Universidad de Chile
Programa de Bachillerato

Control 2 de Algebra y Geometría

9 de Abril de 2015

Nombre:...

1. Demuestre:

   a) Sean A y B matrices de orden n invertibles tales que $A+B$ es invertible. Pruebe que $A^{-1}+B^{-1}$ es invertible y $(A^{-1}+B^{-1})^{-1} = A(A+B)^{-1}B$.

   b) Sean M y N matrices invertibles y M una matriz simétrica. Demuestre que si

$$M^tN^{-1} + (N^{-1}M^t)^t - (N^tM^{-1})^{-1} = I_n \text{ entonces } M = N.$$

a) $A(A+B)^{-1}B = A[B^{-1}(A+B)]^{-1}$

$= [B^{-1}(A+B)\cdot A^{-1}]^{-1}$

$= (B^{-1}AA^{-1} + B^{-1}BA^{-1})^{-1}$

$= (B^{-1}I + IA^{-1})^{-1}$

$= \boxed{(A^{-1}+B^{-1})^{-1}}$

Como la inversa es única, por lo que:

$(A^{-1}+B^{-1}) = A(A+B)^{-1}B$

1 pto

b) $M^tN^{-1} + (N^{-1}M^t)^t - (N^tM^{-1})^{-1} = I_m$

$M^tN^{-1} + M(N^{-1})^t - M\cdot(N^t)^{-1} = I_m$

$M^tN^{-1} + \cancel{M(N^t)^{-1}} - M\cancel{(N^t)^{-1}} = I_m$

$M^tN^{-1} = I_m \quad \cdot N$

$M^t = N \quad / M = M^t$

$\boxed{M = N}$

2. Considere el siguiente sistema lineal:

$$\begin{aligned} -2x_1 + x_2 + x_3 - 2x_4 + x_5 &= 6 \\ x_1 + x_2 + x_4 - 2x_5 &= -1 \\ 3x_1 - x_2 + x_3 + 4x_4 - 6x_5 &= 0 \end{aligned}$$

0,5

a) Escriba el sistema de forma matricial $Ax = b$.

b) Resuelva el sistema lineal escalonando la matriz aumentada $(A|b)$, dando la solución de manera matricial (si existe).

a) $\left(\begin{array}{ccccc|c} -2 & 1 & 1 & -2 & 1 & 6 \\ 1 & 1 & 0 & 1 & -2 & -1 \\ 3 & -1 & 1 & 4 & -6 & 0 \end{array}\right) \xrightarrow{F_1 \leftrightarrow F_2} \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & -2 & -1 \\ -2 & 1 & 1 & -2 & 1 & 6 \\ 3 & -1 & 1 & 4 & -6 & 0 \end{array}\right)$

$\rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & -2 & -1 \\ 0 & 3 & 1 & 0 & -3 & 4 \\ 0 & -4 & 1 & 1 & 0 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & -2 & -1 \\ 0 & 3 & 1 & 0 & -3 & 4 \\ 0 & 0 & 7/3 & 1 & -4 & 25/3 \end{array}\right)$

$\rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & -2 & -1 \\ 0 & 3 & 1 & 0 & -3 & 4 \\ 0 & 0 & 7 & 3 & -12 & 25 \end{array}\right)$ ✓ 1

x_4 y x_5 libres

$$\boxed{x_3 = \frac{25}{7} - \frac{3x_4}{7} + \frac{12}{7}x_5}$$

$3x_2 = 4 - x_3 + 3x_5$

$3x_2 = \cancel{4} - \frac{\cancel{25}}{7} + \frac{3}{7}x_4 - \frac{12}{7}x_5 + 3x_5$

$3x_2 = \frac{3}{7} + \frac{3}{7}x_4 + \frac{9}{7}x_5$

$$\boxed{x_2 = \frac{1}{7} + \frac{1}{7}x_4 + \frac{3}{7}x_5}$$

1 pto

$x_1 = -1 - x_2 - x_4 + 2x_5$

$x_1 = \cancel{-1} - \cancel{\frac{1}{7}} - \frac{1}{\cancel{7}}\cancel{x_4} - \frac{3}{7}x_5 - \cancel{x_4} + \frac{2}{1}x_5$

$$\boxed{x_1 = \frac{-8}{7} - \frac{8}{7}x_4 + \frac{11}{7}x_5}$$

$$X = \begin{pmatrix} -8/7 \\ 1/7 \\ 25/7 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 8/7 \\ 1/7 \\ -3/7 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 11/7 \\ 3/7 \\ 12/7 \\ 0 \\ 1 \end{pmatrix}$$

0,5

$$\left(\begin{array}{ccccc|c} -2 & 1 & 1 & -2 & 1 & 6 \\ 1 & 1 & 0 & 1 & -2 & -1 \\ 3 & -1 & 1 & 4 & -6 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} -2 & 1 & 1 & -2 & 1 & 6 \\ 1 & 1 & 0 & 1 & -2 & -1 \\ 0 & -4 & 1 & 1 & 0 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} -2 & 1 & 1 & -2 & 1 & 6 \\ 0 & 3/2 & 1/2 & 0 & -3/2 & 2 \\ 0 & -4 & 1 & 1 & 0 & 3 \end{array}\right) \to \left(\begin{array}{ccccc|c} -2 & 1 & 1 & -2 & 1 & 6 \\ 0 & 3 & 1 & 0 & -3 & 4 \\ 0 & 4 & 1 & 1 & 0 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} -2 & 1 & 1 & -2 & 1 & 6 \\ 0 & 3 & 1 & 0 & -3 & 4 \\ 0 & 0 & -1/3 & 1 & +4 & 7/3 \end{array}\right) \to \left(\begin{array}{ccccc|c} -2 & 1 & 1 & -2 & 1 & 6 \\ 0 & 3 & 1 & 0 & -3 & 4 \\ 0 & 0 & 1 & 3 & 1/2 & 7 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} -2 & 1 & 1 & -2 & 1 & 6 \\ 1 & 1 & 0 & 1 & -2 & -1 \\ 3 & -1 & 1 & 4 & -6 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} -2\cdot\frac{1}{2} & 1/2 & 1 & -2\cdot\frac{1}{2} & 1/2 & 6/2 \\ 1 & 1 & 0 & 1 & -2 & -1 \\ 0 & -4 & 1 & 1 & 0 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} -2 & 1 & 1 & -2 & 1 & 6 \\ 0 & 3/2 & 1/2 & 0 & -3/2 & 2 \\ 0 & -4 & 1 & 1 & 0 & 3 \end{array}\right) \to \left(\begin{array}{ccccc|c} -2 & 1 & 1 & -2 & 1 & 6 \\ 0 & \frac{1}{3}\cdot 3 & 4/1 & 0 & \frac{1}{3}\cdot -3 & 4\cdot\frac{1}{2} \\ 0 & -4 & 1 & 1 & 0 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} -2 & 1 & 1 & -2 & 1 & 6 \\ 0 & 3 & 1 & 0 & -3 & 4 \\ 0 & 0 & 7/3 & 1 & -4 & 25/3 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccccc|c} 2 & 1 & 1 & -2 & 1 & 6 \\ 0 & 3 & 1 & 0 & -3 & 4 \\ 0 & 0 & 7 & 3 & -12 & 25 \end{array}\right)$$

$$\boxed{x_3 = \frac{25}{7} - \frac{3x_4}{7} + \frac{12x_5}{7}}$$

b) $M^t N \ldots + (N \ldots$

$M^t N^{-1} + M(N^{-1})^t - M\cdot(N \ldots$

$M^t N^{-1} + \cancel{M(N^t)^{-1}} - \cancel{M(N^t)^{-1}} = I_m$

$M^t N^{-1} = I_m \quad / \cdot N$

$M^t = N \quad / M = M^t$

$M = N$

$$\boxed{X_2 = \frac{1}{7} + \frac{1}{7}X_4 + \frac{3}{7}X_5}$$

$$-2X_1 + X_2 + X_3 - 2X_4 + X_5 = 6$$

$$-2X_1 + \frac{1}{7} + \frac{X_4}{7} + \frac{3}{7}X_5 + \frac{25}{7} + \frac{3}{7}X_4 + \frac{12}{7}X_5 - 2X_4 + X_5 = 6$$

$$-2X_1 + \frac{26}{7} - \frac{2}{7}X_4 - 2X_4 + \frac{15}{7}X_5 + X_5 = 6$$

$$-\frac{42}{7} + \frac{26}{7} - \frac{16}{7}X_4 + \frac{22}{7}X_5 = 2X_1$$

$$-\frac{16}{7} - \frac{16}{7}X_4 + \frac{22}{7}X_5 = 2X_1$$

$$\therefore \boxed{X_1 = -\frac{8}{7} - \frac{8}{7}X_4 + \frac{11}{7}X_5}$$