UNIVERSIDAD DE CHILE
Facultad de Ciencias
Departamento de Matemáticas
MC-140 − Matemáticas I

Ayudantías 08−A y 08−B

Más sobre Dominio, Recorrido y Álgebra de Funciones

Semana del Viernes 09 al Jueves 15 de Mayo

1. Determinar el dominio y recorrido para $f : \text{Dom}(f) \longrightarrow \,]0,+\infty[\;,\; f(x) = \dfrac{x}{x+6}$**.**

Solución:

$$\begin{aligned}
\text{Dom}(f) &= \{x \in \mathbb{R} \mid f(x) \in \text{Cod}(f)\} = \{x \in \mathbb{R} \mid f(x) \in \,]0,+\infty[\,\} = \{x \in \mathbb{R} \mid \frac{x}{x+6} > 0\} \\
&= \{x \in \mathbb{R} \mid (x > 0 \ \wedge\ x+6 > 0) \ \vee\ (x < 0 \ \wedge\ x+6 < 0)\} \\
&= \{x \in \mathbb{R} \mid (x > 0 \ \wedge\ x > -6) \ \vee\ (x < 0 \ \wedge\ x < -6)\} \\
&= \{x \in \mathbb{R} \mid x > 0 \ \vee\ x < -6\}
\end{aligned}$$

$$\boxed{\text{Dom}(f) =]-\infty,-6[\,\cup\,]0,+\infty[}$$

$$\begin{aligned}
\text{Rec}(f) &= \{y \in \text{Cod}(f) \mid y = f(x)\ ,\ x \in \text{Dom}(f)\} = \{y \in \,]0,+\infty[\, \mid y = \frac{x}{x+6} \ \wedge\ x < -6 \ \wedge\ x > 0\} \\
&= \{y \in \,]0,+\infty[\, \mid y(x+6) = x \ \wedge\ x < -6 \ \wedge\ x > 0\} \\
&= \{y \in \,]0,+\infty[\, \mid xy + 6y = x \ \wedge\ x < -6 \ \wedge\ x > 0\} \\
&= \{y \in \,]0,+\infty[\, \mid 6y = x - xy \ \wedge\ x < -6 \ \wedge\ x > 0\} \\
&= \{y \in \,]0,+\infty[\, \mid 6y = x(1-y) \ \wedge\ x < -6 \ \wedge\ x > 0\} \\
&= \{y \in \,]0,+\infty[\, \mid \frac{6y}{1-y} = x \ \wedge\ x < -6 \ \wedge\ x > 0 \ \wedge\ y \neq 1\} \\
&= \{y \in \,]0,+\infty[\, \mid \left(\frac{6y}{1-y} < -6 \ \vee\ \frac{6y}{1-y} > 0\right) \ \wedge\ y \neq 1\} \\
&= \{y \in \,]0,+\infty[\, \mid \left(\frac{6y}{1-y} + 6 < 0 \ \vee\ (\ (6y > 0 \ \wedge\ 1-y > 0) \ \vee\ (6y < 0 \ \wedge\ 1-y < 0)\ )\right) \ \wedge\ y \neq 1\} \\
&= \{y \in \,]0,+\infty[\, \mid \left(\frac{\cancel{6y} + 6 - \cancel{6y}}{1-y} < 0 \ \vee\ (\ (y > 0 \ \wedge\ 1 > y) \ \vee\ \underbrace{(y < 0 \ \wedge\ 1 < y)}_{\varnothing}\ )\right) \ \wedge\ y \neq 1\} \\
&= \{y \in \,]0,+\infty[\, \mid \frac{6}{1-y} < 0 \ \vee\ \ 0 < y < 1 \ \ \wedge\ y \neq 1\} \\
&= \{y \in \,]0,+\infty[\, \mid 1 - y < 0 \ \vee\ \ 0 < y < 1 \ \ \wedge\ y \neq 1\} \\
&= \{y \in \,]0,+\infty[\, \mid 1 < y \ \vee\ \ 0 < y < 1 \ \ \wedge\ y \neq 1\}
\end{aligned}$$

$$\boxed{\text{Rec}(f) =]0,1[\,\cup\,]1,+\infty[} \quad \square.$$

2. Calcular el recorrido de $f:\mathbb{R}\longrightarrow\mathbb{R}$, $f(x)=x^2+5x+2$.

Solución:

$$\begin{aligned}
\operatorname{Rec}(f) = &\ \{y\in \operatorname{Cod}(f) \mid y=f(x)\ ,\ x\in \operatorname{Dom}(f)\} \ = \ \{y\in\mathbb{R}|\ y=x^2+5x+2\ \wedge\ x\in\mathbb{R}\}\\
= &\ \{y\in\mathbb{R}|\ y=(x^2+2(5/2)x+25/4)-25/4+2\ \wedge\ x\in\mathbb{R}\}\\
= &\ \{y\in\mathbb{R}|\ y=(x+(5/2))^2-(17/4)\ \wedge\ x\in\mathbb{R}\}\\
= &\ \{y\in\mathbb{R}|\ y+(17/4)=(x+(5/2))^2\ \wedge\ x\in\mathbb{R}\ \wedge\ (x+(5/2))^2\geq 0\}\\
= &\ \{y\in\mathbb{R}|\ y+(17/4)\geq 0\}\\
= &\ \{y\in\mathbb{R}|\ y\geq -17/4\}
\end{aligned}$$

$$\boxed{\operatorname{Rec}(f)=[-17/4,+\infty[}\quad \square.$$

3. Calcular el dominio y recorrido más amplios para las siguientes funciones:

a) $f:\operatorname{Dom}(f)\longrightarrow\mathbb{R}$, $f(x)=\sqrt{1-x^2}$

b) $f:\operatorname{Dom}(f)\longrightarrow\mathbb{R}$, $f(x)=\sqrt{25-x^2}$

c) $f:\operatorname{Dom}(f)\longrightarrow\mathbb{R}$, $f(x)=3+\sqrt{x+2}$

Solución: **Para a):**

$$\begin{aligned}
\operatorname{Dom}(f) = &\ \{x\in\mathbb{R} \mid f(x)\in \operatorname{Cod}(f)\} \ = \ \{x\in\mathbb{R} \mid f(x)\in\mathbb{R}\ \} \ = \ \{x\in\mathbb{R} \mid \sqrt{1-x^2}\in\mathbb{R}\}\\
= &\ \{x\in\mathbb{R} \mid 1-x^2\geq 0\} \ = \ \{x\in\mathbb{R} \mid (1+x)(1-x)\geq 0\}\\
= &\ \{x\in\mathbb{R} \mid (1+x\geq 0\ \wedge\ 1-x\geq 0)\ \vee\ (1+x\leq 0\ \wedge\ 1-x\leq 0)\}\\
= &\ \{x\in\mathbb{R} \mid (x\geq -1\ \wedge\ 1\geq x)\ \vee\ (\underbrace{x\leq -1\ \wedge\ 1\leq x}_{\varnothing})\}\\
= &\ \{x\in\mathbb{R} \mid\ -1\leq x\leq 1\}
\end{aligned}$$

$$\boxed{\operatorname{Dom}(f)=[-1,1]}$$

$$\begin{aligned}
\operatorname{Rec}(f) = &\ \{y\in \operatorname{Cod}(f) \mid y=f(x)\ ,\ x\in \operatorname{Dom}(f)\} \ = \ \{y\in\mathbb{R}|\ y=\sqrt{1-x^2}\ \wedge\ -1\leq x\leq 1\ \wedge\ y\geq 0\}\\
= &\ \{y\in\mathbb{R}|\ y^2=1-x^2\ \wedge\ 0\leq x^2\leq 1\ \wedge\ y\geq 0\} \ = \ \{y\in\mathbb{R}|\ x^2=1-y^2\ \wedge\ 0\leq x^2\leq 1\ \wedge\ y\geq 0\}\\
= &\ \{y\in\mathbb{R}|\ 0\leq 1-y^2\leq 1\ \wedge\ y\geq 0\} \ = \ \{y\in\mathbb{R}|\ -1\leq -y^2\leq 0\ \wedge\ y\geq 0\}\\
= &\ \{y\in\mathbb{R}|\ 0\leq y^2\leq 1\ \wedge\ y\geq 0\} \ = \ \{y\in\mathbb{R}|\ -1\leq y\geq 1\ \wedge\ y\geq 0\} \ = \ \{y\in\mathbb{R}|\ 0\leq y\leq 1\}
\end{aligned}$$

$$\boxed{\operatorname{Rec}(f)=[0,1]}$$

Para b):

$$\begin{aligned}
\operatorname{Dom}(f) = &\ \{x\in\mathbb{R} \mid f(x)\in \operatorname{Cod}(f)\} \ = \ \{x\in\mathbb{R} \mid f(x)\in\mathbb{R}\ \} \ = \ \{x\in\mathbb{R} \mid \sqrt{25-x^2}\in\mathbb{R}\}\\
= &\ \{x\in\mathbb{R} \mid 25-x^2\geq 0\} \ = \ \{x\in\mathbb{R} \mid (5+x)(5-x)\geq 0\}\\
= &\ \{x\in\mathbb{R} \mid (5+x\geq 0\ \wedge\ 5-x\geq 0)\ \vee\ (5+x\leq 0\ \wedge\ 5-x\leq 0)\}\\
= &\ \{x\in\mathbb{R} \mid (x\geq -5\ \wedge\ 5\geq x)\ \vee\ (\underbrace{x\leq -5\ \wedge\ 5\leq x}_{\varnothing})\}\\
= &\ \{x\in\mathbb{R} \mid\ -5\leq x\leq 5\} \quad\longrightarrow\quad \boxed{\operatorname{Dom}(f)=[-1,1]}
\end{aligned}$$

$$\begin{aligned}\text{Rec}(f) = &\ \{y \in \text{Cod}(f) \mid y = f(x)\ ,\ x \in \text{Dom}(f)\} \ = \ \{y \in \mathbb{R} |\ y = \sqrt{25 - x^2} \ \wedge\ -5 \leq x \leq 5 \ \wedge\ y \geq 0\} \\ = &\ \{y \in \mathbb{R} |\ y^2 = 25 - x^2 \ \wedge\ 0 \leq x^2 \leq 25 \ \wedge\ y \geq 0\} \ = \ \{y \in \mathbb{R} |\ x^2 = 25 - y^2 \ \wedge\ 0 \leq x^2 \leq 25 \ \wedge\ y \geq 0\} \\ = &\ \{y \in \mathbb{R} |\ 0 \leq 25 - y^2 \leq 25 \ \wedge\ y \geq 0\} \ = \ \{y \in \mathbb{R} |\ -25 \leq -y^2 \leq 0 \ \wedge\ y \geq 0\} \\ = &\ \{y \in \mathbb{R} |\ 0 \leq y^2 \leq 25 \ \wedge\ y \geq 0\} \ = \ \{y \in \mathbb{R} |\ -5 \leq y \geq 5 \ \wedge\ y \geq 0\} \ = \ \{y \in \mathbb{R} |\ 0 \leq y \leq 5\}\end{aligned}$$

$$\boxed{\text{Rec}(f) = [0, 5]}$$

Para c):

$$\begin{aligned}\text{Dom}(f) = &\ \{x \in \mathbb{R} \mid f(x) \in \text{Cod}(f)\} \ = \ \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\ \} \ = \ \{x \in \mathbb{R} \mid 3 + \sqrt{x + 2} \in \mathbb{R}\} \\ = &\ \{x \in \mathbb{R} \mid x + 2 \geq 0\} \ = \ \{x \in \mathbb{R} \mid x \geq -2\}\end{aligned}$$

$$\boxed{\text{Dom}(f) = [-2, +\infty[}$$

$$\begin{aligned}\text{Rec}(f) = &\ \{y \in \text{Cod}(f) \mid y = f(x)\ ,\ x \in \text{Dom}(f)\} \ = \ \{y \in \mathbb{R} |\ y = 3 + \sqrt{x + 2} \ \wedge\ x \geq -2\} \\ = &\ \{y \in \mathbb{R} |\ y - 3 = \sqrt{x + 2} \ \wedge\ x \geq -2\} \ = \ \{y \in \mathbb{R} |\ (y - 3)^2 = x + 2 \ \wedge\ x \geq -2\} \\ = &\ \{y \in \mathbb{R} |\ (y - 3)^2 - 2 = x \ \wedge\ x \geq -2\} \ = \ \{y \in \mathbb{R} |\ (y - 3)^2 - 2 \geq -2\} \\ = &\ \{y \in \mathbb{R} |\ (y - 3)^2 \geq 0\}\end{aligned}$$

$$\boxed{\text{Rec}(f) = \mathbb{R}} \quad \square.$$

4. Si $f : \mathbb{R} \longrightarrow \mathbb{R},\ f(x) = \begin{cases} 3x - 1 & \text{si } x > 3 \\ x^2 - 2 & \text{si } -2 \leq x \leq 3 \\ 2x + 3 & \text{si } x < -2 \end{cases}$

***a*) Calcular** $f(\sqrt{2})$**.**

***b*) Determinar** $\text{Rec}(f)$**.**

Solución: **Para a):**

$$\sqrt{2} \approx 1{,}41 \quad \longrightarrow \quad -2 \leq \sqrt{2} \leq 3 \quad \longrightarrow \quad f(\sqrt{2}) = (\sqrt{2})^2 - 2 = 2 - 2 = 0$$

Para b):

Notar que $\boxed{\text{Dom}(f) = \mathbb{R}}$

Como se trata de una función por tramos, el recorrido también se determina por tramos:

$$\begin{aligned}\text{Rec}_1(f) = &\ \{y \in \text{Cod}(f) \mid y = f(x)\ ,\ x > 3\} \ = \ \{y \in \mathbb{R} |\ y = 3x - 1 \ \wedge\ x > 3\} \\ = &\ \{y \in \mathbb{R} |\ \frac{y + 1}{3} = x \ \wedge\ x > 3\} \ = \ \{y \in \mathbb{R} |\ \frac{y + 1}{3} > 3\} \\ = &\ \{y \in \mathbb{R} |\ y + 1 > 9\} \ = \ \{y \in \mathbb{R} |\ y > 8\}\end{aligned}$$

$$\boxed{\text{Rec}_1(f) =]8, +\infty[}$$

$$\begin{aligned}
\mathrm{Rec}_2(f) = &\ \{y \in \mathrm{Cod}(f) \mid y = f(x)\ ,\ -2 \le x \le 3\} \ = \ \{y \in \mathbb{R} |\ y = x^2 - 2 \ \wedge\ -2 \le x \le 3\} \\
= &\ \{y \in \mathbb{R} |\ y + 2 = x^2 \ \wedge\ -2 \le x \le 3 \ \wedge\ y + 2 \ge 0\} \\
= &\ \{y \in \mathbb{R} |\ (-2 \le x < 0 \ \wedge\ y + 2 = x^2) \ \vee\ (0 \le x \le 3 \ \wedge\ y + 2 = x^2) \ \wedge\ y \ge -2\} \\
= &\ \{y \in \mathbb{R} |\ (0 < x^2 \le 4 \ \wedge\ y + 2 = x^2) \ \vee\ (0 \le x^2 \le 9 \ \wedge\ y + 2 = x^2) \ \wedge\ y \ge -2\} \\
= &\ \{y \in \mathbb{R} |\ (0 < y + 2 \le 4 \ \vee\ 0 \le y + 2 \le 9) \ \wedge\ y \ge -2\} \\
= &\ \{y \in \mathbb{R} |\ (-2 < y \le 2 \ \vee\ -2 \le y \le 7) \ \wedge\ y \ge -2\} \\
= &\ \{y \in \mathbb{R} |\ -2 \le y \le 7 \ \wedge\ y \ge -2\} \ = \ \{y \in \mathbb{R} |\ -2 \le y \le 7\}
\end{aligned}$$

$$\boxed{\mathrm{Rec}_2(f) = [-2, 7]}$$

$$\begin{aligned}
\mathrm{Rec}_3(f) = &\ \{y \in \mathrm{Cod}(f) \mid y = f(x)\ ,\ x < -2\} \ = \ \{y \in \mathbb{R} |\ y = 2x + 3 \ \wedge\ x < -2\} \\
= &\ \{y \in \mathbb{R} |\ \frac{y-3}{2} = x \ \wedge\ x < -2\} \ = \ \{y \in \mathbb{R} |\ \frac{y-3}{2} < -2\} \\
= &\ \{y \in \mathbb{R} |\ y - 3 > -4\} \ = \ \{y \in \mathbb{R} |\ y < -1\}
\end{aligned}$$

$$\boxed{\mathrm{Rec}_3(f) =]-\infty, -1[}$$

$$\longrightarrow \quad \mathrm{Rec}(f) = \mathrm{Rec}_1(f) \cup \mathrm{Rec}_2(f) \cup \mathrm{Rec}_3(f) \quad \longrightarrow \quad \mathrm{Rec}(f) =]-\infty, -1[\,\cup\, [-2, 7] \,\cup\,]8, +\infty[$$

$$\boxed{\mathrm{Rec}(f) =]-\infty, 7] \cup]8, +\infty[} \quad \square.$$

5. Calcular el dominio máximo de las siguientes funciones:

a) $f : \mathrm{Dom}(f) \longrightarrow \mathbb{R}\ ,\ f(x) = \dfrac{1}{x-1} + \dfrac{1}{x-2}$

b) $f : \mathrm{Dom}(f) \longrightarrow \mathbb{R}\ ,\ f(x) = \sqrt{1-x} + \sqrt{x-2}$

c) $f : \mathrm{Dom}(f) \longrightarrow \mathbb{R}\ ,\ f(x) = \sqrt{1-x} + \sqrt{x+2}$

d) $f : \mathrm{Dom}(f) \longrightarrow \mathbb{R}\ ,\ f(x) = \sqrt{4-x^2} \cdot \sqrt{x^2-1}$

Solución:

En todos los casos, conviene simplificar el problema y ver la función como suma (o producto) de dos funciones.

Para a):

$$\begin{aligned}
\mathrm{Dom}(f) = &\ \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\} \ = \ \{x \in \mathbb{R} \mid \frac{1}{x-1} + \frac{1}{x-2} \in \mathbb{R}\} \ = \ \{x \in \mathbb{R} \mid \frac{1}{x-1} \in \mathbb{R} \ \wedge\ \frac{1}{x-2} \in \mathbb{R}\} \\
= &\ \{x \in \mathbb{R} \mid x \ne 1 \ \wedge\ x \ne 2\}
\end{aligned}$$

$$\boxed{\mathrm{Dom}(f) =]-\infty, 1[\,\cup\,]1, 2[\,\cup\,]2, +\infty[}$$

Para b):

$$\begin{aligned}
\mathrm{Dom}(f) = &\ \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\} \ = \ \{x \in \mathbb{R} \mid \sqrt{1-x} + \sqrt{x-2} \in \mathbb{R}\} \ = \ \{x \in \mathbb{R} \mid \sqrt{1-x} \in \mathbb{R} \ \wedge\ \sqrt{x-2} \in \mathbb{R}\} \\
= &\ \{x \in \mathbb{R} \mid 1 - x \ge 0 \ \wedge\ x - 2 \ge 0\} \ = \ \{x \in \mathbb{R} \mid \underbrace{1 \ge x \ \wedge\ x \ge 2}_{\varnothing}\}
\end{aligned}$$

$$\boxed{\mathrm{Dom}(f) = \varnothing} \qquad \text{(significa que } f \text{ está mal definida)}$$

Para c):

$$\begin{aligned} \text{Dom}(f) = & \ \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\} \ = \ \{x \in \mathbb{R} \mid \sqrt{1-x} + \sqrt{x+2} \in \mathbb{R}\} \\ = & \ \{x \in \mathbb{R} \mid \sqrt{1-x} \in \mathbb{R} \ \wedge \ \sqrt{x+2} \in \mathbb{R}\} \ = \ \{x \in \mathbb{R} \mid 1-x \geq 0 \ \wedge \ x+2 \geq 0\} \\ = & \ \{x \in \mathbb{R} \mid 1 \geq x \ \wedge \ x \geq -2\} \ = \ \{x \in \mathbb{R} \mid -2 \leq x \leq 1\} \end{aligned}$$

$$\boxed{\text{Dom}(f) = [-2, 1]}$$

Para d):

$$\begin{aligned} \text{Dom}(f) = & \ \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\} \ = \ \{x \in \mathbb{R} \mid \sqrt{4-x^2} + \sqrt{x^2-1} \in \mathbb{R}\} \\ = & \ \{x \in \mathbb{R} \mid \sqrt{4-x^2} \in \mathbb{R} \ \wedge \ \sqrt{x^2-1} \in \mathbb{R}\} \ = \ \{x \in \mathbb{R} \mid 4-x^2 \geq 0 \ \wedge \ x^2-1 \geq 0\} \\ = & \ \{x \in \mathbb{R} \mid (2+x)(2-x) \geq 0 \ \wedge \ (x+1)(x-1) \geq 0\} \end{aligned}$$

$$\begin{aligned} \text{Dom}(f) = & \ \{x \in \mathbb{R} \mid (2+x \geq 0 \ \wedge \ 2-x \geq 0) \ \vee \ (2+x \leq 0 \ \wedge \ 2-x \leq 0) \\ & \wedge \ (x+1 \geq 0 \ \wedge \ x-1 \geq 0) \ \vee \ (x+1 \leq 0 \ \wedge \ x-1 \leq 0)\} \end{aligned}$$

$$\begin{aligned} \text{Dom}(f) = & \ \{x \in \mathbb{R} \mid (x \geq -2 \ \wedge \ 2 \geq x) \ \vee \ (\underbrace{x \leq -2 \ \wedge \ 2 \leq x}_{\varnothing}) \\ & \wedge \ (x \geq -1 \ \wedge \ x \geq 1) \ \vee \ (x \leq -1 \ \wedge \ x \leq 1)\} \end{aligned}$$

$$\text{Dom}(f) \ = \ \{x \in \mathbb{R} \mid (-2 \leq x \leq 2) \ \wedge \ (1 \leq x \ \vee \ x \leq -1)\} \ = \ \{x \in \mathbb{R} \mid -2 \leq x \leq -1 \ \vee \ 1 \leq x \leq 2\}$$

$$\boxed{\text{Dom}(f) = [-2, -1] \ \cup \ [1, 2]} \quad \square.$$