UNIVERSIDAD DE CHILE
Facultad de Ciencias
Departamento de Matemáticas
MC-140 − Matemáticas I

Ayudantías 09−A y 09−B

Composición de Funciones

Semana del Viernes 16 al Jueves 22 de Mayo

1. Si $f :]-\infty, 8[\longrightarrow \mathbb{R},\ f(x) = \sqrt{10-x}$ **y** $g :]0, +\infty[\longrightarrow \mathbb{R},\ g(x) = x^2 + 4$**, determinar el dominio y recorrido de:**

a) $f \circ g$ *b)* $g \circ f$

Solución: **Para a)**

$$\begin{aligned} \mathrm{Dom}(f \circ g) &= \{x \in \mathrm{Dom}(g) \mid g(x) \in \mathrm{Dom}(f)\} = \{x \in]0, +\infty[\mid x^2 + 4 \in]-\infty, 8[\ \} \\ &= \{x \in]0, +\infty[\mid x^2 + 4 < 8\} = \{x \in]0, +\infty[\mid x^2 < 4\} = \{x \in]0, +\infty[\mid x < 2\} \end{aligned}$$

$$\boxed{\mathrm{Dom}(f \circ g) =]0, 2[}$$

Una vez encontrado el dominio, conviene escribir la regla de asignación de la función; es decir:

$$f \circ g :]0, 2[\longrightarrow \mathbb{R}\ ,\ f \circ g(x) = f(g(x)) = \sqrt{10 - g(x)} = \sqrt{10 - (x^2 + 4)}$$

$$\boxed{f \circ g :]0, 2[\longrightarrow \mathbb{R}\ ,\ f \circ g(x) = \sqrt{6 - x^2}}$$

$$\begin{aligned} \mathrm{Rec}(f \circ g) &= \{y \in \mathrm{Cod}(f \circ g) \mid y = f \circ g(x) \ \wedge \ x \in \mathrm{Dom}(f \circ g)\} \\ &= \{y \in \mathbb{R} \mid y = \sqrt{6 - x^2} \ \wedge \ 0 < x < 2 \ \wedge \ y \geq 0\} \\ &= \{y \in \mathbb{R} \mid y^2 = 6 - x^2 \ \wedge \ 0 < x^2 < 4 \ \wedge \ y \geq 0\} \\ &= \{y \in \mathbb{R} \mid 6 - y^2 = x^2 \ \wedge \ 0 < x^2 < 4 \ \wedge \ y \geq 0\} \\ &= \{y \in \mathbb{R} \mid 0 < 6 - y^2 < 4 \ \wedge \ y \geq 0\} = \{y \in \mathbb{R} \mid -6 < -y^2 < -2 \ \wedge \ y \geq 0\} \\ &= \{y \in \mathbb{R} \mid 6 > y^2 > 2 \ \wedge \ y \geq 0\} = \{y \in \mathbb{R} \mid \sqrt{6} > y > \sqrt{2}\} \end{aligned}$$

$$\boxed{\mathrm{Rec}(f \circ g) =]\sqrt{2}, \sqrt{6}[}$$

Para b):

$$\begin{aligned} \mathrm{Dom}(g \circ f) &= \{x \in \mathrm{Dom}(f) \mid f(x) \in \mathrm{Dom}(g)\} = \{x \in]-\infty, 8[\mid \sqrt{10 - x} \in]0, +\infty[\ \} \\ &= \{x \in]-\infty, 8[\mid 10 - x > 0\} = \{x \in]-\infty, 8[\mid 10 > x\} \end{aligned}$$

$$\boxed{\mathrm{Dom}(g \circ f) =]-\infty, 8[}$$

$$g \circ f :]-\infty, 8[\longrightarrow \mathbb{R}\ ,\ g \circ f(x) = g(f(x)) = (f(x))^2 + 4 = (\sqrt{10 - x})^2 + 4 = 10 - x + 4$$

$$\boxed{g \circ f :]-\infty, 8[\longrightarrow \mathbb{R}\ ,\ g \circ f(x) = 14 - x}$$

$$\begin{aligned} \mathrm{Rec}(g \circ f) &= \{y \in \mathrm{Cod}(g \circ f) \mid y = g \circ f(x) \ \wedge \ x \in \mathrm{Dom}(g \circ f)\} = \{y \in \mathbb{R} \mid y = 14 - x \ \wedge \ x < 8\} \\ &= \{y \in \mathbb{R} \mid x = 14 - y \ \wedge \ x < 8\} = \{y \in \mathbb{R} \mid 14 - y < 8\} = \{y \in \mathbb{R} \mid 6 < y\} \end{aligned}$$

$$\boxed{\mathrm{Rec}(g \circ f) =]6, +\infty[} \quad \square.$$

2. Si $f :]-\infty, -2[\longrightarrow \mathbb{R},\ f(x) = \sqrt{3-x}$ y $g :]2, +\infty[\longrightarrow \mathbb{R},\ g(x) = x^2 + 2$, determinar el dominio y recorrido de:

a) $f \circ g$ *b)* $g \circ f$

Solución: **Para a)**

$$\begin{aligned} \text{Dom}(f \circ g) &= \{x \in \text{Dom}(g) \mid g(x) \in \text{Dom}(f)\} \ = \ \{x \in]2, +\infty[\mid x^2 + 2 \in]-\infty, -2[\ \} \\ &= \{x \in]2, +\infty[\mid x^2 + 2 < -2\} \ = \ \{x \in]2, +\infty[\mid x^2 < -4\} \end{aligned}$$

$$\boxed{\text{Dom}(f \circ g) = \varnothing} \qquad \text{(significa que ningún } g(x) \text{ está en Dom}(f)\text{)}$$

Para b):

$$\begin{aligned} \text{Dom}(g \circ f) &= \{x \in \text{Dom}(f) \mid f(x) \in \text{Dom}(g)\} \ = \ \{x \in]-\infty, -2[\mid \sqrt{3-x} \in]2, +\infty[\ \} \\ &= \{x \in]-\infty, -2[\mid \sqrt{3-x} > 2 > 0\} \ = \ \{x \in]-\infty, -2[\mid 3 - x > 4 > 0\} \\ &= \{x \in]-\infty, -2[\mid -1 > x\} \quad \longrightarrow \quad \boxed{\text{Dom}(g \circ f) =]-\infty, -2[} \end{aligned}$$

$$g \circ f :]-\infty, -2[\longrightarrow \mathbb{R}\ ,\ g \circ f(x) = g(f(x)) \ = \ (f(x))^2 + 2 \ = \ (\sqrt{3-x})^2 + 2 \ = \ 3 - x + 2$$

$$\boxed{g \circ f :]-\infty, -2[\longrightarrow \mathbb{R}\ ,\ g \circ f(x) = 5 - x}$$

$$\begin{aligned} \text{Rec}(g \circ f) &= \{y \in \text{Cod}(g \circ f) \mid y = g \circ f(x) \ \wedge \ x \in \text{Dom}(g \circ f)\} \ = \ \{y \in \mathbb{R} \mid y = 5 - x \ \wedge \ x < -2\} \\ &= \{y \in \mathbb{R} \mid x = 5 - y \ \wedge \ x < -2\} \ = \ \{y \in \mathbb{R} \mid 5 - y < -2\} \ = \ \{y \in \mathbb{R} \mid 7 < y\} \end{aligned}$$

$$\boxed{\text{Rec}(f) =]7, +\infty[} \ \square.$$

3. Calcular el dominio y recorrido más amplios para $f : \text{Dom}(f) \longrightarrow \mathbb{R}\ ,\ f(x) = \sqrt{1 - \sqrt{1 - x^2}}$

Solución: En este caso conviene simplificar el problema y ver la función como composición de funciones:

$$g(x) = \sqrt{1 - x^2} \quad (\text{Dom}(g) = [-1, 1]) \quad , \quad h(x) = \sqrt{1 - x} \quad (\text{Dom}(h) = [-\infty, 1]) \quad , \quad f(x) = h \circ g(x)$$

$$\begin{aligned} \text{Dom}(f) &= \{x \in \text{Dom}(g) \mid g(x) \in \text{Dom}(h)\} \ = \ \{x \in [-1, 1] \mid \sqrt{1 - x^2} \leq 1 \ \wedge \ \sqrt{1 - x^2} \in \mathbb{R}\} \\ &= \{x \in [-1, 1] \mid 0 \leq 1 - x^2 \leq 1\} \ = \ \{x \in [-1, 1] \mid -1 \leq -x^2 \leq 0\} \\ &= \{x \in [-1, 1] \mid 1 \geq x^2 \geq 0\} \quad \longrightarrow \quad \boxed{\text{Dom}(f) = [-1, 1]} \end{aligned}$$

$$\begin{aligned} \text{Rec}(f) &= \{y \in \text{Cod}(f) \mid y = f(x) \ \wedge \ x \in \text{Dom}(f)\} \\ &= \{y \in \mathbb{R} \mid y = \sqrt{1 - \sqrt{1 - x^2}} \ \wedge \ -1 \leq x \leq 1 \ \wedge \ y \geq 0\} \\ &= \{y \in \mathbb{R} \mid y^2 = 1 - \sqrt{1 - x^2} \ \wedge \ -1 \leq x \leq 1 \ \wedge \ y \geq 0\} \\ &= \{y \in \mathbb{R} \mid 1 - y^2 = \sqrt{1 - x^2} \ \wedge \ -1 \leq x \leq 1 \ \wedge \ 1 - y^2 \geq 0 \ \wedge \ y \geq 0\} \\ &= \{y \in \mathbb{R} \mid (1 - y^2)^2 = 1 - x^2 \ \wedge \ 0 \leq x^2 \leq 1 \ \wedge \ 1 - y^2 \geq 0 \ \wedge \ y \geq 0\} \\ &= \{y \in \mathbb{R} \mid \underline{1 - (1 - y^2)^2 = x^2 \ \wedge \ 0 \leq x^2 \leq 1} \ \wedge \ 1 - y^2 \geq 0 \ \wedge \ y \geq 0\} \\ &= \{y \in \mathbb{R} \mid 0 \leq 1 - (1 - y^2)^2 \leq 1 \ \wedge \ 1 - y^2 \geq 0 \ \wedge \ y \geq 0\} \\ &= \{y \in \mathbb{R} \mid -1 \leq -(1 - y^2)^2 \leq 0 \ \wedge \ 1 - y^2 \geq 0 \ \wedge \ y \geq 0\} \\ &= \{y \in \mathbb{R} \mid \underline{1 \geq (1 - y^2)^2 \geq 0 \ \wedge \ 1 - y^2 \geq 0} \ \wedge \ y \geq 0\} \\ &= \{y \in \mathbb{R} \mid 1 \geq 1 - y^2 \geq 0 \ \wedge \ y \geq 0\} \ = \ \{y \in \mathbb{R} \mid 0 \geq -y^2 \geq -1 \ \wedge \ y \geq 0\} \\ &= \{y \in \mathbb{R} \mid 0 \leq y^2 \leq 1 \ \wedge \ y \geq 0\} \ = \ \{y \in \mathbb{R} \mid 0 \leq y \leq 1\} \end{aligned}$$

$$\boxed{\text{Rec}(f) = [0, 1]} \ \square.$$

4. Si $f : \mathbb{R} \longrightarrow \mathbb{R},\ f(x) = \begin{cases} x^2 + 2 & \text{si } x > 0 \\ x + 2 & \text{si } x \leq 0 \end{cases}$ **y** $g : \mathbb{R} \longrightarrow \mathbb{R}\ ,\ g(x) = 2x + 5$

Determinar el dominio y la regla de asignación de $f \circ g$**.**

Solución:

$$\text{Dom}(f \circ g) = \{x \in \text{Dom}(g) \mid g(x) \in \text{Dom}(f)\} = \{x \in \mathbb{R} \mid 2x + 5 \in \mathbb{R}\}$$

$$\boxed{\text{Dom}(f \circ g) = \mathbb{R}}$$

$$f \circ g : \mathbb{R} \longrightarrow \mathbb{R}\ ,\ f \circ g(x) = f(g(x)) = \begin{cases} (g(x))^2 + 2 & \text{si } g(x) > 0 \\ g(x) + 2 & \text{si } g(x) \leq 0 \end{cases} = \begin{cases} (2x+5)^2 + 2 & \text{si } 2x + 5 > 0 \\ 2x + 5 + 2 & \text{si } 2x + 5 \leq 0 \end{cases}$$

$$\boxed{f \circ g : \mathbb{R} \longrightarrow \mathbb{R}\ ,\ f \circ g(x) = \begin{cases} 4x^2 + 20x + 27 & \text{si } x > -5/2 \\ 2x + 7 & \text{si } x \leq -5/2 \end{cases}} \ \square.$$

5. Si $f : \mathbb{R} \longrightarrow \mathbb{R},\ f(x) = \begin{cases} x^2 - 1 & \text{si } x \leq 1 \\ \dfrac{1}{x} & \text{si } x > 1 \end{cases}$ **y** $g : [0, +\infty[\longrightarrow \mathbb{R}\ ,\ g(x) = \sqrt{x}$

Determinar el dominio y la regla de asignación de:

a) $f \circ g$ *b)* $g \circ f$

Solución: **Para a)** $\text{Dom}(f \circ g) = \{x \in \text{Dom}(g) \mid g(x) \in \text{Dom}(f)\} = \{x \in [0, +\infty[\mid \sqrt{x} \in \mathbb{R}\}$

$$\boxed{\text{Dom}(f \circ g) = [0, +\infty[}$$

$$f \circ g : [0, +\infty[\longrightarrow \mathbb{R}\ ,\ f \circ g(x) = f(g(x)) = \begin{cases} (g(x))^2 - 1 & \text{si } g(x) \leq 1 \\ \dfrac{1}{g(x)} & \text{si } g(x) > 1 \end{cases} = \begin{cases} (\sqrt{x})^2 - 1 & \text{si } \sqrt{x} \leq 1 \\ \dfrac{1}{\sqrt{x}} & \text{si } \sqrt{x} > 1 \end{cases}$$

$$\boxed{f \circ g : [0, +\infty[\longrightarrow \mathbb{R}\ ,\ f \circ g(x) = \begin{cases} x - 1 & \text{si } x \leq 1 \\ \dfrac{1}{\sqrt{x}} & \text{si } x > 1 \end{cases}}$$

Para b):

$$\begin{aligned} \text{Dom}(g \circ f) &= \{x \in \text{Dom}(f) \mid f(x) \in \text{Dom}(g)\} \\ &= \{x \in \mathbb{R} \mid (x^2 - 1 \in [0, +\infty[\wedge x \leq 1) \vee (\frac{1}{x} \in [0, +\infty[\wedge x > 1)\} \\ &= \{x \in \mathbb{R} \mid (x^2 - 1 \geq 0 \ \wedge \ x \leq 1) \vee (\frac{1}{x} \geq 0 \ \wedge \ x > 1)\} \\ &= \{x \in \mathbb{R} \mid (x^2 \geq 1 \ \wedge \ x \leq 1) \vee (\frac{1}{x} > 0 \ \wedge \ x > 1)\} \\ &= \{x \in \mathbb{R} \mid x \leq -1 \ \vee \ x > 1\} \end{aligned}$$

$$\boxed{\text{Dom}(g \circ f) = [-\infty, -1] \cup]1, +\infty[}$$

$$g \circ f : [-\infty, -1] \cup]1, +\infty[\longrightarrow \mathbb{R}\ ,\ g \circ f(x) = g(f(x)) = \sqrt{f(x)} = \begin{cases} \sqrt{x^2 - 1} & \text{si } \overbrace{x \leq 1}^{x \leq -1} \\ \sqrt{\dfrac{1}{x}} & \text{si } x > 1 \end{cases}$$

$$\boxed{g \circ f :] - \infty, -1] \cup]1, +\infty[\longrightarrow \mathbb{R}\ ,\ g \circ f(x) = \begin{cases} \sqrt{x^2 - 1} & \text{si } x \leq -1 \\ \sqrt{\dfrac{1}{x}} & \text{si } x > 1 \end{cases}}$$

6. Si $f:\mathbb{R}\longrightarrow\mathbb{R},\ f(x)=\begin{cases} x & \text{si } x<0 \\ x^2 & \text{si } x\geq 0\end{cases}$ **y** $g:\mathbb{R}\longrightarrow\mathbb{R}\ ,\ g(x)=\dfrac{x+|x|}{2}$

Determinar el dominio y la regla de asignación de:

a) $f\circ g$ *b)* $g\circ f$

Solución: **Para a)** $\text{Dom}(f\circ g)=\ \{x\in\text{Dom}(g)\mid g(x)\in\text{Dom}(f)\}\ =\ \{x\in\mathbb{R}\mid \dfrac{x+|x|}{2}\in\mathbb{R}\}$

$$\boxed{\text{Dom}(f\circ g)=\mathbb{R}}$$

$$f\circ g:\mathbb{R}\longrightarrow\mathbb{R}\ ,\ f\circ g(x)=f(g(x))\ =\ \begin{cases} g(x) & \text{si } g(x)<0 \\ (g(x))^2 & \text{si } g(x)\geq 0\end{cases}\ =\ \begin{cases} \dfrac{x+|x|}{2} & \text{si } \dfrac{x+|x|}{2}<0 \\ \left(\dfrac{x+|x|}{2}\right)^2 & \text{si } \dfrac{x+|x|}{2}\geq 0\end{cases}$$

Se puede verificar que $x+|x|\geq 0$ (hágalo); luego:

$$f\circ g:\mathbb{R}\longrightarrow\mathbb{R}\ ,\ f\circ g(x)\ =\ \left(\frac{x+|x|}{2}\right)^2=\frac{x^2+2x|x|+|x|^2}{4}=\frac{x^2+2x|x|+x^2}{4}=\frac{2x^2+2x|x|}{4}$$

$$\boxed{f\circ g:\mathbb{R}\longrightarrow\mathbb{R}\ ,\ f\circ g(x)=\frac{x^2+x|x|}{2}}$$

Para b):

$$\begin{aligned}\text{Dom}(g\circ f)=&\ \{x\in\text{Dom}(f)\mid f(x)\in\text{Dom}(g)\}\ =\ \{x\in\mathbb{R}\mid (x\in\mathbb{R}\ \wedge\ x<0)\ \vee\ (x^2\in\mathbb{R}\ \wedge\ x\geq 0)\}\\ =&\ \{x\in\mathbb{R}\mid x<0\ \vee\ x\geq 0\}\qquad\longrightarrow\qquad \boxed{\text{Dom}(g\circ f)=\mathbb{R}}\end{aligned}$$

$$g\circ f:\mathbb{R}\longrightarrow\mathbb{R}\ ,\ g\circ f(x)=g(f(x))\ =\ \frac{f(x)+|f(x)|}{2}\ =\ \begin{cases} \dfrac{x+|x|}{2} & \text{si } x<0 \\ \dfrac{x^2+|x^2|}{2} & \text{si } x\geq 0\end{cases}\ =\ \begin{cases} 0 & \text{si } x<0 \\ \dfrac{x^2+x^2}{2} & \text{si } x\geq 0\end{cases}$$

$$\boxed{g\circ f:\mathbb{R}\longrightarrow\mathbb{R}\ ,\ g\circ f(x)=\begin{cases} 0 & \text{si } x<0 \\ x^2 & \text{si } x\geq 0\end{cases}}$$