① ⓐ Pd. V_1 y V_2 subespacios suplementarios

Pd $V = V_1 + V_2$ (i) y $V_1 \cap V_2 = \{0\}$ (ii)

(i) Sea $v \in V$, como B es base de V entonces existen escalares $\alpha_i \in K$ $(i = 1, \dots, n)$ tq

$$v = \sum_{i=1}^{n} \alpha_i v_i = \underbrace{\sum_{i=1}^{K} \alpha_i v_i}_{\in V_1} + \underbrace{\sum_{i=K+1}^{n} \alpha_i v_i}_{\in V_2}$$

$$\Rightarrow v_1 = \sum_{i=1}^{K} \alpha_i v_i, \quad v_1 \in V_1$$

$$v_2 = \sum_{i=K+1}^{n} \alpha_i v_i, \quad v_2 \in V_2$$

$\Rightarrow v = v_1 + v_2$, $v_1 \in V_1$ y $v_2 \in V_2$ $\Rightarrow V \subset V_1 + V_2$ y como $V_1 + V_2 \subset V$.

$\therefore V = V_1 + V_2$

(ii) Sup. que $v \in V_1 \cap V_2 \Rightarrow v \in V_1$ y $v \in V_2$

$\Rightarrow v = \sum_{i=1}^{K} \beta_i v_i$ ciertos escalares $\beta_i \in K$ y

$v = \sum_{i=K+1}^{n} \gamma_i v_i$ ciertos escalares $\gamma_i \in K$

Restando,

$$0 = \sum_{i=1}^{K} \beta_i v_i - \sum_{i=K+1}^{n} \gamma_i v_i = \beta_1 v_1 + \dots + \beta_K v_K + -\gamma_{K+1} v_{K+1} + \dots + -\gamma_n v_n$$

Como B es base de V $\Rightarrow \beta_i = \gamma_i = 0 \quad \forall i$

$\therefore v = 0$.

ⓑ Si $U = V$, existe el subespacio $\{0\} = S$ tq $V = V \oplus \{0\}$.

Si $U = \{0\}$, existe el subespacio $S = V$ tq $V = \{0\} \oplus V$.

Sea $U \neq \{0\}, V$, U subespacio de V

Si $B_1 = \{v_1, v_2, \dots, v_K\}$ es una base cualquiera de U, por teorema de la base incompleta, existe el conjunto $\{v_{K+1}, \dots, v_m\} = B_2$ de V tq $B = B_1 \cup B_2$ es una base de V.

Tomando $S = [v_{k+1}, \dots, v_n]$, por la parte (a) tenemos que U y S son subespacios suplementarios de V, es decir, $V = U \oplus S$.

(2) $T: \mathbb{R}^4 \to \mathbb{M}_{2\times 2}(\mathbb{R})$

$\operatorname{Im}(T) = \{A \in \mathbb{M}_{2\times 2}(\mathbb{R}) : A^t = A\}$

Sea $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ base canónica de $\mathbb{R}^4$.

Im(T): $A = A^t$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \Rightarrow \begin{array}{l} a=a \\ b=c \\ c=b \\ d=d \end{array}$$

$$\operatorname{Im}(T) = \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} / a,b,d \in \mathbb{R} \right\}$$

$$= \left\{ a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} / a,b,d \in \mathbb{R} \right\}$$

$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]$$

$$T(1,0,0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T(0,1,0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$T(0,0,1,0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T(0,0,0,1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\therefore T(x,y,z,w) = x\,T(1,0,0,0) + y\,T(0,1,0,0) + z\,T(0,0,1,0) + w\,T(0,0,0,1)$$

$$= x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + w\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T(x,y,z,w) = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$$

Sabemos que,

$$\dim \mathbb{R}^4 = \dim(\ker T) + \dim(\operatorname{Im} T)$$

$$4 = \dim(\ker T) + 3$$

$$\dim \ker(T) = 4 - 3 = 1.$$

③ $F: \mathbb{P}_n[x] \to \mathbb{P}_{n+1}[x]$

$F(p(x)) = \int_0^x p(t)\,dt + p(0)$

ⓘ Sean $p_1(x), p_2(x) \in \mathbb{P}_n[x]$ y $\lambda \in \mathbb{R}$.

$(\lambda p_1 + p_2)(x) = \lambda p_1(x) + p_2(x)$

$$F((\lambda p_1+p_2)(x)) = \int_0^x (\lambda p_1 + p_2)(x)\,dx + (\lambda p_1 + p_2)(0)$$
$$= \int_0^x \lambda p_1(x) + p_2(x)\,dx + \lambda p_1(0) + p_2(0)$$
$$= \lambda \int_0^x p_1(x)\,dx + \int_0^x p_2(x)\,dx + \lambda p_1(0) + p_2(0)$$
$$= \lambda\left[\int_0^x p_1(x)\,dx + p_1(0)\right] + \left[\int_0^x p_2(x)\,dx + p_2(0)\right]$$
$$= \lambda F(p_1(x)) + F(p_2(x)) \quad \therefore F \text{ es lineal}$$

ⓘⓘ F inyectiva?

$\ker(F) = \{p(x) \in \mathbb{P}_n[x] \,/\, F(p(x)) = 0\}$

$F(p(x)) = \int_0^x p(t)\,dt + p(0) = 0$

Si $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

$$F(p(x)) = \int_0^x a_0 + a_1t + a_2t^2 + \cdots + a_nt^n\,dt + a_0$$
$$= a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + \cdots + a_n\frac{x^{n+1}}{n+1} + a_0 = 0$$

$\Rightarrow a_0 = a_1 = a_2 = \cdots = a_n = 0$.

$\therefore \ker(F) = \{0_{\mathbb{P}_n[x]}\}$

$\therefore$ F es inyectiva

F epiyectiva?

$\dim(\mathbb{P}_n[x]) = \dim(\ker F) + \dim(\operatorname{Im} F)$

$n+1 = 0 + \dim(\operatorname{Im} F)$

$\dim(\operatorname{Im} F) = n+1$

Como $\dim(\operatorname{Im} F) \neq n+2 = \dim(\mathbb{P}_{n+1}[x])$

$\Rightarrow \operatorname{Im} F \subset \mathbb{P}_{n+1}[x]$ $\therefore$ F no es epiyectiva.

(b) $A^2 + A + I_n = 0$

$A^2 + A = -I_n$

$-(A^2 + A) = I_n$

$-A(A + I_n) = I_n$

$A(-A - I_n) = I_n$ (*)

Si $B = -A - I_n$, de (*) $A \cdot B = I_n$ y

$(-A - I_n)A = -A^2 - A = I_n \Rightarrow BA = I_n$

$\therefore$ $A \cdot B = BA = I_n$ y $B = -A - I_n$ es la matriz inversa (A es invertible)

(4)

(a) $$\left(\begin{array}{cccc|c} 1 & 2 & 4 & -3 & 2 \\ 3 & 7 & 5 & -5 & 3 \\ 5 & 12 & 6 & -7 & \alpha \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 4 & -3 & 2 \\ 0 & 1 & -7 & 4 & -3 \\ 0 & 2 & -14 & 8 & \alpha-10 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 4 & -3 & 2 \\ 0 & 1 & -7 & 4 & -3 \\ 0 & 0 & 0 & 0 & \alpha-4 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 18 & -11 & 8 \\ 0 & 1 & -7 & 4 & -3 \\ 0 & 0 & 0 & 0 & \alpha-4 \end{array}\right) (**)$$

Si $\alpha = 4$: Rang $(A) = 2 =$ Rang $(A|b) < 4 =$ nº de variables

$\therefore$ Tiene infinitas soluciones

Si $\alpha \neq 4$: Rang $(A) = 2$ y Rang $(A|b) = 3$

$\therefore$ Rang $(A) \neq$ Rang $(A|b)$

$\Rightarrow$ No hay solución.

(5) $$\ker(T) = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; A\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} = \left\{ z\begin{pmatrix} -18 \\ 7 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} 11 \\ -4 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; z, w \in \mathbb{R} \right\}$$

De (**) (b=0): $x = -18z + 11w$
$y = 7z - 4w$

$$= \left[\begin{pmatrix} -18 \\ 7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 11 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right]$$

Base de $\ker(T)$

$\operatorname{Im}(T) = \left[\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 12 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} -3 \\ -5 \\ -7 \end{pmatrix} \right]$ y $\dim(\operatorname{Im} T) =$ Rango de $A = 2$.

Como las dos primeras columnas presentan los pivotes

Base de $\operatorname{Im}(T) = \left\{ \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 12 \end{pmatrix} \right\}$.