Ayudantía 15
(09/01/2020)

P1) Encuentre el polinomio de Taylor de grado n, centrado en c para:

a) $f(x) = \sqrt{x}$, $n=4$, $c=4$ y aproxime el valor de $f(4,1)$

Sol: Buscamos

$$P_{4,4}(x) = f(4) + \frac{f'(4)}{1!}(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3 + \frac{f''''(4)}{4!}(x-4)^4$$

Calculemos las derivadas:

- $f'(x) = \frac{1}{2\sqrt{x}}$, $f''(x) = \left(\frac{1}{2\sqrt{x}}\right)' = \frac{1}{2}(x^{-1/2})' = \frac{1}{2}\cdot\frac{-1}{2}x^{-3/2} = \frac{-1}{4}x^{-3/2}$

- $f'''(x) = \left(\frac{-1}{4}x^{-3/2}\right)' = \frac{-1}{4}(x^{-3/2})' = \frac{-1}{4}\cdot\frac{-3}{2}x^{-5/2} = \frac{3}{8}x^{-5/2}$

- $f''''(x) = \left(\frac{3}{8}x^{-5/2}\right)' = \frac{3}{8}(x^{-5/2})' = \frac{3}{8}\cdot\frac{-5}{2}x^{-7/2} = \frac{-15}{16}x^{-7/2}$

Luego:

- $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$, $f''(4) = \frac{-1}{4}(4)^{-3/2} = \frac{-1}{4}\frac{1}{(\sqrt{4})^3} = \frac{-1}{32}$

- $f'''(4) = \frac{3}{8}(4)^{-5/2} = \frac{3}{8}\frac{1}{(\sqrt{4})^5} = \frac{3}{8\cdot 2^5} = \frac{3}{256}$

- $f''''(4) = \frac{-15}{16}(4)^{-7/2} = \frac{-15}{16(\sqrt{4})^7} = \frac{-15}{16\ 2^7} = \frac{-15}{2048}$

Así, el polinomio de Taylor de grado 4 centrado en 4 para $f(x) = \sqrt{x}$ es:

$$P_{4,4}(x) = 2 + \frac{(x-4)}{4} - \frac{(x-4)^2}{2\cdot 32} + \frac{3\cdot(x-4)^3}{6\cdot 256} - \frac{15(x-4)^4}{24\cdot 2048}$$

$$= 2 + \frac{(x-4)}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512} - \frac{5(x-4)^4}{16384}$$

$$\Rightarrow \sqrt{x} \approx 2 + \frac{(x-4)}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512} - \frac{5(x-4)^4}{16384} \text{ si } x \text{ está cerca de } 4$$

Sea $x = 4{,}1$ donde $4{,}1$ está cerca de 4, así:

$$f(4,1) = \sqrt{4,1} \approx 2 + \frac{(4,1-4)}{4} - \frac{(4,1-4)^2}{64} + \frac{(4,1-4)^3}{512} - \frac{5(4,1-4)^4}{16384}$$

$$\approx 2 + \frac{(0,1)}{4} - \frac{(0,1)^2}{64} + \frac{(0,1)^3}{512} - \frac{5(0,1)^4}{16384}$$

$$\approx 2{,}02484567$$

· <u>OBS</u>: $f(4,1) = \sqrt{4,1} = 2{,}02484567$ ✓

b) $f(x) = x^2 \operatorname{sen}(x)$, $n=3$, $c=\pi$ y aproxime el valor de $f(3)$

<u>Sol</u>. Buscamos

$$P_{3,\pi}(x) = f(\pi) + \frac{f'(\pi)(x-\pi)}{1!} + \frac{f''(\pi)(x-\pi)^2}{2!} + \frac{f'''(\pi)(x-\pi)^3}{3!}$$

Calculemos las derivadas:

- $f'(x) = (x^2 \operatorname{sen}(x))' = 2x \operatorname{sen}(x) + x^2 \cos(x)$
- $f''(x) = (2x\operatorname{sen}(x) + x^2\cos(x))' = 2\operatorname{sen}(x) + 2x\cos(x) + 2x\cos(x) - x^2\operatorname{sen}(x) = (2-x^2)\operatorname{sen}(x) + 4x\cos(x)$
- $f'''(x) = ((2-x^2)\operatorname{sen}(x) + 4x\cos(x))'$
$= -2x\operatorname{sen}(x) + (2-x^2)\cos(x) + 4\cos(x) - 4x\operatorname{sen}(x)$
$= -6x\operatorname{sen}(x) + (2-x^2+4)\cos(x)$
$= -6x\operatorname{sen}(x) + (6-x^2)\cos(x)$

Luego:

- $f(\pi) = \pi^2 \operatorname{sen}(\pi) = 0$
- $f'(\pi) = 2\cdot\pi \cancel{\operatorname{sen}(\pi)} + \pi^2\cos(\pi) = -\pi^2$
- $f''(\pi) = (2-\pi^2)\cancel{\operatorname{sen}(\pi)} + 4\pi\cos(\pi) = -4\pi$

- $f'''(\pi) = -6\pi \cancel{\operatorname{sen}(\pi)} + (6-\pi^2)\cos(\pi) = -(6-\pi^2) = \pi^2 - 6$

Así, el polinomio de Taylor de grado 3 centrado en π para $f(x) = x^2 \operatorname{sen}(x)$ es:

$$P_{3,\pi}(x) = -\pi^2(x-\pi) - 4\pi\frac{(x-\pi)^2}{2} + \frac{(\pi^2-6)(x-\pi)^3}{6}$$

$$\Rightarrow x^2 \operatorname{sen}(x) \approx -\pi^2(x-\pi) - 4\pi\frac{(x-\pi)^2}{2} + \frac{(\pi^2-6)(x-\pi)^3}{6} \quad \text{si } x \text{ está cerca de } \pi$$

Como 3 está cerca de π entonces:

$$f(3) \approx P_{3,\pi}(3)$$

Luego $$f(3) \approx -\pi^2(3-\pi) - 4\pi\frac{(3-\pi)^2}{2} + \frac{(\pi^2-6)(x-\pi)^3}{6}$$

$$\approx 1.269664379\,32$$

y $f(3) = 1.27008007254$ //

P2 Encuentre el desarrollo en serie indicado, para:

a) $f(x) = \frac{1}{1+x}$, Maclaurin.

Sol: Nos piden la serie de Taylor de f centrada en 0, es decir

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Calculemos las derivadas:

- $f^{(1)}(x) = \left(\frac{1}{1+x}\right)' = \frac{-1}{(1+x)^2} \quad \Rightarrow f^{(1)}(0) = -1$

- $f^{(2)}(x) = \left(\dfrac{-1}{(1+x)^2}\right)' = \dfrac{2(1+x)}{(1+x)^4} = \dfrac{2}{(1+x)^3} \Rightarrow f^{(2)}(0) = 2$
- $f^{(3)}(x) = \left(\dfrac{2}{(1+x)^3}\right)' = \dfrac{-2\cdot 3}{(1+x)^4} \Rightarrow f^{(3)}(0) = -6$
- $f^{(4)}(x) = \left(\dfrac{-6}{(1+x)^4}\right)' = \dfrac{2\cdot 3\cdot 4}{(1+x)^5} \Rightarrow f^{(4)}(0) = 24$

$\therefore\ f^{(n)}(x) = \dfrac{(-1)^n\, n!}{(1+x)^{n+1}} \Rightarrow f^{(n)}(0) = \dfrac{(-1)^n\, n!}{(1+0)^{n+1}} = (-1)^n\, n!,\ \forall n \geq 0$

Así, $\quad f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cancel{n!}}{\cancel{n!}} x^n = \sum_{n=0}^{\infty} (-1)^n x^n$ //

b) $f(x) = e^{-x^2}$, Maclaurin

Sol: Nuevamente, buscamos:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Vamos a buscar la de $f(x) = e^x$ (es más sencilla)
Así:

$$f^{(1)}(x) = f^{(2)}(x) = \dots = f^{(n)}(x)$$

$$\Rightarrow \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ pues } f^{(n)}(0) = 1$$

luego $\quad e^{-x^2} = \displaystyle\sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

c) $f(x) = \ln(x)$, Taylor en $c=1$

Sol Nos piden:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

Veamos las derivadas:

- $f^{(1)}(x) = \frac{1}{x} \Rightarrow f^{(1)}(1) = 1$
- $f^{(2)}(x) = -\frac{1}{x^2} \Rightarrow f^{(2)}(1) = -1$
- $f^{(3)}(x) = \frac{2}{x^3} \Rightarrow f^{(3)}(1) = 2$
- $f^{(4)}(x) = -\frac{6}{x^4} \Rightarrow f^{(4)}(1) = -6$
- $f^{(5)}(x) = \frac{24}{x^5} \Rightarrow f^{(5)}(1) = 24$

$\vdots$

Luego: $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}, \forall n \geq 1 \Rightarrow f^{(n)}(1) = (-1)^{n-1}(n-1)!, \forall n \geq 1$

y $f(1) = \ln(1) = 0$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!(x-1)^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$$

d) $f(x) = \ln(1-x)$, Maclaurin

Sol: Buscamos:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Veamos las derivadas:

- $f^{(1)}(x) = \frac{-1}{1-x} \Rightarrow f^{(1)}(0) = -1$
- $f^{(2)}(x) = \frac{-1}{(1-x)^2} \Rightarrow f^{(2)}(0) = -1$
- $f^{(3)}(x) = \frac{-2}{(1-x)^3} \Rightarrow f^{(3)}(0) = -2$
- $f^{(4)}(x) = \frac{-6}{(1-x)^4} \Rightarrow f^{(4)}(0) = -6$
- $f^{(5)}(x) = \frac{-24}{(1-x)^5} \Rightarrow f^{(5)}(0) = -24$

Luego $f^{(n)}(x) = -(n-1)!, \forall n \geq 1 \Rightarrow f^{(n)}(0) = -(n-1)!, \forall n \geq 1$

y $f(0) = \ln(1-0) = \ln(1) = 0$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=1}^{\infty} \frac{-(n-1)!}{n!} x^n = \sum_{n=1}^{\infty} -\frac{x^n}{n} \quad //$$