use their term, so that it is applicable to young learners is a process whose difficulty should not be underestimated, and it is one very much connected with mathematical understanding; as they put it, "one needs to be able to deconstruct one's own mathematical knowledge into less polished and final form, where elemental components are accessible and visible" (Ball & Bass, 2000, p. 98). In this way, the issues suggested by the first turns of the 'spiral' could be seen in such a light to make them equally significant for the teachers at all the levels and thus a basis for dialogue among them. The focus on content, then, is in line with Zaslavsky and Leikin's (2004) emphasis on mathematical tasks and challenges in forming what they and others have called a 'community of practice' (see also, Lave & Wenger, 1991; Roth, 1998).

The later turns of the 'spiral' were naturally designed to give the teachers a sense of the mathematical significance of the tasks at all levels as well. However, in these turns of the 'spiral', the activity is also meant to bring out the meaning and importance of mathematical 'depth'. Teachers all too often present problem situations to their students as if they are one-dimensional entities; problems are presented as if they embody a single technique or a single concept. Demonstrating how a single problem situation may contain simpler ideas and more complex ideas provides a view of problems as multidimensional entities full of potential for further development. This is, in some ways, the message of Brown and Walter's (1990) idea of 'problem posing', but it is not always put in terms of mathematical 'depth', even though, in our opinion, this is an essential part of the message.

Before we look at the details of the 'spiral activity', one more remark is in order. As with most of the *Kidumatica* activities, discussion was an essential component of the 'spiral activity'. This involved not only discussion between the teachers and the teacher-educators but, more importantly, also among the teachers themselves. This was not a formal part of the design of the activity; that is, there was no special time set aside for discussion, rather it was a general guideline that the teacher-educators keep a watchful eye for the seeds of discussion and that, when it happens, encourage it. Such discussions are what Britt et al. have called 'professional conversations', which, more precisely, they define to be "...discussions among those who share a complex task or profession in order to improve their understanding of, and efficacy in what they do" Britt, Irwin, & Ritchie, 2001. Within the school setting, Stigler and Hiebert noted the importance of this kind of reflective, but focused, discussion among Japanese teachers to sharpen the quality of particular lessons (Stigler & Hiebert, 1999, chap. 7); other writers too (e.g., Horn, 2000; Jenlink & Carr, 1996) have pointed out the importance of conversation as a means of educational improvement. What is important about 'professional conversations' in the context of the 'spiral activity' is that they go beyond the immediate issues connected with a given lesson or unit or even grade level; they concern the links between grade levels or achievement levels. So, 'professional conversations', in this way, are an essential means of fulfilling our final goal of improving and knitting together a broad mathematics teaching community.

THE 'SPIRAL' TASK

The Basic Problem and its Transformations

The activity began with an initial problem situation, the center of the 'spiral': *Flowers and grass are to be planted on a rectangular plot whose dimensions are 6×10 m. Grass is to be planted in four right triangles whose right angles are those of the rectangle. The right triangles at D and B are also congruent isosceles triangles* (Figure 1). *Flowers are to be planted in the remaining parallelogram.*

Of course, this problem situation was not yet a problem; there was still no question. The activity was designed to evolve not only with respect to content area – pre-algebra, algebra and functions, beginning analysis – but also with respect to the kinds of questions each content area suggested and allowed. Throughout the activity, we tried to emphasize the importance of questions, that different questions, perhaps more than answers, reflect different ways of thinking, and that it is questions, more than answers, that drive discussion and exploration.

Having presented the initial problem situation, each of the teacher-educators in turn offered developments of the situation. These developments took the form of questions and tasks flowing from different content areas. At every point of the activity, teachers were invited to think about the questions, *as if they were students*, and also think about other questions that they thought worthy of exploration. The progression of developments from one content area and one level to another was occasionally interrupted by short, but crucial, digressions, which we initiated and called 'syntheses'. As will be seen shortly, these 'syntheses' had the effect of directing the discussion towards the relationship between different turns of the 'spiral'; they were meant to be foci for reflection. Pictorially, the structure of the activity can be represented as follows (Figure 2).

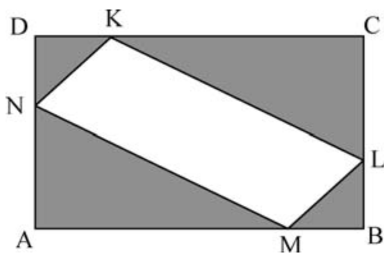


Figure 1. The basic problem situation.

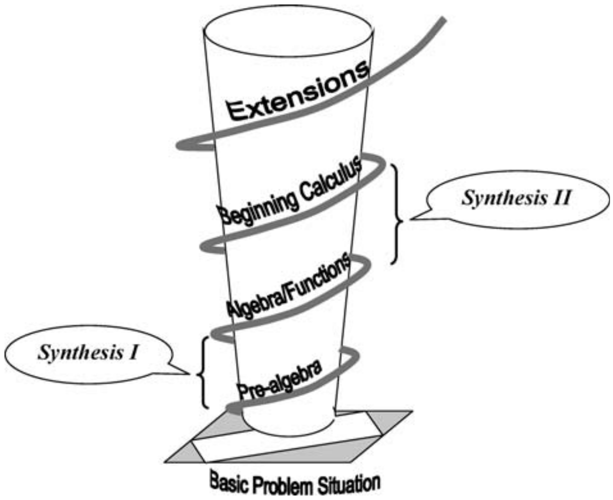


Figure 2. The 'Spiral' activity.

Pre-Algebra

The first point of discussion, before any specific question was posed by the teacher-educators, was how to verify that NKLM was, in fact, a parallelogram. This set the tone for the entire exercise, for teachers teaching in middle schools had to approach this question in a way different from that of high school teachers whose students already have some training in elementary geometry.

From here, tasks and questions appropriate for students at the pre-algebra level were formulated. An initial task at this level was simply to find the coordinates of K, L, M, N with the picture situated in a coordinate system, as in Figure 3:

Following this, the teacher-educators suggested questions about the original figure (Figure 1) of a quasi computational character

- Which of the following are possible lengths for DN: 0, 9,-3, 1, 5, 2.3?
- How big can the flower area be?
- How small can it be? (In particular, can it be 0, or is there some limit?)
- How does the flower area change when the length of DN is doubled?

While the discussion of these questions was to be in terms of pre-algebra concepts and skills, such as coordinate plotting and simple area calculation, the questions themselves pointed beyond pre-algebra;

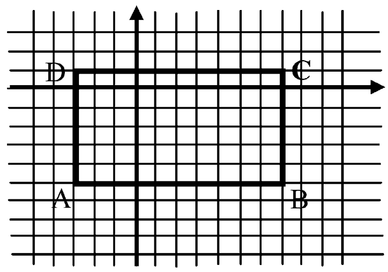


Figure 3. A pre-algebra task.

it was hard to avoid bringing in such ideas as constants, variables, and functional dependence – precisely the ideas which would be focus of the next turn of the spiral.

Algebra and Functions

Next, then, tasks and questions appropriate for students learning algebra and rudimentary ideas of functions were formulated, and teachers teaching algebra and functions were addressed directly. In this case, a variable could be introduced: DK (= DN) was given to be equal to x. The following problems were then considered.

- Find all other lengths equal to x.
- Find expressions in terms of x for the lengths AN, AM, CL, and CK.
- Find the areas of triangles BML and DNK, ANM and CKL in terms of x.
- Find the area of parallelogram KLMN in terms of x.
- For what value of x will KLMN be a rhombus?

Having done this, the teachers were then asked to interpret the expressions in terms of functions: Think of the expression Area (KLMN) = $16x-2x^2$ as a function, $f(x)=16x-2x^2$ (see Figure 4).

Consider your students' approach to the following tasks

- Find the values of x such that $f(x)$ will be less than 24. Interpret your result.
- Find the domain and range of f.
- Find value of x for which $f(x)$ will be maximal.
- Sketch the graph of f.
- As x increases, when does the flower area increase most quickly, and when least quickly?"

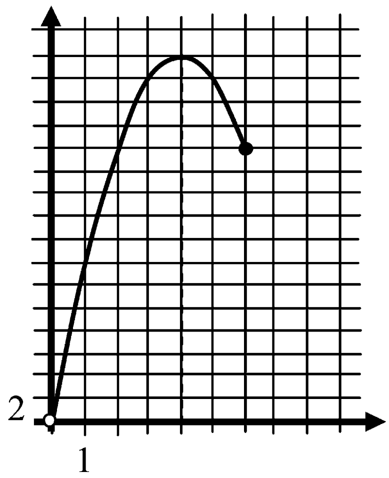
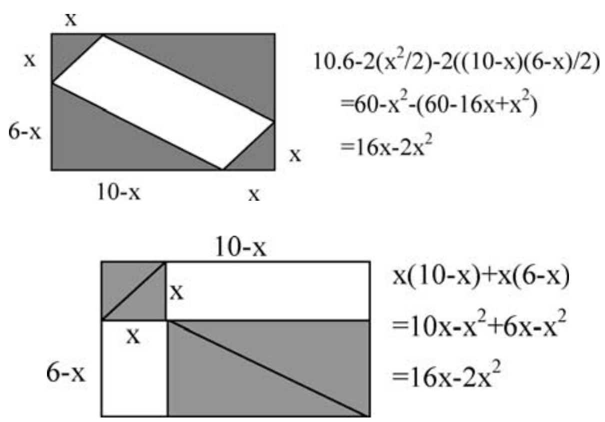


Figure 4. Introducing functions.

'Synthesis' I

Teachers used to teaching pre-algebra students sometimes had trouble moving on to the algebraic approach. But more often, and more telling, was the fact that teachers used to teaching the upper grades could not always find an interpretation of the problem situation *without* using algebra or functional notions; algebra had become, for them, a crutch that was hard for them to do without. Teachers of the upper grades had to work out explanations with the teachers of the lower grades appropriate for the latter's students. To give some structure to the discussion, we suggested diagrams such as Figure 5, which showed how geometrically intuitive arguments appropriate to 7^{th} grade teachers might lead to the algebraic arguments used by the other teachers.



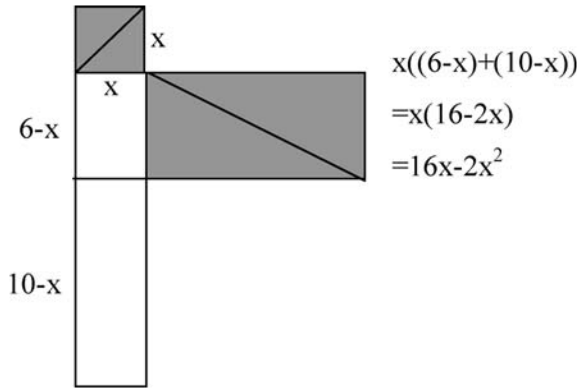


Figure 5. The pre-algebra, algebra synthesis: different ways of finding an expression for the area of KLMN.

We called these junctures in the activity, "syntheses." These syntheses, being somewhat non-routine, were not in the camp of one group of teachers more than the other, and, in this way, they were meant to encourage free discussion between them.

Beginning Calculus

The next level in the activity addressed teachers whose business was to help students make the first steps in calculus (these were, generally, teachers of the 10^{th} and 11^{th} grades). The main question here was, as one might guess, "Find the maximum of the function relating the length of DN, x, to the area of the flower bed." A slight generalization was also discussed: Let DN = BL = x and DK = BM = px ($p > 0$)

- Find the function relating x to the area of the flower bed.
- Find the maximum of the function for different values of p.
- Draw the family of functions obtained by letting p vary.
- Show that every function in this family passes through the point (3,30)

'Synthesis' II

Once again, a synthesis was proposed between the teaching approaches for the algebra stage and the early calculus stage. In this synthesis, questions such as these were asked:

- *Without the use of derivatives*, find the maximum of each of the functions in the family described above.

- Compare the calculus and non-calculus methods.
- Interpret the steps of the methods in terms of diagrams like those in the pre-post algebra synthesis above.
- Give an algebraic and geometrical explanation for why the graphs of the functions all pass through the point (3,30), regardless of the value of p.

It is important to realize that this synthesis was meant to take in the earlier synthesis as well. This is evident in the formulation of the third question, from which, after considerable discussion, the following method was suggested for finding the maximum of the function without resorting to derivatives:

Clearly, the graph of each function $f(x) = (10+6p)x - 2px^2$ (or $f(x) = x((10+6p) - 2px)$) is a parabola, and, thus, the maximum is obtained along the axis of symmetry, which is half way between 0 and $(10+6p)/2p$ (the two zeros of the function), or

$$x_{\text{sym.}} = (10 + 6p)/4p$$

Geometrically, this follows from the fact that the greatest of all the rectangles inscribed in a triangle with one side A along the base of the triangle is that whose width is half the altitude of the triangle. Consider the case when $p = 1$, the case of the original problem.

Draw again the second diagram in the pre-post algebra synthesis above. The original figure is ACGE, where the sides of the square AD are x, AC = 10 and AE = 6, and the flower-bed area is rectangle DC + rectangle DE.

Draw a copy of the 6×6 square ABFE on the other side of CG (Figure 6). Then the flower area is equal to rectangle JD – this is just a variation on the third figure in the pre-post algebra synthesis.

This rectangle is inscribed in the isosceles right triangle AHL whose altitude is half AH = 1/2(10 + 6) = 1/2(16) = 8. The greatest rectangle JD, then, is such that KD is half the altitude of AHL, or 4.

Extensions

Finally, we considered extensions of the problem, for example, questions about "envelopes": With the length DK (= BL) ranging over all real values p, we ask the teachers to consider the family of lines KL. They are all tangent to a certain curve (see Figure 7). Conjecture what the curve is. Try and prove your conjecture.

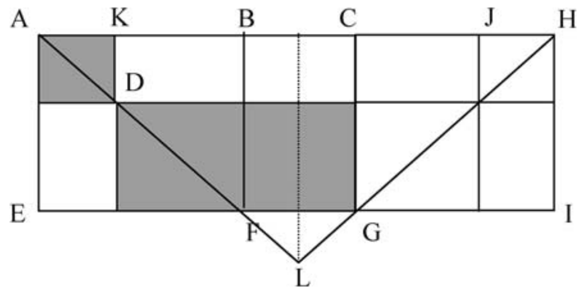


Figure 6. The algebra, early calculus synthesis.

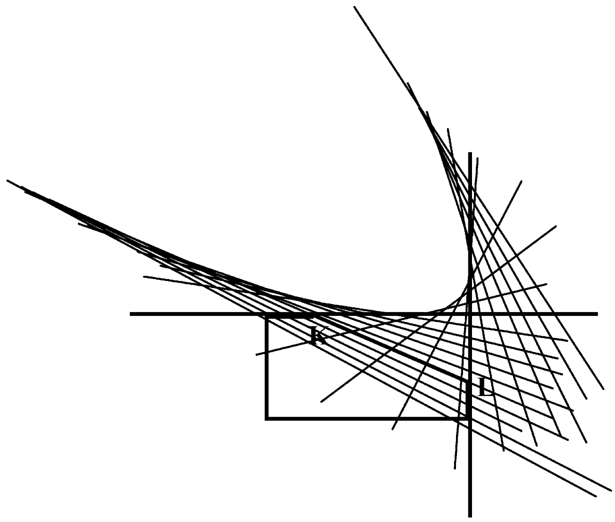
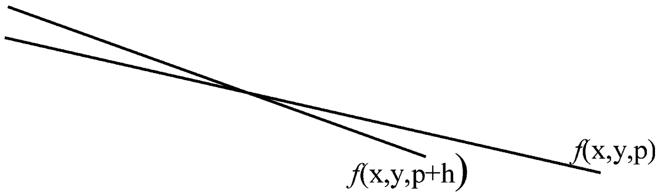


Figure 7. An extension: 'envelopes'.

A few teachers with strong mathematics backgrounds knew that the equation of the envelope (which turns out to be $x^2+2xy+y^2-24x+40y-240=0$ or a parabola rotated through 45°) by solving the system

$$\begin{cases} f(x,y,p)=0 \\ \frac{\partial}{\partial p}(f(x,y,p)=0 \end{cases}$$

where $f(x,y,p)=(p-10)(y-6)-(6-p)(x-p)$. But this left out the teachers with a more elementary background and, therefore, defeated our purpose. For this reason, we suggested the alternate approach via the intersection of 'neighboring' lines:



where again $f(x,y,p)$ represents the family of lines: $f(x,y,p) = (p-10)(y-6)-(6-p)(x-p) = 0$. Thus, we could begin with the system

$$f(x,y,p) = (p-10)(y-6) - (6-p)(x-p) = 0$$
$$f(x,y,p+h) = (p+h-10)(y-6) - (6-(p+h))(x-(p+h)) = 0$$

From which it follows that $x+y=2p+h$. Letting $h \to 0$, we have, $x+y=2p$ or $p=(x+y)/2$ Substituting this in $f(x,y,p)=(p-10)(y-6)-(6-p)(x-p)=0$, we could again find the equation of the envelope

$$x^2 + 2xy + y^2 - 24x - 40y + 240 = 0$$

See appendix for further mathematical details.

Admittedly, the method of 'neighboring' lines (which is really only a *heuristic* method) was still difficult for many of the teachers, and, indeed, it involves some subtleties. But because it *also* involves procedures such as finding a line through two points and solving systems of linear equations (with parameters), which were familiar to all the teachers, we felt that most of the teachers could grasp the general idea behind the method, even when they did not always follow the details. In this way, an extension such as this, with or without the actual derivation of the envelope, was important in pointing to directions where explorations with parameters, suggested in the calculus turn of the spiral, might lead. In fact, we found there are advantages to be gained by presenting the extension *without* finally providing the derivation of the envelope. This was done with at least one group of teachers who subsequently engaged in a very fruitful discussion of how the equation of the envelope *might* be found, illustrating our remark above regarding the value of questions over answers.

DISCUSSION AND CONCLUDING THOUGHTS

The spiral approach described in this paper aimed to encourage discussion among groups of teachers who in general would pursue in-service training independently of one another. Based upon our experience, activities like it have the potential to lead not only to a mathematics teacher population possessing a deeper understanding of the mathematics it teaches, but also to a population charged with a sense of cooperation, both within each individual school and within the region as a whole. In general, our experience in the *Kidumatica* program with this and other such activities was that the simultaneous presence of middle school teachers, high school teachers, and department chair people contributed to very fruitful and vigorous conversations (the sometimes surprising results of these interactions has been reported previously in Louzoun et al. (2000)). The positive reactions of teachers to such discussion-centered activities over the 7 years during which the *Kidumatica* program was in existence, was also clearly consistent with results on teacher–teacher, teacher–researcher co-learner partnerships (e.g., Britt et al., 2001, p.31; Horn, 2000; Jenlink & Carr, 1996)

The teachers' own impressions of the spiral task and their own grasp of what we were trying to do could be gauged from the feedback sheets which the teachers filled out immediately after the spiral task workshop. Feedback sheets such as these were given following every *Kidumatica* activity. On them, teachers were asked to write the name of the activity, react to the content and quality of the activity and rate its relevance to the teachers' classroom practice and personal enrichment, using the categories, 'highly relevant', 'has possibility', 'not relevant'. The teachers regarded these feedback sheets seriously, and our experience was that they generally answered honestly and even critically. Thus, it was notable that with almost no exceptions teachers rated the spiral task as 'highly relevant' to their personal enrichment. As for their classroom practice, here too most gave the spiral task a rating of 'highly relevant, though a fair number gave the slightly lower rating of 'has possibility'. No teacher rated the task as 'not relevant'. Comments on the content of the activity made it clear that the teachers understood the aim of the activity and saw it as relevant to their teaching practice. For example: '[The workshop was] relevant to what goes on in different classes and different levels'; '[The workshop was] interesting, especially the possibility of presenting [the activity] in several classes and at different levels'.

One revealing detail in these feedback sheets was the various ways teachers listed the name of the activity. Officially, it was called 'Flowers and Grass' because of the opening problem situation. Many of the teachers indeed listed the activity in this way, but more than a few wrote other names such as 'The Spiral Task', 'A Subject Running Through Every Level' and 'Problems: 7^{th} to 12^{th} Grade'. These names (the recording of which was not *supposed* to supply us with interesting information) demonstrated to us, almost more than the explicit comments and ratings, teachers' appreciation of our intention to find common ground for communication among the teachers teaching at different grade and aptitude levels.

Finally, although we have emphasized the importance of the 'spiral activity' as a spur to communication and mutual appreciation among teachers of different grade and achievement levels, its other goals should not be forgotten. For we should also want to stress its potential to help teachers concretely in the design of mathematics lessons (in pre-algebra, algebra, and analysis subject areas) which have depth and which link smoothly to what the students have learned and will learn. In this way, the 'spiral activity' presented here – and this explains why it was important to present the activity in so much mathematical detail – can serve as a paradigm, a model, which mathematics teaching staffs can use to develop activities appropriate for the specific concerns in their own schools. Thus, the 'spiral activity' can be reproduced in the practice of the teaching staff as well as that of the individual teacher, and this is very important for an in-service program, such as *Kidumatica*, dedicated not only to each teacher individually, but to the teaching community as an integral whole.

APPENDIX

Further Details of the Envelope of Lines

As p (= DK = BL) ranges over all real values a family of lines KL is produced. We assume that these lines are all tangent to a certain curve – the envelope of the family. Now, the general idea of the 'neighboring' lines method for finding the envelope is this: suppose line KL in the family is tangent to the envelope at point P and another line in the family K′L′ meets KL at point Q. Then, provided the envelope is sufficiently smooth, point Q will approach point P as K′L′ approaches KL (as in the Figure A1).

Thus, we find the points on the envelope by consideration of the points of intersection of the two lines K′L′ and KL – the 'neighboring' lines.

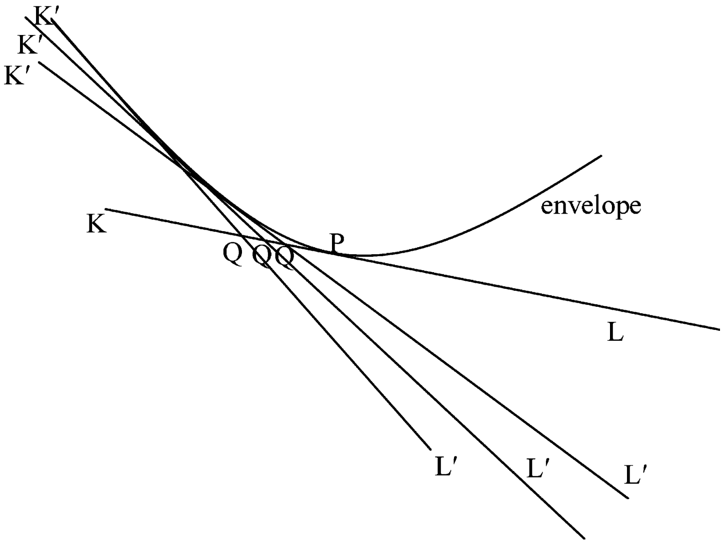


Figure A1.

In the particular case described in the paper, each line KL passes through the points K(p,6) and L(10,p), so that the family of lines KL is given by the expression

$f(x, y, p) = (y - 6)(p - 10) - (x - p)(6 - p) = 0$(see the figure below).

By altering p slightly, by adding to p a small increment h, we obtain the neighboring line K′L′ , which is, accordingly,

$$f(x, y, p + h) = ((p + h) - 10)(y - 6) - (6 - (p + h))(x - (p + h)) = 0.$$

Hence, the point Q is given by the solution of the system

$$\begin{cases} f(x, y, p) = (p - 10)(y - 6) - (6 - p)(x - p) = 0 \\ f(x, y, p + h) = (p + h - 10)(y - 6) - ((6 - (p + h))(x - (p + h)) = 0 \end{cases}$$

Subtracting the upper from the lower equation, we find

$$h[x + y - (2p + h)] = 0$$

which, after dividing through by h, gives us

$$x + y = 2p + h.$$

Now comes the slightly ‘illegal’ move: we let $h \rightarrow 0$ to obtain $x + y = 2p$ or

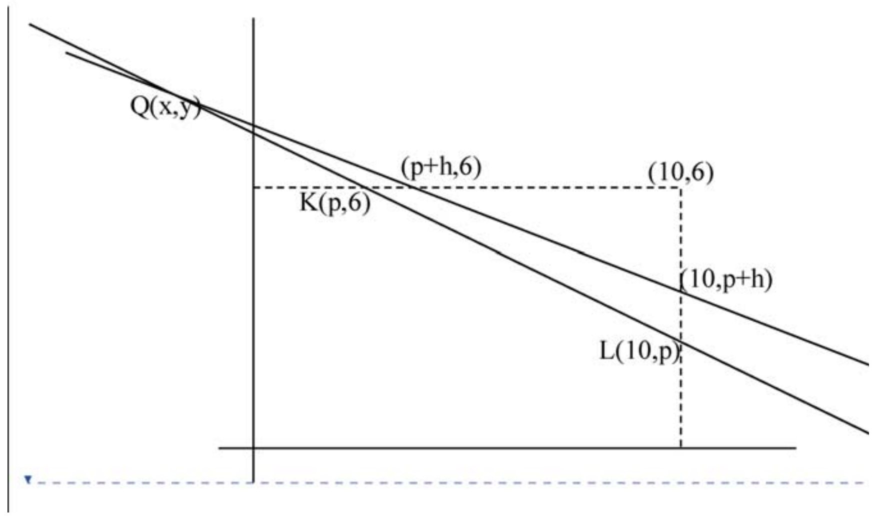


Figure A2.

$$p = (x + y)/2$$

It is in this move that we are allowing K′L′ to approach KL until it coincides with KL (in fact, by finding the difference between the two equations and dividing by h, we are finding $[f(x,y,p+h) - f(x,y,p)]/h$, which, in the limit, is the partial derivative $\partial f / \partial p$ appearing the more mathematical correct derivation of the envelope).

Substituting $p = (x + y)/2$ in the equation $f(x,y,p) = (p-10)\ (y - 6) - (6 - p)(x - p) = 0$ and expanding and simplifying, we obtain the equation of the envelope

$$x^2 + 2xy + y^2 - 24x - 40y + 240 = 0.$$

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