Demuestre la igualdad usando propiedades de sumatorias.

$$\sum_{k=1}^{n} kr^{k-1} = \frac{1}{(1-r)^2}[1-(n+1)r^n + nr^{n+1}], r \neq 0$$

<u>Primera forma de hacerlo</u>

$$S = \sum_{k=1}^{n} kr^{k-1} \quad = \quad 1 + 2r + 3r^2 + \cdots + nr^{n-1}$$

Ahora si multplicamos ambos lados de la igualdad por r, se tiene

$$rS = \sum_{k=1}^{n} kr^{k} \quad = \quad r + 2r^2 + 3r^3 + \cdots + nr^{n}$$

si tomamos la diferencia entre la primera expresion y la segunda se tiene

$$\begin{aligned}
S(1-r) = S - rS = \quad &= 1 + 2r + 3r^2 + \cdots + nr^{n-1} - (r + 2r^2 + 3r^3 + \cdots + nr^n) \\
&= 1 + r + r^2 + \cdots + r^{n-1} - nr^n \\
&= 1 + \sum_{k=1}^{n-1} r^k - nr^n \\
&= 1 - nr^n + \frac{r - r^n}{1-r} \\
&= \frac{1 - r - nr^n(1-r) + r - r^n}{1-r} \\
&= \frac{1 - nr^n + nr^{n+1} - r^n}{1-r} \\
&= \frac{1 - r^n(n+1) + nr^{n+1}}{1-r}
\end{aligned}$$

y ahora despejando S, se tiene

$$S = \sum_{k=1}^{n} kr^{k-1} = \frac{1 - r^n(n+1) + nr^{n+1}}{(1-r)^2}$$

Segunda forma de hacerlo Primero escribiremos la suma en extenso

$$
\begin{aligned}
\sum_{k=1}^{n} kr^{k-1} &= 1 + 2r + 3r^2 + \cdots + nr^{n-1} \\
&= 1 + r + r + r^2 + r^2 + r^2 + \cdots r^{n-1} + r^{n-1} + \cdots r^{n-1} \\
&= (1 + r + r^2 + \cdots + r^{n-1}) + (r + r^2 + \cdots + r^{n-1}) + (r^2 + \cdots + r^{n-1}) + \cdots r^{n-1} \\
&= (\sum_{k=0}^{n-1} r^k) + (\sum_{k=1}^{n-1} r^k) + (\sum_{k=2}^{n-1} r^k) + \cdots (\sum_{k=n-1}^{n-1} r^k) \\
&= \sum_{j=0}^{n-1} (\sum_{k=j}^{n-1} (r^k)) \\
&= \sum_{j=0}^{n-1} (\frac{r^j - r^n}{1 - r}) \\
&= \frac{1}{1-r} \left(\sum_{j=0}^{n-1} r^j - \sum_{j=0}^{n-1} r^n \right) \\
&= \frac{1}{1-r} \left(\frac{1 - r^n}{1 - r} - nr^n \right) \\
&= \frac{1 - r^n(n+1) + nr^{n+1}}{(1-r)^2}
\end{aligned}
$$

tercera forma de hacerlo

$$\sum_{k=1}^{n} kr^{k-1} = \sum_{k=1}^{n}(k-1+1)r^{k-1} = \sum_{k=1}^{n}(k-1)r^{k-1}+r^{k-1} = r\sum_{k=1}^{n}(k-1)r^{k-2}+\sum_{k=1}^{n} r^{k-1}$$

ahora pasando la expresion $r\sum_{k=1}^{n}(k-1)r^{k-2}$ a la izquierda, se tiene

$$\begin{aligned}
\frac{1-r^n}{1-r} = \sum_{k=1}^{n} r^{k-1} &= \sum_{k=1}^{n} kr^{k-1} - r\sum_{k=1}^{n}(k-1)r^{k-2} \\
&= (r+1-r)\sum_{k=1}^{n} kr^{k-1} - r\sum_{k=1}^{n}(k-1)r^{k-2} \\
&= r\sum_{k=1}^{n}\left(kr^{k-1}-(k-1)r^{k-2}\right) + (1-r)\sum_{k=1}^{n} kr^{k-1} \\
&= r(nr^{n-1}-0) + (1-r)\sum_{k=1}^{n} kr^{k-1}
\end{aligned}$$

DE esta ultima igualdad podemos despejar la ultima sumatoria, asi obtenemos

$$(1-r)\sum_{k=1}^{n} kr^{k-1} = \frac{1-r^n}{1-r} - nr^n = \frac{1-r^n(n+1)+nr^{n+1}}{1-r}$$

por lo tanto

$$\sum_{k=1}^{n} kr^{k-1} = \frac{1-r^n(n+1)+nr^{n+1}}{1-r}$$