MATHEMATICS AS THE ART OF SEEING THE INVISIBLE…

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The role of metaphors and the switch in cognitive modes in relation to visualization in learning and teaching mathematics is discussed, based on examples and case studies with students and teachers. We present some preliminary evidence supporting our claims that visualization requires the activation of various metaphors, that it is rather hampered than facilitated by traditional teaching in mathematics, but it is however a trainable capacity in teachers and students.

INTRODUCTION

When teaching mathematics to first year undergraduates who intend to major in Humanities or Social Sciences at the University of Chile, we usually ask them to concoct aphorisms to describe their ongoing experience of mathematics, as the course unfolds.

An often emerging aphorism is: "Mathematics is the art of seeing the invisible", a close kin to "Visualization offers a method of seeing the unseen" (McCormick, DeFanti, & Brown, 1987, p. 3), taken up as leitmotiv in Arcavi (2003) when discussing the role of visual representations in the learning of mathematics.

At the other end of the spectrum, in a recent paper (Faux & Gates, 2004) on supersymmetric representation theory, in theoretical physics, we read:

> The use of symbols to connote ideas which defy simple verbalization is perhaps one of the oldest of human traditions. The Asante people of West Africa have long been accustomed to using simple yet elegant motifs known as Adinkra symbols, to serve just this purpose. With a nod to this tradition, we christen our graphical symbols as "Adinkras."

Later, Prof. Gates (the first Afro-American to be awarded an endowed Chair in Physics in a major US university), referring to the unsolved mathematical problems posed by superstring theory, tells us further (Shy science star shines, 2006):

> I have been worried about them and having some ideas about new ways to approach these problems. And I have developed the math as topology, using pictures with coloured

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strings and dots (looking just like macramé). All the information from the equations is in these pictures which we have called *adinkras*. Adinkra is an Ashanti (West African) word meaning symbol with hidden meaning. Once you create these pictures from the equations, there are things you can see more clearly than you can from a page of figures and numbers

In Fig. 1 we see some examples of classical Adinkras and in Fig. 2, their analogues designed by Gates and collaborators.

Figure 1

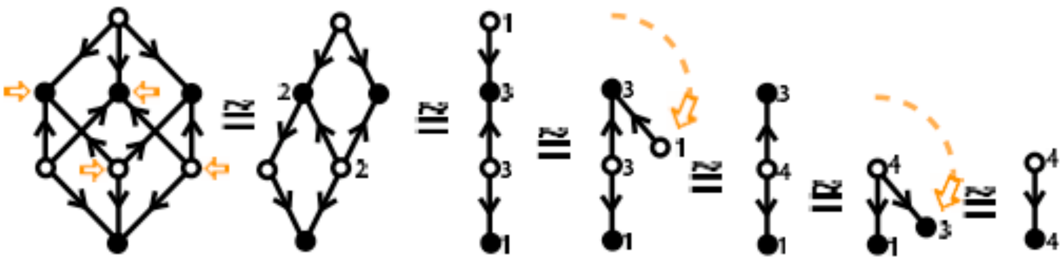


Figure 2

In the latter, which shows the folding of a cube, white nodes stand for bosons, black nodes for fermions.

Better late than never, the key role played by visualization in the teaching and learning of mathematics has received slow but increasing recognition in the last decades. See Presmeg (2006) for a comprehensive review. Nowadays its rather low status in mathematics and mathematics education is being progressively upgraded.

In this paper we intend to explore, through examples, the didactical role of visualization in mathematics together with its interplay with metaphors and transiting amongst cognitive modes. We build for that on our previous research and experimental findings (Soto-Andrade, 2006, 2007) concerning the role of metaphors and cognitive modes in the teaching-learning of mathematics.

After setting up our tentative theoretical framework, we put forward our main working hypotheses related to necessity of metaphor activation in visualization and we proceed to report on some specific examples of the role of metaphors and cognitive modes in the process of visualization. In this way we get some preliminary experimental evidence to support our working hypotheses and also suggestions for further research along these lines, that we discuss in the final section.

THEORETICAL FRAMEWORK

Nature and Role of Metaphors

It has been progressively recognized during the last decade (Araya, 2000; English, 1997; Lakoff & Núñez, 2000; Presmeg, 1997; Sfard, 1997; Soto-Andrade, 2006, 2007 and many others) that metaphors are not just rhetorical devices, but also powerful cognitive tools that help us to build or to grasp new concepts, as well as to solve problems in an efficient and friendly way. Those are indeed "conceptual metaphors" (Lakoff & Núñez, 2000) , that appear as mappings from a "source domain" into a "target domain" which carry the inferential structure of the first domain into that one of the second, and enable us to understand the second, usually more abstract and opaque, in terms of the first, more down-to-earth and transparent.

Cognitive modes: Transiting for understanding

A cognitive mode (sometimes called cognitive style) is defined nowadays as one's preferred way to think, perceive and recall, in short, to cognize. It reveals itself, for instance, when we try to solve problems. This concept emerged from work by Luria (1973) and de La Garanderie (1989).

Flessas and Lussier (2005), building on this previous work, have given a first classification of cognitive modes ("styles cognitifs|" in French), including an operational description of what they call the four basic cognitive modes and have pointed out to their impact on the teaching-learning process.

To generate the 4 basic cognitive modes, they combine 2 dichotomies: verbal – non verbal and sequential – non sequential (or simultaneous), closely related to the left – right brain hemisphere dichotomy and to the frontal – occipital dichotomy. (Luria, 1973). This gives the following table

Table 1

The 4 cognitive modes	VERBAL	NONVERBAL
SEQUENTIAL	S-V	S-NV
NONSEQUENTIAL	NS-V	NS-NV

Example: How can you check that you have the same number of fingers in your hands?

You can readily realize how to check this fact using the 4 cognitive modes (Soto-Andrade, 2007)

Visualization as an emergent process

According to Arcavi (2003):

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings.

In a systemic approach to the classroom, it may be argued that learning itself appears as an emergent process, that may encompass visualization to a greater or lesser degree.

Nowadays there is widespread interest in understanding how visualization interacts with the didactics of mathematics, and how we could enhance its usefulness in mathematics education (Woolner, 2004; Presmeg, 2006)

In what follows we adhere mainly to the framework laid by Lakoff and Núñez (2000) for metaphors, Flessas and Lussier (2005) for cognitive modes and Arcavi (2003) and Presmeg (2006) for visualization.

PROBLEMATICS AND OPEN QUESTIONS

We claim that most in-service teachers are not familiar with either visualization, metaphors or cognitive modes, in relation with their teaching of mathematics. More precisely:

Most teachers privilege the sequential-verbal cognitive mode, and devaluate visualization. They are usually unable to switch from one cognitive mode to another and "to see the invisible".

Students seem to be reluctant to visualize and to tackle a problem in a different cognitive mode that the one it came wrapped in. They have big trouble visualizing, when they try to, after having been "robotized" to calculate blindly.

The spectrum of available metaphors of most teachers is too narrow. They would rarely have more than one metaphor for each mathematical concept or process and so they have trouble creating "unlocking metaphors" or eliciting visualization by their students.

Evidence for these claims arises from observation of group work and interviewing of more than 3000 in – service primary and secondary school teachers, since 2000. The most significant reports have come out, in 2006 – 2008, of 5 classes of 30 primary school teachers each, enrolled in a 15 months program aiming at upgrading their command of elementary mathematics. On the other hand these primary school teachers have made the recurrent observation in their classrooms that from 5th grade onwards the children are quite unable to take advantage of visualization when tackling a problem. Their feeling is that this is a result of the instruction the children have received from previous teachers, which tend to repress spontaneous visualization in favour of rote learning of algorithms.

This problematics leads naturally to some key questions regarding visualization (cf. Presmeg, 2006):

- What conversion processes are involved in a smooth transit amongst different cognitive modes, specially from sequential-verbal to simultaneous-nonverbal, as an antidote to unimodal cognition?
- What is the role of metaphors in the switching from one cognitive mode to another, necessary to visualization?
- How can teachers help learners to transit between visual and verbal cognitive modes, i.e. between the iconic and the symbolic domain, for specific mathematical notions|?

Research hypotheses

Our main working hypothesis is that metaphors and cognitive modes are key ingredients in a meaningful teaching-learning process. In particular, we claim that visualization is elicited by the activation of previous metaphors that entail a switch in the cognitive mode of the learner.

This applies specifically to visualization understood as the ability to "see the invisible", typically in the case of arithmetical or algebraic objects and processes. The situation may be less clear cut in cases like the "rope around the earth problem" (Arcavi, 2003, p. 29 -30), for instance, where we imagine a rope laid around the circumference of the earth and ask how much longer should it be if we want to place it on six-foot-high poles all the way. You could argue that when you visualize both ropes around the earth and you deform their shape, from circular to square, to see better what is going on and to solve the problem, you are not activating any metaphor, because you remain in the geometric realm. However, we can also point out, that usually we transit first from the "concrete" setting of the problem to a symbolic one, by writing the formula for both perimeters involved. One calculates the answer but one is not really convinced unless one really sees it. Then a metaphor activates that allows you to transit from the algebraic formulation to the "square with rounded-off corners" visualization.

It is also one of our working hypothesis that the competences regarding multimodal cognition, use and creation of metaphors and visualization are trainable and that measurable progress can be achieved in one semester course or even in a one week workshop. This, in spite of the strongly unimodal (sequential-verbal) training most teachers have been exposed to.

Regarding students, our working hypothesis is that they would significantly improve the quality and robustness of their learning if they can draw from a suitable spectrum of metaphors and approach problems in a multimodal way, emphasizing visualization.

We present below some examples, tested in teaching mathematics courses for several years, to various audiences of students, to illustrate the didactical use of different cognitive modes, in particular visual ones, for approaching mathematical objects and their interplay with the use of metaphors.

RESEARCH BACKGROUND AND METHODOLOGY

The background for our experimental research consisted in several courses, to wit:

- Mathematics 0: A one semester general mathematics course, given since 2000 to first year students of the Bachelor Program in Humanities and Social Sciences at the University of Chile. This course aims at introducing the students to the mathematical way of thinking and cognitive attitudes of mathematics. The class has 30 to 40 students. Experiments are carried out during the lessons (twice 90 minutes weekly) and during exams (2 hours each, 4 in a semester). Lessons are interactive, although students do not usually engage in group work.

- Random walks in "Metaphorland" (Paseos al azar en el país de las metáforas): A one semester optional mathematics course, open to all students of the University of Chile. The aim of the course is to introduce students of all suits to the power of metaphorical thinking in mathematics, while performing a "random walk" through several key topics, like randomness, symmetry, infinity and the systemic approach The class had 70 students in 2006 and 40 in 2007. Lessons consist mainly of group work, in small groups of 4 to 5 students. Activities and problematic situations - à la Brousseau (1998) - are proposed, to be tackled in groups by the students.

- Numbers: One yearly module (220 hrs approx.), for 5 classes of 30 primary school teachers, in the post-graduate program of the University of Chile, for in-service primary teachers who did not major in mathematics in their training. The aim of this module is to review the mathematics as well as the didactics of numbers, specially fractions, ratios, decimal and binary description of numbers. The teachers usually work in interactive sessions, forming small groups of 3 to 4 each.

The methodology consisted in observing the students and teachers, as they carried out various activities, as in the examples described below, that were proposed during lessons, group work sessions and as a part of exams and diagnostics. Records of this observation comprised videos, written and drawn production of the students and some transcriptions. Observations include those reported by the teachers, of their own classes.

In the following, "students and teachers" refers to students in the first two courses and primary school teachers in the third course.

EXPERIMENTAL ACTIVITIES AND PRELIMINARY RESULTS:

Example 1: Sum of powers of 1/4

How much is: $1/4 + (1/4)^2 + (1/4)^3 + (1/4)^4$ *?*

Could you calculate or estimate its value, without much toil?

Of course, you could use your calculator, to get 0.3320 with four digits (sequential-verbal mode).

But do yo really "see" that the sum is very close to 1/3?

How could the students tackle this problem in a nonverbal - sequential mode first?

An implicit or underlying metaphor would be: "Product is area".

So, they could try to represent, or visualize, 1/4 as a sub-region or portion of a region that stands for the whole, the unit. Most students, and in-service teachers, try this visualization (Fig. 1).

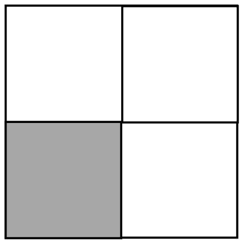


Figure 3

Then, as observed by our teachers in their classes, 5th and 6th graders tend to place $(1/4)^2$ side by side with the marked fourth of the square and so on (Figs. 4 and 5). Only roughly one out of ten teachers "see" spontaneously in this way that you are taking one out of three "all the time". Some of them have a hard time "seeing" this, even after being told so.

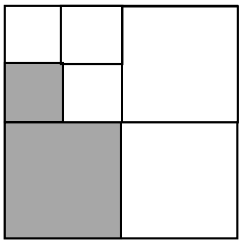


Figure 4

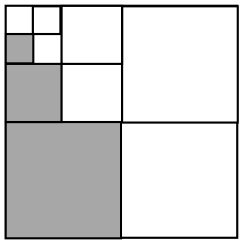


Figure 5

Older students and in-service teachers are more prone to place the smaller squares along the diagonal, as in Fig. 6. That way, they suddenly "see". Notice that this last figure affords a truly nonsequential (simultaneous) nonverbal approach to our sum. In fact, if you want to couch it in words you could say that this affords a transparent way for trisecting the square using only squares, including an efficient estimate for the "remainder", in case you stop short of infinity...

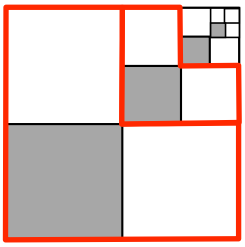


Figure 6

In our "Random Walks" course some students concocted various visualizations, like the one illustrated in Fig. 7, where succesive transversal partitions reveal a joint region that can be seen to comprise almost 1/3 of the total area, if you also mark its symmetrical twin as in Fig. 8.

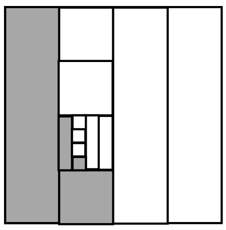


Figure 7

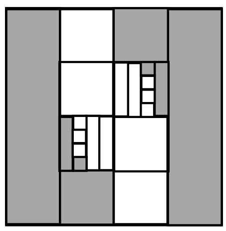


Figure 8

In Arcavi (2003) a different visualization – whose stage is the equilateral triangle - is recalled (Fig. 9). In our case studies with first year university students, in-service

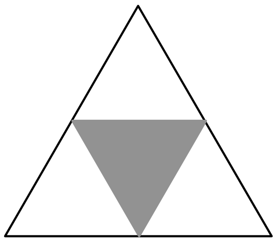


Figure 9

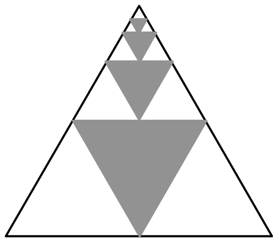


Figure 10

primary school teachers and their students (from 5th to 8th grade), this visualization didn't appear spontaneously as an alternative to the square partition. It only did after prompting. Anyway, once students and teachers got the idea of the equilateral triangle, the whole procedure unfolded very smoothly (see Fig. 10).

Example 2: Sum of powers of 1/3.

Then we suggested the case of the sum of powers of 1/3 instead of 1/4.

Students and teachers quickly visualized stepwise the triangles in Fig. 11.

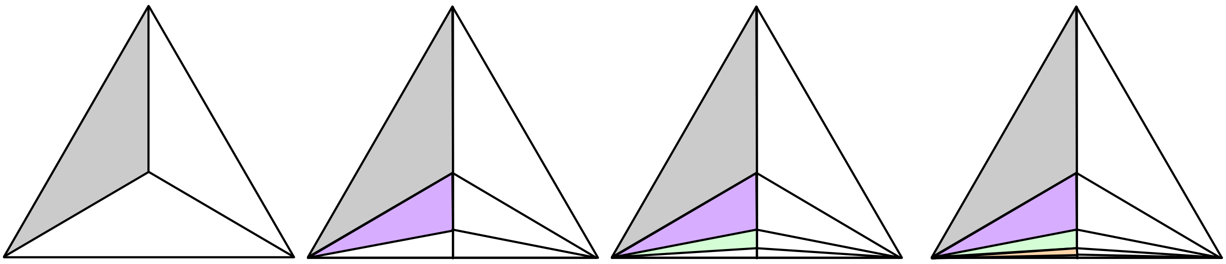


Figure 11

Which are the ingredients of these visualization processes, up to now?

Our viewpoint is that these visualizations emerge thanks to the activation of several metaphors. The crucial one is the "Product is Area", or "Fractions are Portions" metaphor, which develops into the "Powers of a Fraction arise from Iterated Partitions" metaphor. Then the "Adding is putting together" metaphor comes into play. We argue that visualization is grounded on these metaphors, which entail a switch in cognitive mode, from verbal to nonverbal (visual) and finally, from sequential to simultaneous.

Indeed, when the metaphor is activated, or "runs", a switch in cognitive mode arises: first from sequential-verbal to sequential-nonverbal when we represent our powers of 1/4 one by one, stepwise, and then to simultaneous-nonverbal, when we "see" that our sum is almost 1/3. Notice that when we compute it, we come to this conclusion stepwise, but when we visualize it, we see it at a glance, truly in a simultaneous-nonverbal mode. It is a "aha!" experience, that is unlikely to occur when we compute stepwise with the calculator.

Example 2: Sum of powers of 1/5.

How much is $1/5 + (1/5)^2 + (1/5)^3 + (1/5)^4 + (1/5)^5$?

Can you calculate, or estimate the value of this sum?

Can you "see" how much it should be?

Practically minded students would fetch their calculator and get:

$$.2 + .2\text{^}2 + .2\text{^}3 + .2\text{^}4 + .2\text{^}5 = 0.24992,$$

which is approximately .25 with an error less than .0001.

Those who like fractional yoga (a minority) would get

$1/5 + 1/25 + 1/125 + 1/625 + 1/3125 = (625 + 125 + 25 + 5 + 1)/3125 = 781/3125$,

which looks rather opaque as a result. After some pondering, they would realize however that *781 times 4 is 3124,* so that our unyielding fraction is very close to $1/4 = 781/3124$.

Or the other way around, since $3125/4 = 781\frac{1}{4}$, they can get

$$781/3125 = (781\tfrac{1}{4} - \tfrac{1}{4})/3125 = 1/4 - 1/3125 = 1/4 - 1/12500$$

so that the value of our sum is 1/4, up to an error smaller than 1/10000.

So, either the decimal way or the fractional way, we can calculate our sum. Both ways are however relentlessly verbal-sequential and they definitely disgust many students, particularly those motivated by the humanities or the social sciences.

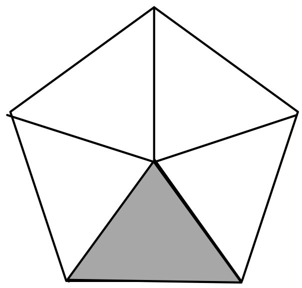


Figure 12

How could you "see" what is not visible in the preceding calculation?

When so prompted, most students try to visualize 1/5 in a pentagon, as in Fig. 12. But then, they are stuck! They see no transparent way to partition an isosceles triangle into fifths! What to do?

To be able to visualize what is less visible, we need to change our underlying metaphor!

A good idea is to revisit the more favourable case of the sum of powers of 1/4:

How could we visualize our sum with the help of a different metaphor?

After some pondering, Karel, a student in our "Random walks" course, said that the whole geometric visualisation in the square reminded him of a fern…

From this, after some discussion, the hydraulic tree in Fig. 13 emerged.

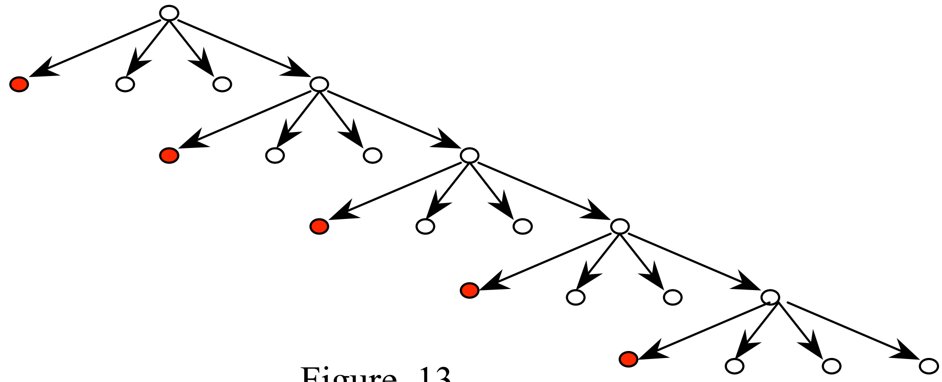


Figure 13

Indeed, the root of the tree stands for the unit square, the coloured "son", for the coloured 1/4 of the square, the coloured grandson, for the coloured $(1/4)^2$, etc. We imagine that 1 litre of water is poured into the root, and that it flows downwards, splitting with equanimity into four fourths at each node. We collect the water coming out the leftmost duct, at all stages. There are many possible ways of representing $(1/4)^2$ in this hydraulic tree, the problem is choosing the most transparent one for visualizing the value of our sum of powers of 1/4. The one shown in Fig. 13 was preferred by most students.

It is quite clear now, that this visualization carries over to the case of 1/5, or 1/3, or any 1/n, and even further…

What does this example show us?

Notice that our "Product is area " metaphor forsakes in the case of 1/5 because it doesn't help us much to see that our sum is almost 1/ 4.

To visualize further we need to switch to another metaphor for the product, one that we have "met before":

"Product is concatenation", i. e.: " To multiply is to concatenate branchings."

Notice that what we have met before is – for instance - successive enlargements or reductions of a photographic print or image.

Specially when dealing with multiplication of fractions, this metaphor is very helpful. It is closely related to the hydraulic metaphor or the pedestrian metaphor (Soto-Andrade, 2006, 2007)

So eventually, visualization of closely related mathematical objects or processes, may require the activation of different kinds of metaphors, with different scopes.

Example 4: The number sequence: a Far East visualization…

In Soto-Andrade (2007) we reported on some preliminary results concerning the following challenge, posited to students and teachers:

Is it possible to represent the usual numerical sequence, up to some number, in a non verbal and non sequential way? Let us say the sequence 0, 1, 2, 3, …,63.

We suggested first to try to represent this number sequence in a less verbal way than usual. As a preliminary, we proposed the binary activity suggested in Soto-Andrade (2006), i. e. we ask the students to get the binary description of their number in the classroom, without counting themselves first. To do this, they just stand up, try to match up in pairs, check whether there is or not one “odd man out”. Then the pairs do the same, and so on.

When the pairing game is over, we just ask whether there is an unmatched person, an unmatched pair, an unmatched quadruplet, and so on. If the answers are: YES – NO – NO – YES – NO – YES, for instance, we encourage the students to codify them in a less verbal way. Eventually they re-discover the I Ching (Yi Jing) codification: a broken line for NO, a continuous line for YES, or something equivalent. If they draw the lines from top to bottom, they get the hexagram in the second column (from the left), 6th row, in the square arrangement shown in Fig. 14. This arrangement, due to the Chinese philosopher and mathematician Shao Yong (1011-1077) represents exactly the binary sequence of numbers 0 to 63. Students and teachers successfully completed this activity and most of them reported later having understood for the first time the binary description of a number.

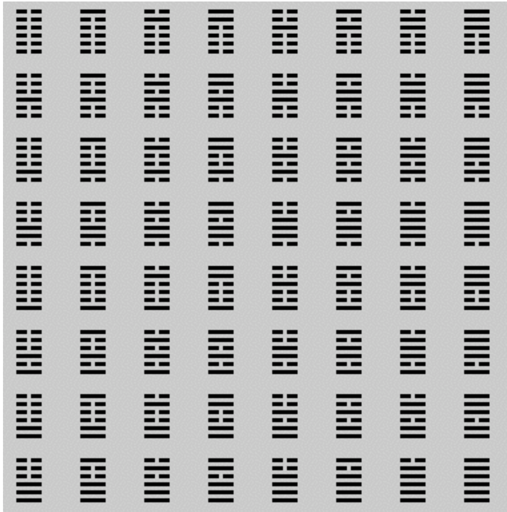


Figure 14

We are then able to visualize our number sequence as a sequence of diagrams; however we would like to push this visualization even further:

Could we draw a picture that describes the whole situation in a non verbal and simultaneous way? So that just a glance of it should be enough to reconstruct it correctly, and to get back from it the whole binary 64 hexagram sequence?

On this challenge, which they tackled in small groups, the primary school teachers did better than the first year Bachelor students: In a first class of 30 primary school teachers (25 female, 5 male), after 30 min. working in groups, when prompted to switch to non sequential cognitive style to describe the binary hexagram sequence, 5 teachers out of 30 came up with diagrams equivalent to the famous Shao Yong’s Xiantian (Before Heaven) diagram, or its inverted form, illustrated in Figs. 15 and 16, respectively.

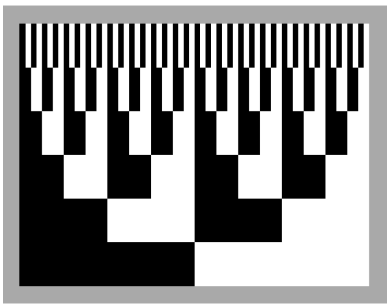


Figure 15

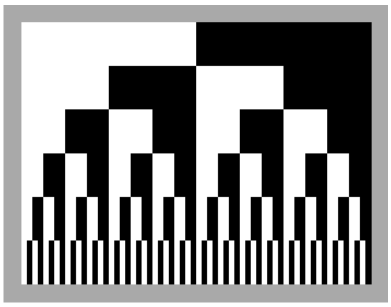
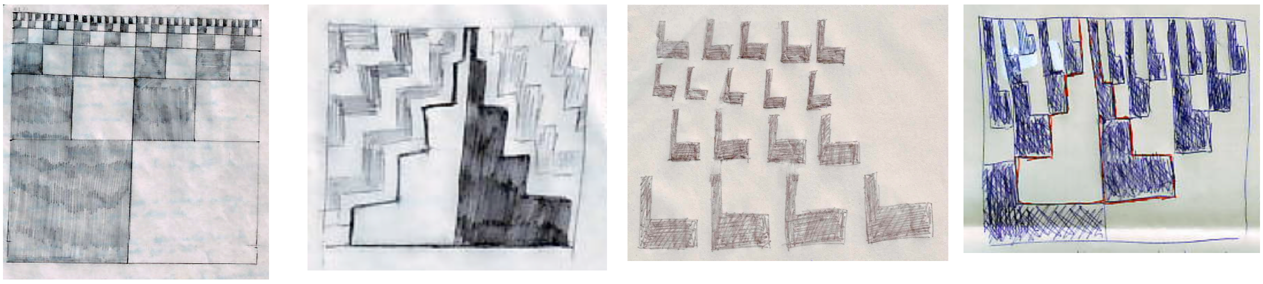


Figure 16

Later, we tried to check to what extent this image could be perceived in a glance and reconstructed thereafter. We tested it with students of our first year Math 0 course (in a final exam), and with third year M. Sc. Students (in an anonymous test), who had been already exposed to I Ching and Shao Yong's binary square.



Bárbara Math 0 Myriam Math 0 Francisca Math 0 Gerardo Math 0

Figure 17: Drawings by Math. 0 students

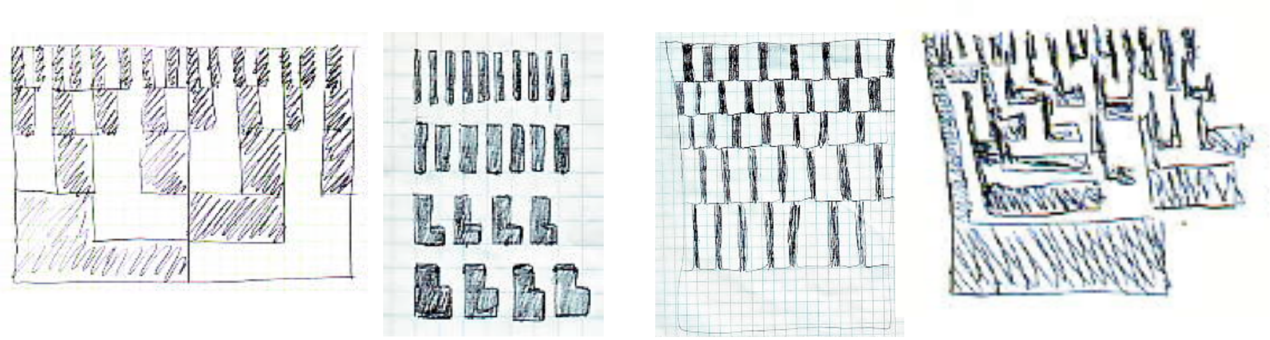


Figure 18: Drawings by anonymous M.Sc. students

As the sample in Figs 17 and 18 shows, we look at the same image but we see quite different thing. Apparently what we see, in this visualization, depends heavily on our previous history and training. Some case study with professional mathematicians and researchers in didactics gave also disparate reactions and performances. Among mathematicians, algebraists had a harder time than geometers, but one researcher in didactics of mathematics instantly "saw" the binary tree in Xian Tian. In fact, roughly one out of 10 of our students did so, showing that this image may be the oldest explicit emergence of the binary tree in the history of mankind...

Discussion

We have shown, through several activities carried out in the classroom, how some classical mathematical objects and problematic situations may be unexpectedly approached through various cognitive modes, in particular with different visualizations grounded in alternative metaphors.

We have seen that it is possible to facilitate the activation of these less usual cognitive modes and the emergence of visualization, even for in-service teachers having been trained in a unimodal (verbal-sequential) way. In fact, a high percentage of students as well as teachers were able, after some prompting, and initial effort, to begin to visualize in ways hitherto unknown to them. In this way, some of them even rediscovered visualizations of familiar mathematical objects, developed in other cultures (like the ancient Chinese, for instance), that favoured visualization more than ours : “One image is worth 10 000 words” …they said.

Our teachers noticed frequently that 3rd or 4th graders were more prone to visualize spontaneously, than 7th or 8th graders, who seemed to be much more “robotized” to calculate, with little drive to visualize. This suggests that the ability to visualize is rather repressed in children than stimulated, by traditional mathematical education.

Our observations seem to indicate however that multimodal cognitive approaches, visualizing, in particular, and transiting from one cognitive style to another, are trainable, in students as well as in teachers, specially by working in groups. First person reports by students and teachers bear witness of the impact and meaningfulness that this sort of cognitive experience has for them. Moreover, we should mention that some recent reports from our primary school teachers point out to dramatic improvements in the scores of their students in the last national assessment test (from 281 to 304 points, for instance, national average stagnating at 246 points), after several months working with explicit metaphors and visualizations, but with no specific test oriented training.

It would be interesting then to test and measure systematically the depth of learning that students may achieve when taught with ample use of visualization, with the help of a broad spectrum of metaphors and various cognitive modes.

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